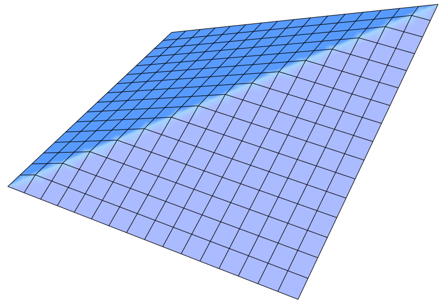

# ACTSC 231: Mathematics of Finance

Prof. Javid Ali • Fall 2013 • University of Waterloo
MWF 10:30–11:20AM in AL 116

Transcribed, LaTeXed, and edited by Rob Zimmerman

**Note:** These are personal lecture notes, not official course materials. They may contain errors (which by default you should attribute to me, Rob Zimmerman, rather than the course instructor). The permanent home of these — and all of my other lecture notes — is `https://rob-zimmerman.github.io/coursenotes`.

## Contents

# 1. The Time Value of Money

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### Lecture 1 — Monday, September 09, 2013

**Interest** is the compensation received for lending an asset. In financial settings, the asset is typically money. If you deposit money in a bank account, the bank can use that money for lending or investment, and in return you receive interest. The same idea appears outside finance: if you lend someone a car with its tank one quarter full and they return it with a full tank, then the additional fuel can be viewed as interest.

At time 0, you lend **capital**, and at a later time $T$ you receive your capital plus interest. In this course, the capital is usually monetary and is often called the **principal**. The basic transaction is summarized in Figure 1.


FIGURE 1

Money has a time value: a fixed amount today is worth more than the same amount received later. This chapter introduces the basic language and conventions for interest calculations. A timeline is often the most reliable way to organize information before solving a problem.

## Accumulation and Amount Functions

The **accumulation function**, denoted by $a(t)$, is the dollar value at time $t$ of \$1 invested at time 0. Therefore,

$$a(0) = 1.$$

The timeline with two dates in Figure 2 makes the definition explicit.


FIGURE 2

If an initial principal of $P$ is invested at time 0, the **amount function** $A(t)$ gives the value at time $t$. Thus,

$$A(0) = P \qquad \text{and} \qquad A(t) = P\,a(t).$$

This scaling by the principal is shown in Figure 3.


FIGURE 3

**Example 1** Given that

$$A(t) = 2 + e(t)\,t$$

for some function $e(t)$, find the accumulation function at times $t = 0, 1, 2, 2.5$ if

$$e(0) = 1, \qquad e(1) = 0.02, \qquad e(2) = -0.02, \qquad \text{and} \qquad e(2.5) = 0.04.$$

**Solution.** Since $A(0) = P$, we have

$$P = A(0) = 2 + e(0) \cdot 0 = 2.$$

Hence

$$a(t) = \frac{A(t)}{P} = \frac{2 + e(t)\,t}{2}.$$

Now evaluate at each time.

$$t = 0: \quad A(0) = 2, \qquad\qquad a(0) = \frac{2}{2} = 1,$$

$$\begin{aligned}
t = 1: &\quad A(1) = 2 + 0.02(1) = 2.02, & a(1) &= \frac{2.02}{2} = 1.01, \\
t = 2: &\quad A(2) = 2 - 0.02(2) = 1.96, & a(2) &= \frac{1.96}{2} = 0.98, \\
t = 2.5: &\quad A(2.5) = 2 + 0.04(2.5) = 2.10, & a(2.5) &= \frac{2.10}{2} = 1.05.
\end{aligned}$$

Therefore,

$$a(0) = 1, \qquad a(1) = 1.01, \qquad a(2) = 0.98, \qquad \text{and} \qquad a(2.5) = 1.05.$$

Let $I_{[t_1,t_2]}$ denote the amount of interest earned over $(t_1, t_2)$. Then

$$I_{[t_1,t_2]} = A(t_2) - A(t_1).$$

The **effective interest rate** from time $t_1$ to $t_2$ is

$$i_{[t_1,t_2]} = \frac{I_{[t_1,t_2]}}{A(t_1)} = \frac{A(t_2) - A(t_1)}{A(t_1)} = \frac{a(t_2) - a(t_1)}{a(t_1)}.$$

For integer periods, write $I_n$ for interest earned over $(n-1, n)$ and $i_n$ for the corresponding effective rate.

$$i_n = \frac{I_n}{A(n-1)} = \frac{a(n) - a(n-1)}{a(n-1)}.$$

$\square$

**Example 2** In Example 1, calculate the effective interest rates over $(0, 1)$, $(1, 2)$, $(2, 2.5)$, and $(0, 2.5)$.

**Solution.** From Example 1,

$$A(0) = 2, \qquad A(1) = 2.02, \qquad A(2) = 1.96, \qquad \text{and} \qquad A(2.5) = 2.10.$$

Compute each rate using

$$i_{[t_1,t_2]} = \frac{A(t_2) - A(t_1)}{A(t_1)}.$$

$$i_{[0,1]} = \frac{2.02 - 2}{2} = 0.01 = 1\%,$$

$$\begin{aligned}
i_{[1,2]} &= \frac{1.96 - 2.02}{2.02} \approx -0.02970297 \approx -2.97\%, \\
i_{[2,2.5]} &= \frac{2.10 - 1.96}{1.96} = 0.07142857 \approx 7.14\%, \\
i_{[0,2.5]} &= \frac{2.10 - 2}{2} = 0.05 = 5\%.
\end{aligned}$$

□

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### Lecture 2 — Wednesday, September 11, 2013

## Simple Interest

If an investment accumulates at **simple interest** rate $i$, then

$$a(t) = 1 + i\,t.$$

Consequently, the amount function is

$$A(t) = P(1 + i\,t),$$

and the interest earned on principal $P$ over a period of length $t$ is $P\,i\,t$.

Under simple interest, interest is earned only on the original principal, not on previously earned interest. Therefore,

$$a(t+s) \neq a(t)a(s)$$

in general.

**Example 3** \$1000 is deposited in an account that earns simple interest at 5% per year for 3 years.

(1) Calculate the account balance at the end of 3 years.

(2) Calculate the effective interest rates over $(0, 1)$, $(1, 2)$, and $(2, 3)$.

**Solution.** Here $P = 1000$, $i = 0.05$, and

$$A(t) = 1000(1 + 0.05t).$$

For (1),

$$A(3) = 1000(1 + 0.15) = 1150.$$

So the account balance after 3 years is \$1150. For (2), first compute balances at integer times.

$$A(0) = 1000, \qquad A(1) = 1050, \qquad A(2) = 1100, \qquad \text{and} \qquad A(3) = 1150.$$

Hence,

$$\begin{aligned} i_{[0,1]} &= \frac{1050 - 1000}{1000} = 0.05 = 5\%, \\ i_{[1,2]} &= \frac{1100 - 1050}{1050} = \frac{50}{1050} \approx 4.7619\%, \\ i_{[2,3]} &= \frac{1150 - 1100}{1100} = \frac{50}{1100} \approx 4.5455\%. \end{aligned}$$

The effective rate decreases over time when simple interest is used. Simple interest is quoted as an annual rate and is commonly used for short periods. With calendar dates, two standard day-count conventions are used.

- **Exact simple interest:** use the exact number of days and a 365-day year,

$$t = \frac{\text{exact number of days}}{365}.$$

- **Ordinary simple interest:** use the exact number of days and a 360-day year,

$$t = \frac{\text{exact number of days}}{360}.$$

Unless stated otherwise, simple interest problems in these notes use exact simple interest.

□

**Example 4** On August 1, 2013, \$1000 is deposited in an account earning simple interest at 5%. Calculate the account balance on September 11, 2013 using each of the following conventions:

(1) exact interest.

(2) ordinary interest.

**Solution.** The exact number of days from August 1 to September 11 is 41.

Using simple interest,

$$A = 1000\,(1 + 0.05t)\,.$$

For exact interest,

$$t = \frac{41}{365} \qquad \text{and} \qquad A_{\text{exact}} = 1000\left(1 + 0.05 \cdot \frac{41}{365}\right) \approx 1005.62.$$

For ordinary interest,

$$t = \frac{41}{360} \qquad \text{and} \qquad A_{\text{ordinary}} = 1000\left(1 + 0.05 \cdot \frac{41}{360}\right) \approx 1005.69.$$

□

## Compound Interest

If an investment accumulates at a **compound interest** rate $i$, then

$$a(t) = (1 + i)^t.$$

With compounding, previously earned interest is reinvested, so

$$a(t + s) = a(t)a(s).$$

**Example 5** Redo Example 3 assuming the account earns an annual compound interest rate of 5%.

**Solution.** Here

$$A(t) = 1000(1.05)^t.$$

At $t = 3$,

$$A(3) = 1000(1.05)^3 = 1157.625 \approx 1157.63.$$

Now compute one-year effective rates.

$$\begin{aligned}
i_{[0,1]} &= \frac{A(1) - A(0)}{A(0)} = \frac{1000(1.05) - 1000}{1000} = 5\%, \\
i_{[1,2]} &= \frac{A(2) - A(1)}{A(1)} = \frac{1000(1.05)^2 - 1000(1.05)}{1000(1.05)} = 5\%, \\
i_{[2,3]} &= \frac{A(3) - A(2)}{A(2)} = \frac{1000(1.05)^3 - 1000(1.05)^2}{1000(1.05)^2} = 5\%.
\end{aligned}$$

So with a constant annual compound rate, the effective one-year rate is constant. Simple interest can be viewed as a first-order approximation to compound interest for small rates and short horizons. In this example, after three years, simple interest gives \$1150 while compound interest gives approximately \$1157.63.

□

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### Lecture 3 — Friday, September 13, 2013

## Present Value (PV)

So far we have computed accumulated values. We also need the reverse operation: determining the value today of a payment made in the future.

If $a(t)$ is the accumulation function and an initial amount $X$ grows to 1 by time $t$, then

$$X\,a(t) = 1 \Longrightarrow X = a(t)^{-1}.$$

Thus $a(t)^{-1}$ is the **discount function**, which maps a future amount to present value. Equivalently, Figure 4 shows the amount at time 0 that accumulates to one dollar at time $t$.


FIGURE 4

Under simple interest,

$$a(t)^{-1} = \frac{1}{1 + i\,t}.$$

Under compound interest,

$$a(t)^{-1} = (1 + i)^{-t}.$$

In actuarial notation, $v = (1 + i)^{-1}$ is the **discount factor**, so $a(t)^{-1} = v^t$.

**Example 6** For each accumulation function below, determine how much should be invested now so that the account contains $1000 in five years.

(1) $a(t) = 1.05^t$,

(2) $a(t) = 1 + 0.05t$.

**Solution.** Let $X$ be the amount invested now. We need

$$X\,a(5) = 1000 \Longrightarrow X = \frac{1000}{a(5)}.$$

For (1),

$$X = \frac{1000}{1.05^5} \approx 783.53.$$

For (2),

$$X = \frac{1000}{1 + 0.05 \cdot 5} = \frac{1000}{1.25} = 800.$$

$\square$

**Example 7** You may choose one of the following lottery payment options.

(1) Option 1: \$5 million at the end of year 2 and \$5 million at the end of year 5.

(2) Option 2: \$10 million at the end of year 3.

Assuming an annual compound interest rate of 4%, which option is preferable?

**Solution.** Compare present values at time 0. In millions of dollars,

Option 1 gives

$$PV_1 = \frac{5}{1.04^2} + \frac{5}{1.04^5} \approx 8.7324,$$

while Option 2 gives

$$PV_2 = \frac{10}{1.04^3} \approx 8.8900.$$

Since $PV_2 > PV_1$, Option 2 is better.

□

**Example 8** ABC Ltd must choose between two projects.

- Project 1: initial outlay \$8000; inflows \$10000, \$4000, and \$1000 at the ends of years 1, 2, and 3.
- Project 2: initial outlay \$10000; inflows \$3500, \$6000, and \$8000 at the ends of years 1, 2, and 3.

Assume the effective annual compound rate is $i = 10\%$. Which project should be chosen?

**Solution.** Compute net present value (NPV) for each project.

For Project 1,

$$\text{NPV}_1 = -8000 + \frac{10000}{1.1} + \frac{4000}{1.1^2} + \frac{1000}{1.1^3} \approx 5148.01.$$

For Project 2,

$$\text{NPV}_2 = -10000 + \frac{3500}{1.1} + \frac{6000}{1.1^2} + \frac{8000}{1.1^3} \approx 4151.01.$$

Both projects have positive NPV, so both are profitable at a 10% benchmark rate. Since $\text{NPV}_1 > \text{NPV}_2$, Project 1 is preferred. Unless otherwise specified, we will assume a compound interest framework in subsequent sections.

□

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### Lecture 4 — Monday, September 16, 2013

## Effective Discount Rate

A **discount rate** (not to be confused with the discount factor) can sometimes be quoted instead of an interest rate. It is important to understand how these two rates differ.

Suppose you must choose between receiving \$1000 today or \$1100 in one year. If the effective interest rate is $i = 10\%$, you would be indifferent between these options. Now suppose you must choose between paying \$900 today or \$1000 in one year. If the effective discount rate is $d = 10\%$, you would again be indifferent.

In the interest rate scenario, interest is calculated on the initial balance and paid at the end of the period. In the discount rate scenario, the discount is calculated on the end-of-period balance and paid (or deducted) at the beginning of the period.

The **effective discount rate** from time $t_1$ to $t_2$ is defined as

$$d_{[t_1,t_2]} = \frac{I_{[t_1,t_2]}}{A(t_2)} = \frac{A(t_2) - A(t_1)}{A(t_2)} = \frac{a(t_2) - a(t_1)}{a(t_2)}.$$

For integer periods, write $d_n$ for the effective discount rate over $(n-1, n)$.

$$d_n = \frac{I_n}{A(n)} = \frac{a(n) - a(n-1)}{a(n)}.$$

Thus, the discount rate measures interest earned relative to the ending balance, whereas the interest rate measures interest earned relative to the initial balance.

With an effective interest rate $i$, \$1 at time 0 grows to $\$(1+i)$ at time 1. With an effective discount rate $d$, \$$(1-d)$ at time 0 grows to \$1 at time 1, as shown in Figure 5.


FIGURE 5

What is the relationship between the effective interest rate and the effective discount rate? When we say two rates are "equivalent" over a period, we mean they produce the same accumulation function over that period.

Given an effective discount rate $d$, we have \$$(1-d)$ at time 0 growing to \$1 at time 1. Under compound interest where $a(t) = (1+i)^t$, we must have

$$(1-d)\,a(1) = 1 \qquad \text{or} \qquad (1-d)(1+i) = 1.$$

Solving,

$$d = \frac{i}{1+i} = iv.$$

Thus, the discount can be interpreted as the present value of the interest earned over the period. This is intuitive: the discount is paid upfront (at the beginning), whereas interest is paid at the end.

Under the simple discount scheme, the accumulation function is

$$a(t) = (1-dt)^{-1}, \quad 0 \le t < \frac{1}{d}.$$

Under the compound discount scheme,

$$a(t) = (1-d)^{-t}, \quad t \ge 0.$$

**Example 9** Find the annual rate of compound interest equivalent to a compound discount rate of 6% per year for each of the following investment horizons:

(1) 1 year.

(2) 2 years.

Repeat for simple interest and simple discount.

**Solution.** For compound rates with horizon 1 year, use $d = iv$:

$$d = \frac{i}{1+i} \Longrightarrow i = \frac{d}{1-d} = \frac{0.06}{0.94} \approx 0.063830 = 6.383\%.$$

For a 2-year horizon, we require

$$(1-d)^{-2} = (1+i)^2 \Longrightarrow i = \left(\frac{1}{1-d}\right) - 1 \approx 0.063830 = 6.383\%.$$

The equivalence is the same for any horizon under compound rates. For simple rates over 1 year, we have

$$(1-d\cdot 1)^{-1} = 1 + i\cdot 1 \Longrightarrow i = \frac{d}{1-d} \approx 0.063830 = 6.383\%.$$

For simple rates over 2 years,

$$(1-d\cdot 2)^{-1} = 1 + i\cdot 2 \Longrightarrow i = \frac{(1-2d)^{-1}-1}{2} = \frac{1/0.88-1}{2} \approx 0.068182 = 6.818\%.$$

□

**Example 10** To encourage early payment, ABC Ltd includes in invoices the term "4/30, n/90": a 4% discount is granted if payment is made within 30 days; otherwise, full payment is due within 90 days.

Suppose you receive an invoice for \$1000, but your cash is tied up for 90 days. What is the highest exact simple interest rate at which you would borrow to take advantage of the discount?

**Solution.** Paying early costs $\$1000(1-0.04) = \$960$. Waiting 90 days costs \$1000. The period between payment dates is $90 - 30 = 60$ days.

If you borrow \$960 at rate $i$ for 60 days, the amount owed after 60 days is

$$960\left(1 + i\cdot\frac{60}{365}\right).$$

We are indifferent if this equals the 90-day price of \$1000.

$$960\left(1+i\cdot\frac{60}{365}\right)=1000 \Longrightarrow i=\frac{1000-960}{960}\cdot\frac{365}{60}\approx 0.253472=25.35\%.$$

□

**Example 11** You are required to make payments of \$200 in 1 year and \$600 in 2 years. Assuming an annual compound discount rate of 4%, find the present value.

**Solution.** First convert $d = 0.04$ to an interest rate:

$$i=\frac{d}{1-d}=\frac{0.04}{0.96}\approx 0.041667.$$

Then

$$PV = 200(1-d)+600(1-d)^2 = 200(0.96)+600(0.96)^2 = 744.96.$$

□

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**Lecture 5 — Wednesday, September 18, 2013**

## Nominal Rates of Interest and Discount

The term "effective" means that interest is paid once per measurement period. For example, an **effective annual rate** of 5% means that every dollar at the beginning of the year earns \$0.05 in interest after one year.

A **nominal rate** is an annualized rate, but interest is paid multiple times per year. Suppose a bank quotes a rate of 12% per year compounded quarterly. This is a nominal rate. The effective rate per quarter is $\frac{12\%}{4} = 3\%$.

In general, the **nominal interest rate compounded $m$-thly** is denoted by $i^{(m)}$. The effective rate per period (of length $1/m$ year) is $\frac{i^{(m)}}{m}$.

Common values for $m$ are as follows.

(1) $m = 2$ (semiannual)

(2) $m = 4$ (quarterly)

(3) $m = 12$ (monthly)

(4) $m = 365$ (daily)

You may encounter the terms **Annual Percentage Rate (APR)** and **Annual Percentage Yield (APY)**. The APR refers to the nominal rate $i^{(m)}$, while the APY refers to the annual effective rate $i$.

If \$1000 is deposited today at a nominal rate of 12% compounded quarterly, then at the end of each quarter the values are as follows.

$$\begin{aligned}
&\text{Quarter 1:} \quad 1000(1.03) = 1030,\\
&\text{Quarter 2:} \quad 1030(1.03) = 1060.90,\\
&\text{Quarter 3:} \quad 1060.90(1.03) = 1092.73,\\
&\text{Quarter 4:} \quad 1092.73(1.03) = 1125.51.
\end{aligned}$$

Interest earned over the year is \$125.51. Had the 12% been an annual effective rate, interest would be only \$120.

**Example 12** \$1000 is borrowed today for two years at a 6% nominal rate compounded monthly. Calculate the amount to be paid at the end of two years.

**Solution.** The rate per month is

$$r = \frac{0.06}{12} = 0.005.$$

Over two years (24 months),

$$A = 1000(1.005)^{24} \approx 1127.16.$$

□

**Example 13** Interest compounds monthly with an APR of 12% on a \$1000 loan. How much interest is paid at each of the following times?

(1) after 1 month?

(2) after 1 year?

**Solution.** The monthly rate is $r = \frac{0.12}{12} = 0.01$.

After 1 month,

$$\text{Interest} = 1000 \cdot 0.01 = 10.$$

After 1 year,

$$\text{Interest} = 1000\left[(1.01)^{12} - 1\right] \approx 126.83.$$

When are a nominal rate $i^{(m)}$ and an effective annual rate $i$ equivalent? They are equivalent if an initial principal $P$ grows to the same amount after one year under both rates.

$$P\left(1 + \frac{i^{(m)}}{m}\right)^m = P(1+i).$$

Hence,

$$i = \left(1 + \frac{i^{(m)}}{m}\right)^m - 1.$$

Conversely,

$$i^{(m)} = m\left[(1+i)^{1/m} - 1\right].$$

The accumulation function over $(0, t)$ is

$$a(t) = \left(1 + \frac{i^{(m)}}{m}\right)^{mt}.$$

$\square$

**Example 14** Calculate the nominal rate compounded monthly that is equivalent to an annual effective rate of 5%.

**Solution.**

$$i^{(12)} = 12\left[(1.05)^{1/12} - 1\right] \approx 0.048889 = 4.889\%.$$

□

**Example 15** Which option is preferable?

(1) 5% compounded semiannually,

(2) 5.1% compounded annually,

(3) 4.97% compounded monthly.

**Solution.** Compare effective annual rates.

For Option (A), the effective annual rate is

$$i_A = \left(1 + \frac{0.05}{2}\right)^2 - 1 \approx 0.050625 = 5.0625\%.$$

For Option (B), the effective annual rate is

$$i_B = 0.051 = 5.1\%.$$

For Option (C), the effective annual rate is

$$i_C = \left(1 + \frac{0.0497}{12}\right)^{12} - 1 \approx 0.050848 = 5.0848\%.$$

Since $i_B > i_C > i_A$, Option (B) is best. □

The nominal discount rate $d^{(m)}$ is defined similarly. The effective discount rate per period is $\frac{d^{(m)}}{m}$, paid at the beginning of each period.

Given an effective annual discount rate $d$, the equivalent **nominal discount rate compounded *m*-thly** is

$$d^{(m)} = m\left[1 - (1-d)^{1/m}\right],$$

and the accumulation function is

$$a(t) = \left(1 - \frac{d^{(m)}}{m}\right)^{-mt}.$$

**Example 16** What is the present value of \$1000 due in 3 years at a nominal discount rate of 8% compounded quarterly?

**Solution.** The rate per quarter is

$$\frac{d^{(4)}}{4} = \frac{0.08}{4} = 0.02.$$

Over 3 years (12 quarters),

$$PV = 1000\,(1-0.02)^{12} \approx 1000(0.98)^{12} \approx 784.72.$$

□

## Force of Interest

The **force of interest** is the nominal rate compounded continuously, denoted $\delta$ and sometimes written as $i^{(\infty)}$. Effective and nominal rates are more common in practice, but the force of interest is useful analytically and approximates interest that compounds very frequently (e.g., daily).

For compound interest,

$$i^{(m)} = m\left[(1+i)^{1/m} - 1\right].$$

Taking the limit as $m \to \infty$,

$$\delta = \lim_{m\to\infty} m\left[(1+i)^{1/m} - 1\right] = \log(1+i).$$

The first equality is a definition, and the second is a result.

**Proposition** Under compound interest, the force of interest is constant with respect to time.

**Proof.** Let $t = 1/m$ and $f(t) = (1+i)^t$, so the limit becomes

$$\begin{aligned}
\lim_{t\to 0^+} \frac{(1+i)^t - 1}{t} &= \lim_{t\to 0^+} \frac{f(t) - f(0)}{t - 0} \\
&= f'(0) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(1+i)^t\Big|_{t=0} \\
&= (1+i)^t \log(1+i)\Big|_{t=0} \\
&= \log(1+i).
\end{aligned}$$

□

We can describe the same accumulation in any of the following ways:

(1) a force of interest of $\delta$.

(2) a rate of interest of $\delta$ compounded continuously.

(3) a continuously compounded rate of $\delta$.

(4) a continuous rate of $\delta$.

All of these statements are equivalent.

**Example 17** What is the nominal rate compounded monthly if the force of interest is 5%?

**Solution.** We have $\delta = 0.05$. Then

$$i = e^{\delta} - 1 \approx 0.051271,$$

and

$$i^{(12)} = 12\left[(1+i)^{1/12} - 1\right] = 12\left[e^{\delta/12} - 1\right] \approx 0.049896 = 4.990\%.$$

$\square$

For a positive differentiable accumulation function $a(t)$, we can derive the force of interest at time $t$. Over a small period $\left[t, t+\frac{1}{m}\right]$ where $m$ is large, the effective rate is

$$i_{[t,t+\frac{1}{m}]} = \frac{a\left(t+\frac{1}{m}\right) - a(t)}{a(t)}.$$

The corresponding nominal rate (converting to an annual rate) is

$$i^{(m)} = m \cdot \frac{a\left(t+\frac{1}{m}\right) - a(t)}{a(t)} = \frac{1}{a(t)} \cdot \frac{a\left(t+\frac{1}{m}\right) - a(t)}{1/m}.$$

Thus, the force of interest at time $t$ is

$$\delta_t = \lim_{m\to\infty} i^{(m)} = \frac{a'(t)}{a(t)}.$$

**Proposition** We can retrieve the accumulation function over $[0, t]$ as

$$a(t) = \exp\left(\int_0^t \delta_u \,\mathrm{d}u\right).$$

**Proof.** Let $h = \frac{1}{m}$. Then

$$\delta_t = \lim_{m\to\infty} \frac{1}{a(t)} \cdot \frac{a\left(t+\frac{1}{m}\right) - a(t)}{1/m} = \frac{1}{a(t)} \cdot \lim_{h\to 0} \frac{a(t+h) - a(t)}{h} = \frac{a'(t)}{a(t)}.$$

That is,

$$\delta_t = \frac{\mathrm{d}}{\mathrm{d}t} \log(a(t)).$$

Integrating both sides from 0 to $t$ and exponentiating, we obtain

$$a(t) = \exp\left(\int_0^t \delta_u \,\mathrm{d}u\right).$$

□

**Example 18** Assume a constant force of interest $\delta$.

(1) Find the equivalent annual effective interest rate.

(2) Express $a(t)$ in terms of $i$.

(3) Show that the earlier integral formula yields the same $a(t)$.

**Solution.** (1) From $\delta = \log(1+i)$, we have

$$i = e^{\delta} - 1.$$

(2) Under compound interest,

$$a(t) = (1+i)^t = \left(e^{\delta}\right)^t = e^{\delta t}.$$

(3) Using the integral formula,

$$a(t) = \exp\left(\int_0^t \delta \, \mathrm{d}u\right) = \exp(\delta t) = \exp(\log(1+i)t) = (1+i)^t.$$

Under a constant force of interest, $a(t) = e^{\delta t}$ and $a(t)^{-1} = e^{-\delta t}$.

□

**Example 19** A project produces cash flows of \$1000 in 5 years and \$4000 in 11 years. Assuming a continuously compounded rate of 10%, what is the present value of the project's cash flows?

**Solution.** We have $\delta = 0.10$. Then

$$PV = 1000\, e^{-0.10 \cdot 5} + 4000\, e^{-0.10 \cdot 11} \approx 1000\, e^{-0.5} + 4000\, e^{-1.1} \approx 606.53 + 1331.59 = 1938.12.$$

□

The **force of discount** is obtained by taking the limit as $m \to \infty$ of the nominal discount rate, and happens to be equal to the force of interest. In general, we have

$$i > i^{(2)} > i^{(4)} > i^{(12)} > i^{(365)} > \delta = d^{(\infty)} > d^{(365)} > d^{(12)} > d^{(4)} > d^{(2)} > d.$$

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### Lecture 6 — Friday, September 20, 2013

## Varying Rates of Interest

In practice, the effective annual rate may vary over time. For instance, an investment might earn 5% for the first five years and 3% thereafter. The **annual effective rate of return** over a specified period is the single constant annual rate $i$ that produces the same accumulated value over that period.

**Example 20** A savings account earns an annual effective rate of 5% for the first 5 years and 2% thereafter.

(1) If \$1000 is invested today, what is the accumulated value after 8 years?

(2) Find the effective annual rate of return over the 8 years.

**Solution.** (1) After 5 years,

$$A(5) = 1000(1.05)^5 \approx 1276.28.$$

After 3 more years,

$$A(8) = 1276.28(1.02)^3 \approx 1354.35.$$

(2) We seek $i$ such that

$$1000(1+i)^8 = 1354.35 \Longrightarrow i = (1354.35/1000)^{1/8} - 1 \approx 0.03850 = 3.85\%.$$

□

**Example 21** You are given that

$$\delta_t = \begin{cases} 0.05, & 0 \le t \le 4, \\ \dfrac{0.05}{\sqrt{t}}, & 4 < t \le 8, \\ 0.002t + 0.05, & 8 < t \le 12, \\ \dfrac{0.05}{1+0.05t}, & t > 12. \end{cases}$$

(1) What is the value at time $t = 12$ of \$1000 invested today?

(2) What is the present value of a payment of \$1000 at time 14?

**Solution.** (1)

$$\begin{aligned} \int_0^{12} \delta_t \, \mathrm{d}t &= \int_0^4 0.05 \, \mathrm{d}t + \int_4^8 \frac{0.05}{\sqrt{t}} \, \mathrm{d}t + \int_8^{12} (0.002t + 0.05) \, \mathrm{d}t \\ &= 0.05(4) + 0.05 \left[2\sqrt{t}\right]_4^8 + (0.001t^2 + 0.05t)\Big|_8^{12} \\ &= 0.2 + 0.05(2\sqrt{8} - 2\sqrt{4}) + (0.001(144 - 64) + 0.05(12 - 8)) \\ &= 0.2 + 0.05(5.6569 - 4) + 0.08 + 0.2 \approx 0.56284. \end{aligned}$$

Hence,

$$A(12) = 1000\, e^{0.56284} \approx 1755.66.$$

(2) Continue integrating:

$$\begin{aligned} \int_{12}^{14} \delta_t \, \mathrm{d}t &= \int_{12}^{14} \frac{0.05}{1 + 0.05t} \, \mathrm{d}t = [\log(1 + 0.05t)]_{12}^{14} \\ &= \log(1.7) - \log(1.6) \approx 0.06062. \end{aligned}$$

The total integral from 0 to 14 is

$$\int_0^{14} \delta_t \, \mathrm{d}t \approx 0.56284 + 0.06062 = 0.62346.$$

Hence,

$$PV = 1000\, e^{-0.62346} \approx 536.08.$$

$\square$

**Lecture 7 — Monday, September 23, 2013**

## 2. Equivalence Equations

We want to consider the value of a stream of cash flows at some convenient date or time point. The date to which all cash flows are brought (either accumulated or discounted) is called the **focal date**. In the previous section, our focal dates were usually either at the beginning (for present values) or at the end (for accumulated values).

Consider the timeline in Figure 6.

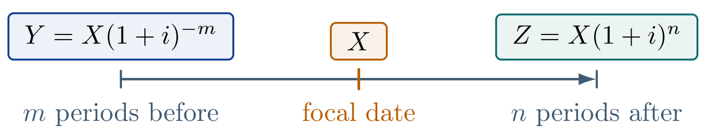


FIGURE 6

In this case, $X$, $Y$, and $Z$ are **equivalent payments** at compound interest rate $i$. A payment of $X$ at the focal date is equivalent to either of the following:

(1) an early (smaller) payment of $Y = X(1+i)^{-m}$ $m$ periods before the focal date.

(2) a late (larger) payment of $Z = X(1+i)^n$ $n$ periods after the focal date.

When working under compound interest, we have two important properties.

(1) If $X$ is equivalent to $Y$ and $Y$ is equivalent to $Z$, then $X$ is equivalent to $Z$ (**transitivity**).

(2) If two sets of payments are equivalent at one focal date, then they are equivalent at *any* focal date.

**Example 22** You must choose one of the following payment plans.

(1) Plan 1: Pay \$1000 at the end of 8 years.

(2) Plan 2: Pay \$100 today, \$200 at the end of 4 years, and $\$X$ at the end of 10 years.

You are indifferent between the two plans at an effective annual rate of 4%.

(1) Find $X$ using $t = 0$ as the focal date.

(2) Find $X$ using $t = 5$ as the focal date.

(3) Find $X$ using $t = 10$ as the focal date.

Redo this question using a simple interest rate of 4% instead.

**Solution.** Under compound interest with $i = 0.04$:

For (1), the present value of Plan 1 is

$$PV_1 = \frac{1000}{1.04^8} \approx 730.69.$$

The present value of Plan 2 is

$$PV_2 = 100 + \frac{200}{1.04^4} + \frac{X}{1.04^{10}}.$$

Setting $PV_1 = PV_2$ gives

$$\begin{aligned} 730.69 &= 100 + \frac{200}{1.04^4} + \frac{X}{1.04^{10}}. \\ 730.69 &= 100 + 170.96 + \frac{X}{1.04^{10}}. \\ X &= 459.73 \cdot 1.04^{10} \approx 680.58. \end{aligned}$$

For (2), the value of Plan 1 at $t = 5$ is

$$V_1 = \frac{1000}{1.04^3} \approx 889.00.$$

The value of Plan 2 at $t = 5$ is

$$V_2 = 100 \cdot 1.04^5 + 200 \cdot 1.04 + \frac{X}{1.04^5}.$$

Setting $V_1 = V_2$ and solving yields $X \approx 680.58$.

For (3), the value of Plan 1 at $t = 10$ is

$$V_1 = 1000 \cdot 1.04^2 = 1081.60.$$

The value of Plan 2 at $t = 10$ is

$$V_2 = 100 \cdot 1.04^{10} + 200 \cdot 1.04^6 + X.$$

Setting them equal gives

$$1081.60 = 148.02 + 253.00 + X \Longrightarrow X \approx 680.58.$$

All three focal dates yield the same $X \approx 680.58$ (minor differences due to rounding). Under simple interest at $i = 0.04$, the value of $X$ will depend on the choice of focal date because the transitivity property does not hold. Using $t = 10$ as the focal date (to find accumulated value), the value of Plan 1 at $t = 10$ is

$$V_1 = 1000(1 + 0.04 \cdot 2) = 1080.$$

The value of Plan 2 at $t = 10$ is

$$V_2 = 100(1 + 0.04 \cdot 10) + 200(1 + 0.04 \cdot 6) + X = 140 + 248 + X.$$

Setting them equal gives

$$1080 = 388 + X \Longrightarrow X = 692.$$

Under compound interest, we arrive at the same value for $X$ regardless of focal date. In the simple-interest calculation above, each cash flow is moved by applying the factor $1+i|t_2-t_1|$ over its own interval. These interval factors do not compose multiplicatively, so the resulting equation of value depends on the focal date. As a guideline for that convention, use the end of the period as focal date when finding accumulated values, or $t=0$ when finding present values.

By contrast, if a model is specified by one positive global accumulation function $a(t)$ from a common origin, the consistent value at time $t_2$ of a cash flow $X$ at time $t_1$ is

$$X \cdot \frac{a(t_2)}{a(t_1)}.$$

Those ratios are transitive by construction. They should not be confused with reapplying the simple-interest duration factor separately on every interval.

□

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### Lecture 8 — Wednesday, September 25, 2013

## Equations of Value

The general interest problem has four components.

(1) the principal invested,

(2) the length of the investment period,

(3) the interest rate, and

(4) the accumulated value at the end of the period.

Given any three, we can solve for the fourth.

When solving interest problems, use the following approach. Use a timeline to illustrate cash flows (inflows + or outflows −).

Select a convenient focal date and accumulate/discount all cash flows to that date.

The equation of value sets the sum of all discounted/accumulated cash flows at the focal date equal to the balance at that date.

**Example 23** Jennifer purchases items worth \$2000 from Hamza. She will pay Hamza \$500 at the end of 4 months, \$$X$ at the end of 8 months, and \$600 at the end of 1 year. If Hamza charges interest at $i^{(12)} = 15\%$ on the unpaid balance, what is the value of $X$?

**Solution.** The monthly rate is

$$r = \frac{0.15}{12} = 0.0125.$$

Using $t = 0$ as the focal date, the present value of Jennifer's payments must equal \$2000.

$$2000 = \frac{500}{(1.0125)^4} + \frac{X}{(1.0125)^8} + \frac{600}{(1.0125)^{12}}.$$

$$2000 = 475.76 + \frac{X}{1.1045} + 516.91.$$

$$X = 1007.33 \cdot 1.1045 \approx 1112.58.$$

□

**Example 24** In Example 23, suppose instead that Jennifer pays \$500 at the end of 4 months, \$1100 at the end of $t$ months, and \$600 at the end of 1 year. Find $t$.

**Solution.** Using $i^{(12)} = 0.15$, so $r = 0.0125$:

$$2000 = \frac{500}{(1.0125)^4} + \frac{1100}{(1.0125)^t} + \frac{600}{(1.0125)^{12}}.$$

$$2000 = 475.76 + \frac{1100}{(1.0125)^t} + 516.91.$$

$$\frac{1100}{(1.0125)^t} = 1007.33 \Longrightarrow (1.0125)^t = \frac{1100}{1007.33} \approx 1.0920.$$

$$t = \frac{\log(1.0920)}{\log(1.0125)} \approx 7.08 \text{ months.}$$

□

**Example 25** Giancarlo makes yearly payments of \$100 into an account that compounds interest annually at rate $i$. The first payment is made at the beginning of the first year. If the balance at the end of 6 years is \$829, find $i$.

**Solution.** Payments are made at times $t = 0, 1, 2, 3, 4, 5$. Using $t = 6$ as the focal date:

$$829 = 100(1+i)^6 + 100(1+i)^5 + 100(1+i)^4 + 100(1+i)^3 + 100(1+i)^2 + 100(1+i).$$

$$829 = 100(1+i)\left[(1+i)^5 + (1+i)^4 + (1+i)^3 + (1+i)^2 + (1+i) + 1\right].$$

$$829 = 100(1+i)\cdot\frac{(1+i)^6 - 1}{i}.$$

This is a nonlinear equation requiring numerical methods or a financial calculator. Rearranging gives

$$8.29i = (1+i)^7 - (1+i).$$

By trial or calculator, $i \approx 0.093158 = 9.3158\%$. In an exam, if the solution is nonlinear, you may be asked to show that $i$ satisfies a given value (by substitution) or to use linear interpolation.

□

**Example 26** If the balance at the end of 6 years is \$829, show that $9\% < i < 10\%$. Then solve for $i$ approximately using linear interpolation.

**Solution.** Define

$$f(i) = 100(1+i)\cdot\frac{(1+i)^6 - 1}{i}.$$

At $i = 0.09$, the value is

$$f(0.09) = 100(1.09)\cdot\frac{(1.09)^6 - 1}{0.09} \approx 820.04 < 829.$$

At $i = 0.10$, the value is

$$f(0.10) = 100(1.10) \cdot \frac{(1.10)^6 - 1}{0.10} \approx 848.72 > 829.$$

Since $f(0.09) < 829 < f(0.10)$, we have $0.09 < i < 0.10$. Linear interpolation gives

$$i \approx 0.09 + \frac{829 - 820.04}{848.72 - 820.04} \cdot (0.10 - 0.09) \approx 0.0931 = 9.31\%.$$

$\square$

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### Lecture 9 — Friday, September 27, 2013

## Inflation

Consider two options: receive \$100 now, or \$110 in one year. If the annual effective rate is 10%, you could invest \$100 now at 10% to receive \$110 in one year, making the options equivalent.

What if inflation is 15% while the investment still earns 10%? Suppose an item currently costs \$100 per unit and will cost \$115 per unit in one year. With option 1, you can buy one unit today. With option 2, you can buy only $\frac{110}{115} \approx 0.9565$ units in one year. The item increased by 15%, but your investment increased by only 10%. If the item has the same value to you next year, you prefer option 1.

A commonly used index for measuring inflation is the **Consumer Price Index (CPI)**, which gives the cost of a specified basket of consumer items (food, shelter, clothing, etc.). An increase in the CPI is **inflation**; a decrease is **deflation**.

The rate of return after accounting for inflation is the **real rate of return** (or **real interest rate**). It measures how much more of the basket your money can purchase. In the example, the real rate is

$$i_{\text{real}} = \frac{110 - 115}{115} \approx -0.043478 = -4.35\%.$$

A negative real rate suggests it is better to spend now than invest. However, evidence shows people often save more when returns are low.

Let $r$ denote the annual effective inflation rate and $i$ the annual effective interest rate. If one basket costs \$1 today, it costs \$$(1+r)$ in one year. If we invest our \$1, it grows to \$$(1+i)$, allowing us to purchase $\frac{1+i}{1+r}$ baskets. The real rate is therefore

$$i_{\text{real}} = \frac{1+i}{1+r} - 1 = \frac{i-r}{1+r}.$$

**Example 27** You invest \$1000 for one year at an annual effective rate of 10%. The annual inflation rate is 3%. What is your annual real rate of return?

**Solution.**

$$i_{\text{real}} = \frac{0.10 - 0.03}{1.03} \approx 0.06796 = 6.80\%.$$

□

**Example 28** The annual effective rate is 8% and the inflation rate is 6%. Let $X$ be the accumulated value at time 10 of \$1 invested at time 0, measured in time-0 dollars. Let $Y$ be the accumulated value at time 10 of \$1 invested at time 0, computed at the real rate. Find the ratio $\frac{X}{Y}$.

**Solution.** In time-0 dollars,

$$X = \frac{(1.08)^{10}}{(1.06)^{10}} = \left(\frac{1.08}{1.06}\right)^{10} \approx 1.2055.$$

The real rate is

$$i_{\text{real}} = \frac{0.08 - 0.06}{1.06} = \frac{1.08}{1.06} - 1 \approx 0.018868.$$

$$Y = (1 + i_{\text{real}})^{10} = \left(\frac{1.08}{1.06}\right)^{10} = X \approx 1.2055.$$

Thus the two calculations are simply different descriptions of the same inflation-adjusted accumulation, and

$$\frac{X}{Y} = 1.$$

□

**Example 29 (SOA Exam FM Question)** An insurance company must pay medical costs for a claimant. Average annual claims costs today are \$5000, and medical inflation is expected to be 7% per year. The claimant is expected to live an additional 20 years. Claim payments are made at yearly intervals, with the first payment one year from today.

Find the present value of the obligation if the annual interest rate is 5%.

**Solution.** The first payment (in one year) is 5000(1.07). The second payment (in two years) is $5000(1.07)^2$, and so on. The $k$-th payment is $5000(1.07)^k$ made at time $k$.

The present value is

$$\begin{aligned} PV &= \sum_{k=1}^{20} \frac{5000(1.07)^k}{(1.05)^k} = 5000 \sum_{k=1}^{20} \left(\frac{1.07}{1.05}\right)^k \\ &= 5000 \cdot \frac{1.07}{1.05} \cdot \frac{1 - \left(\frac{1.07}{1.05}\right)^{20}}{1 - \frac{1.07}{1.05}} \\ &= 5000 \cdot \frac{1.07}{1.05} \cdot \frac{1 - \left(\frac{1.07}{1.05}\right)^{20}}{-\frac{0.02}{1.05}} \approx 122633.60. \end{aligned}$$

□

## Treasury Bills

A **Treasury bill (T-bill)** is a short-term government security sold at a discount from its face value. The investor buys the bill for less than face value and receives the full face value at maturity.

Governments issue T-bills with maturities less than one year, and T-bill yields are often used as a proxy for the risk-free rate. Let $F$ be face value, $P$ be purchase price, and $n$ be the number of days to maturity.

In the U.S., the quoting convention uses a simple discount rate (bank discount yield) and a 360-day year.

$$d = \frac{F-P}{F} \cdot \frac{360}{n} \iff P = F\left(1 - d \cdot \frac{n}{360}\right).$$

In Canada, the quoting convention uses a simple interest rate and a 365-day year.

$$i = \frac{F-P}{P} \cdot \frac{365}{n} \iff P = \frac{F}{1 + i \cdot \frac{n}{365}}.$$

**Example 30** Find the price, $P$, of a U.S. T-bill quoted at 5% with 100 days to maturity and a face value of \$200.

**Solution.** For a U.S. T-bill,

$$P = F\left(1 - d \cdot \frac{n}{360}\right) = 200\left(1 - 0.05 \cdot \frac{100}{360}\right) = 200(0.986111\ldots) \approx 197.22.$$

$\square$

**Example 31** Find the price, $P$, of a Canadian T-bill quoted at 5% with 100 days to maturity and a face value of \$200.

**Solution.** For a Canadian T-bill,

$$P = \frac{F}{1 + i \cdot \frac{n}{365}} = \frac{200}{1 + 0.05 \cdot \frac{100}{365}} \approx 197.30.$$

$\square$

# 3. Annuities

### Lecture 10 — Monday, September 30, 2013

An **annuity** is a sequence of periodic (equally spaced) payments over a specified term. The **payment period/interval** is the length of time between successive payments. In this course, we focus on **annuities certain**, where the payment schedule is deterministic. A **contingent annuity** has a random term (for example, many life-contingent pension payments), and those will be treated in later courses.

The geometric series formulas are the main algebraic tools used to derive annuity expressions. If $a \in \mathbb{R}$ and $r \neq 1$, then

$$\sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + \cdots + ar^{n-1} = a\,\frac{1-r^n}{1-r}.$$

If $|r| < 1$, then

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}.$$

## Ordinary Annuity (Annuity Immediate)

An **ordinary annuity** (or **annuity immediate**) has payments at the *end* of each period. The timeline for level unit payments in Figure 7 makes this convention visible.


Figure 7

Let $i$ be the effective interest rate per payment period and $v = (1+i)^{-1}$.

The present value at time 0 is

$$a_{\overline{n|}}^{i} = v + v^2 + \cdots + v^n = \frac{1-v^n}{i}.$$

The future value at time $n$ is

$$s_{\overline{n|}}^{i} = 1 + (1+i) + \cdots + (1+i)^{n-1} = \frac{(1+i)^n - 1}{i}.$$

Also,

$$s_{\overline{n}|}^{i} = (1+i)^n a_{\overline{n}|}^{i}.$$

**Example 32** At the end of each year for the next 20 years, you deposit \$1000 into an account earning an annual effective rate of 5%. Find the present value of the deposits.

**Solution.** This is an annuity immediate with payment $R = 1000$, $n = 20$, and $i = 0.05$:

$$PV = 1000\, a_{\overline{20}|}^{0.05} = 1000 \cdot \frac{1-(1.05)^{-20}}{0.05} \approx 12462.21.$$

□

**Example 33** When purchasing a computer, you are offered two payment plans.

(1) Pay \$50 at the end of each month for 24 months.

(2) Pay \$200 at the end of each quarter for 6 quarters.

If the nominal annual rate is $i^{(12)} = 12\%$, find the present value of each plan.

**Solution.** Under $i^{(12)} = 0.12$, the effective monthly rate is

$$i_m = \frac{0.12}{12} = 0.01.$$

For (1),

$$PV_1 = 50\, a_{\overline{24}|}^{0.01} = 50 \cdot \frac{1-(1.01)^{-24}}{0.01} \approx 1062.17.$$

For (2), convert to an effective quarterly rate.

$$i_q = (1.01)^3 - 1 = 0.030301.$$

Hence,

$$PV_2 = 200\, a_{\overline{6}|}^{0.030301} = 200 \cdot \frac{1-(1.030301)^{-6}}{0.030301} \approx 1082.36.$$

So the present values are approximately \$1062.17 and \$1082.36, respectively.

□

**Example 34** Starting on his 40th birthday, Jason deposits \$2000 each year into an investment account. His last deposit is on his 65th birthday. The account earns an effective quarterly rate of 0.8%.

(1) What is the account balance just after his last deposit?

(2) Jason is now age 65 and withdraws \$300 at the end of each month for the next 10 years. If the monthly effective rate is 0.12%, find the account balance just after his last withdrawal.

**Solution.** For (1), deposits are made on birthdays $40, 41, \ldots, 65$, so there are 26 deposits at times $0, 1, \ldots, 25$. First convert to an annual effective rate.

$$i = (1+0.008)^4 - 1 \approx 0.032386.$$

Thus the balance at age 65 (just after the last deposit) is

$$B_{65} = 2000\, s_{\overline{26}|}^{i} = 2000 \cdot \frac{(1+i)^{26}-1}{i}.$$

Numerically,

$$B_{65} \approx 79683.42.$$

For (2), over the next 10 years there are 120 monthly withdrawals of \$300, with monthly rate $j = 0.0012$. The balance just after the final withdrawal is

$$B_{75} = B_{65}(1+j)^{120} - 300\, s_{\overline{120}|}^{j}.$$

Therefore,

$$B_{75} \approx 79683.42(1.0012)^{120} - 300 \cdot \frac{(1.0012)^{120}-1}{0.0012} \approx 53321.06.$$

$\square$

### Lecture 11 — Wednesday, October 02, 2013

## Annuity Due

An **annuity due** has level payments made at the *beginning* of each period. Their placement is shown in Figure 8.


FIGURE 8

At an effective interest rate of $i$ per period, the present value of the annuity due is

$$\ddot{a}^{i}_{\overline{n}|} = \frac{1 - v^n}{d}$$

and the future value is

$$\ddot{s}^{i}_{\overline{n}|} = \frac{(1+i)^n - 1}{d},$$

where

$$d = \frac{i}{1+i}.$$

Useful identities are

$$\ddot{a}^{i}_{\overline{n}|} = (1+i)a^{i}_{\overline{n}|} \qquad \text{and} \qquad \ddot{s}^{i}_{\overline{n}|} = (1+i)s^{i}_{\overline{n}|},$$

$$\ddot{a}^{i}_{\overline{n}|} = 1 + a^{i}_{\overline{n-1}|} \qquad \text{and} \qquad \ddot{s}^{i}_{\overline{n}|} = s^{i}_{\overline{n+1}|} - 1.$$

**Example 35** At the beginning of each year for the next 15 years, Mary deposits \$1000 into a savings account earning 5% effective annually. At the end of 15 years, her son starts college and tuition is paid at the beginning of each year for 4 years.

(1) Using focal date $t = 15$, find the maximum constant yearly tuition.

(2) Redo part (1) using focal date $t = 14$.

**Solution.** For (1), the savings at $t = 15$ are

$$F_{15} = 1000\, \ddot{s}_{\overline{15}|}^{0.05} \approx 22657.49.$$

If the yearly tuition is $T$, then the value at $t = 15$ of the four beginning-of-year tuition payments is

$$T\, \ddot{a}_{\overline{4}|}^{0.05}.$$

The equation of value is

$$1000\, \ddot{s}_{\overline{15}|}^{0.05} = T\, \ddot{a}_{\overline{4}|}^{0.05}.$$

Hence,

$$T = \frac{1000\, \ddot{s}_{\overline{15}|}^{0.05}}{\ddot{a}_{\overline{4}|}^{0.05}} \approx 6085.41.$$

For (2), move both streams one year earlier. The fund value is

$$F_{14} = \frac{F_{15}}{1.05} = \frac{1000\, \ddot{s}_{\overline{15}|}^{0.05}}{1.05} \approx 21578.56.$$

The tuition stream is one period deferred relative to $t = 14$, so its value is

$$T\, v\, \ddot{a}_{\overline{4}|}^{0.05}.$$

Thus,

$$\frac{1000\, \ddot{s}_{\overline{15}|}^{0.05}}{1.05} = T\, v\, \ddot{a}_{\overline{4}|}^{0.05},$$

which again gives

$$T \approx 6085.41.$$

$\square$

## Lecture 12 — Friday, October 04, 2013

### Deferred Annuity

A **deferred annuity** is an annuity whose first payment occurs later than one payment period from now. Figure 9 separates the deferral period from the subsequent unit payments.

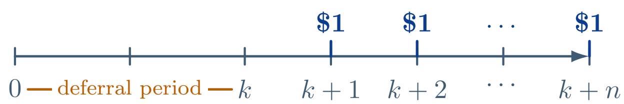


FIGURE 9

For a $k$-period deferred annuity-immediate with $n$ level payments, the present value is

$$ {}_{k|}a^i_{\overline{n}|} = v^k a^i_{\overline{n}|} = a^i_{\overline{k+n}|} - a^i_{\overline{k}|}. $$

The first equality holds because

$$ {}_{k|}a^i_{\overline{n}|} = v^{k+1} + v^{k+2} + \cdots + v^{k+n} = \sum_{j=k+1}^{k+n} v^j = v^k \sum_{j=1}^{n} v^j = v^k a^i_{\overline{n}|}, $$

and the second holds because

$$ {}_{k|}a^i_{\overline{n}|} = v^{k+1} + v^{k+2} + \cdots + v^{k+n} = \sum_{j=1}^{k+n} v^j - \sum_{j=1}^{k} v^j = a^i_{\overline{k+n}|} - a^i_{\overline{k}|}. $$

**Example 36** Joe has just turned 60. He will receive a pension of \$5000 per year beginning when he turns 65, with payments made at the beginning of each year. He is expected to die just before turning 75. Assuming an annual effective rate of 3%, find the present value of the pension at age 60.

**Solution.** Payments occur at ages $65, 66, \ldots, 74$, so there are 10 annual payments of \$5000 in an annuity due, deferred 5 years.

The value at age 65 is

$$V_{65} = 5000\, \ddot{a}^{0.03}_{\overline{10|}}.$$

Discounting to age 60 gives

$$PV = 5000\, v^5\, \ddot{a}^{0.03}_{\overline{10|}} = 5000(1.03)^{-5} \ddot{a}^{0.03}_{\overline{10|}} \approx 37894.87.$$

$\square$

**Example 37** Consider the following payment stream.

(1) \$1 at the end of each year for years 1 to 10,

(2) \$5 at the end of each year for years 11 to 20,

(3) \$7 at the end of each year for years 21 to 35.

Find the present value at an annual effective rate of 5%, assuming that payments are made at the end of the year.

**Solution.** Let $v = (1.05)^{-1}$. Break the stream into three annuities:

$$PV = a^{0.05}_{\overline{10|}} + 5v^{10} a^{0.05}_{\overline{10|}} + 7v^{20} a^{0.05}_{\overline{15|}}.$$

Therefore,

$$PV \approx 58.808.$$

$\square$

This final example illustrates how annuity formulas can be used to solve for either the term or the interest rate.

**Example 38** Rent of \$800 is paid monthly at the beginning of each month for $n$ years.

(1) If the monthly effective interest rate is 1% and the present value at the beginning of

the lease is \$30682.96, find $n$.

(2) If $n = 3$ years and the present value is \$22250.82, show that the monthly effective rate $i$ satisfies $1\% < i < 2\%$, and then approximate $i$ by linear interpolation.

**Solution.** For (1), there are $12n$ beginning-of-month payments, so

$$800\,\ddot{a}_{\overline{12n}|}^{0.01} = 30682.96.$$

Using

$$\ddot{a}_{\overline{12n}|}^{0.01} = \frac{1 - v^{12n}}{d}, \qquad d = \frac{0.01}{1.01}, \qquad \text{and} \qquad v = \frac{1}{1.01},$$

we get

$$800 \cdot \frac{1 - v^{12n}}{d} = 30682.96.$$

Solving for $n$,

$$n = \frac{\log\left(1 - \dfrac{30682.96\,d}{800}\right)}{12\log(v)} = 4.$$

For (2), now there are 36 payments and

$$800\,\ddot{a}_{\overline{36}|}^{i} = 22250.82.$$

Evaluate the left-hand side at two nearby rates.

$$800\,\ddot{a}_{\overline{36}|}^{0.01} \approx 24326.864070 \qquad \text{and} \qquad 800\,\ddot{a}_{\overline{36}|}^{0.02} \approx 20798.895466.$$

Since

$$20798.895466 < 22250.82 < 24326.864070,$$

it follows that

$$1\% < i < 2\%.$$

Linear interpolation gives

$$i \approx 1\% + \frac{24326.864070 - 22250.82}{24326.864070 - 20798.895466}(2\% - 1\%) \approx 1.588453\%.$$

So the monthly effective rate is approximately 1.59%.

$\square$

**Lecture 13 — Monday, October 07, 2013**

# 4. General Annuities

## Perpetuities (Annuities with Infinite Term)

If we let $n \to \infty$ in finite annuity formulas, we obtain annuities with infinitely many payments, called **perpetuities**. Since the term is infinite, it is not meaningful to compute a future value at the end of the term, but the present value can still be finite when $i > 0$.

For a **perpetuity-immediate**, the present value is

$$a^{i}_{\overline{\infty}|} = \lim_{n\to\infty} a^{i}_{\overline{n}|} = \lim_{n\to\infty} \frac{1-v^n}{i} = \frac{1}{i}.$$

Equivalently, by geometric series,

$$a^{i}_{\overline{\infty}|} = v + v^2 + v^3 + \cdots = \frac{v}{1-v} = \frac{1}{i}.$$

For a **perpetuity-due**, the present value is

$$\ddot{a}^{i}_{\overline{\infty}|} = \lim_{n\to\infty} \ddot{a}^{i}_{\overline{n}|} = \frac{1}{d} \qquad \text{and} \qquad d = \frac{i}{1+i}.$$

**Example 39** Joey purchases an asset that pays \$1000 at the end of each year in perpetuity. Assume the annual effective rate is $i = 5\%$.

(1) Determine the purchase price.

(2) Suppose Joey has just received a \$1000 payment and decides to sell the asset immediately. What is the selling price at that time?

**Solution.** For (1), the price is the present value of a perpetuity-immediate.

$$P = 1000\, a_{\overline{\infty}|}^{0.05} = 1000 \cdot \frac{1}{0.05} = 20000.$$

For (2), immediately after a payment date, the remaining cash flow stream is still a perpetuity-immediate of \$1000 per year. Therefore,

$$P_{\text{sell}} = 1000\, a_{\overline{\infty}|}^{0.05} = 20000.$$

□

## Annuities Where Payments Are Less Frequent Than Interest Compounding

When payments are less frequent than interest conversion, the standard approach is to convert the given interest information to an effective rate over the **payment period**. Then we apply the usual annuity formulas with that effective rate.

**Example 40** Monthly deposits of \$200 are made for 10 years into an account earning an effective daily rate of 0.05%. Deposits are made at the end of each month. Find the accumulated value at the end of the 10-year period.

**Solution.** Let the effective daily rate be $i_d = 0.0005$. Using a 365-day year, the equivalent monthly effective rate is

$$i_m = (1 + i_d)^{365/12} - 1 = (1.0005)^{365/12} - 1 \approx 0.0153207.$$

There are 120 end-of-month deposits, so this is an annuity immediate.

$$S = 200\, s_{\overline{120}|}^{i_m} = 200 \cdot \frac{(1 + i_m)^{120} - 1}{i_m} \approx 67881.54.$$

□

### Lecture 14 — Wednesday, October 09, 2013

## Annuities Where Payments Are More Frequent Than Interest Compounding

If payments are more frequent than compounding, one approach is still the **change of rate approach**: convert to an effective rate per payment period and apply annuity formulas directly.

**Example 41** Monthly deposits of \$200 are made for 10 years into an account earning $i^{(1)} = 6\%$. Deposits are made at the end of each month. Use the change of rate approach to find the accumulated value at the end of 10 years.

**Solution.** The equivalent monthly effective rate is

$$j = (1.06)^{1/12} - 1 \approx 0.00486755.$$

Hence,

$$S = 200\, s_{\overline{120}|}^{j} = 200 \cdot \frac{(1+j)^{120} - 1}{j} \approx 32494.69.$$

Another approach is the **change of payment approach**: replace many smaller payments during a compounding period by one equivalent payment at the end of that period.

$\square$

**Example 42** Repeat the previous problem using the change of payment approach.

(1) Find the single deposit at the end of each year equivalent to the 12 monthly deposits made during that year.

(2) Use that yearly equivalent payment to find the accumulated value at the end of 10 years.

**Solution.** For (1), with $j = (1.06)^{1/12} - 1$, the value at each year-end of the 12 monthly deposits is

$$Y = 200\, s_{\overline{12}|}^{j} = 200 \cdot \frac{(1+j)^{12}-1}{j} \approx 2465.31.$$

For (2), now treat this as 10 yearly end-of-year deposits at annual effective rate 0.06.

$$S = Y\, s_{\overline{10}|}^{0.06} \approx 2465.31 \cdot \frac{(1.06)^{10}-1}{0.06} \approx 32494.69.$$

As expected, this matches the change of rate approach. $\square$

**Proposition** For an annual effective rate $i$, the present and future values of a standard $m$-thly annuity-immediate are

$$a_{\overline{n}|}^{(m)} := \frac{1-v^n}{i^{(m)}} \qquad \text{and} \qquad s_{\overline{n}|}^{(m)} := \frac{(1+i)^n-1}{i^{(m)}},$$

respectively, where

$$i^{(m)} = m\left((1+i)^{1/m}-1\right).$$

Similarly, for an $m$-thly annuity-due,

$$\ddot{a}_{\overline{n}|}^{(m)} := \frac{1-v^n}{d^{(m)}} \qquad \text{and} \qquad \ddot{s}_{\overline{n}|}^{(m)} := \frac{(1+i)^n-1}{d^{(m)}},$$

with

$$d^{(m)} = m\left(1-(1+i)^{-1/m}\right).$$

This notation is standard.

**Proof.** Let $v = (1+i)^{-1}$. For an $m$-thly annuity-immediate paying $\frac{1}{m}$ at the end of each $\frac{1}{m}$-year period for $n$ years,

$$PV = \frac{1}{m}v^{1/m} + \frac{1}{m}v^{2/m} + \cdots + \frac{1}{m}v^{nm/m} = \frac{1}{m}v^{1/m} \cdot \frac{1-v^n}{1-v^{1/m}}.$$

Now,

$$\frac{v^{1/m}}{1-v^{1/m}} = \frac{(1+i)^{-1/m}}{1-(1+i)^{-1/m}} = \frac{1}{(1+i)^{1/m}-1} = \frac{m}{i^{(m)}}.$$

Hence,

$$PV = \frac{1-v^n}{i^{(m)}} = a_{\overline{n}|}^{(m)}.$$

At focal date $t = 0$, the value of the payment stream is $a_{\overline{n}|}^{(m)}$. Therefore, at focal date $t = n$, the value is

$$s_{\overline{n}|}^{(m)} = (1+i)^n a_{\overline{n}|}^{(m)} = (1+i)^n \cdot \frac{1-v^n}{i^{(m)}} = \frac{(1+i)^n - 1}{i^{(m)}}.$$

The $m$-thly annuity-due formulas are derived in the same way, replacing $i^{(m)}$ with $d^{(m)}$. $\square$

Observe that when $m = 1$, we recover our earlier expressions for the annuity-immediate and annuity-due: for example, $\ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{n}|i}^{(1)}$ and $a_{\overline{n}|} = a_{\overline{n}|i}^{(1)}$.

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## Lecture 15 — Friday, October 11, 2013

A **mortgage** is a loan with real estate as collateral. If the borrower cannot make the payments agreed to in the contract, they default on the mortgage and the lender may repossess the collateral. The **amortization period** gives the number of years (usually 20 or 25) over which payments will be made. However, the interest rate may fluctuate over this period; it is guaranteed only over a shorter **term** (usually 4 or 5 years). At the end of the term, the outstanding balance is either repaid in full or the mortgage is renegotiated at the market interest rates then in effect.

**Example 43** Alex is purchasing a house for \$200,000 and pays \$100,000 down, so the mortgage principal is \$100,000. The mortgage is to be amortized over 25 years with equal monthly payments made at the end of each month. Assume $i^{(12)} = 9\%$.

(1) Find the monthly payment.

(2) Find the outstanding principal balance at the end of 5 years.

**Solution.** The monthly effective rate is

$$j = \frac{0.09}{12} = 0.0075,$$

and there are 300 monthly payments.

For (1), if $R$ is the level monthly payment,

$$100000 = R\, a_{\overline{300}|}^{0.0075} \Longrightarrow R = \frac{100000}{a_{\overline{300}|}^{0.0075}} \approx 839.20.$$

For (2), after 5 years, 60 payments have been made and 240 payments remain. The outstanding balance is the present value at that time of remaining payments.

$$B_{60} = R\, a_{\overline{240}|}^{0.0075} \approx 93272.44.$$

As a check, the retrospective formula gives the same value.

$$B_{60} = 100000(1.0075)^{60} - R\, s_{\overline{60}|}^{0.0075} \approx 93272.44.$$

□

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### Lecture 16 — Wednesday, October 16, 2013

## Annuities with Payments in Geometric Progression

We now consider annuities where payments follow geometric growth.

> **Example 44** An annuity provides 20 annual payments. The first payment is \$1000, and each subsequent payment is 8% greater than the previous payment. If the annual effective interest rate is 7%, find the present value when the first payment is at the beginning of year 1.

**Solution.** Here $P = 1000$, $n = 20$, $g = 0.08$, and $i = 0.07$. Compute

$$i^* = \frac{0.07 - 0.08}{1.08} = -0.00925926.$$

Hence,

$$PV = 1000\,\ddot{a}_{\overline{20}|}^{i^*} = 1000 \sum_{k=0}^{19} \left(\frac{1.08}{1.07}\right)^k \approx 21879.35.$$

□

This idea generalizes. Let the first payment be $P$, and let each subsequent payment be multiplied by $(1+g)$. If the annual effective interest rate is $i$ and $i \neq g$, then a useful transformed rate is

$$i^* = \frac{i-g}{1+g}.$$

**Proposition** For $n$ annual payments, the formulas are as follows:

$$PV = \begin{cases} P\,\ddot{a}_{\overline{n}|}^{i^*}, & \text{if the first payment is at the beginning of year 1,} \\ \dfrac{P}{1+g}\,a_{\overline{n}|}^{i^*}, & \text{if the first payment is at the end of year 1.} \end{cases}$$

**Proof.** Let $v = (1+i)^{-1}$ and define

$$v^* = (1+g)v = \frac{1+g}{1+i}.$$

If $i^*$ is defined by $v^* = (1+i^*)^{-1}$, then

$$i^* = \frac{1}{v^*} - 1 = \frac{1+i}{1+g} - 1 = \frac{i-g}{1+g}.$$

If the first payment is at the beginning of year 1,

$$PV = P + P(1+g)v + \cdots + P(1+g)^{n-1}v^{n-1} = P\left(1 + v^* + \cdots + (v^*)^{n-1}\right) = P\,\ddot{a}_{\overline{n}|}^{i^*}.$$

If the first payment is at the end of year 1,

$$PV = Pv + P(1+g)v^2 + \cdots + P(1+g)^{n-1}v^n = \frac{P}{1+g}\left(v^* + (v^*)^2 + \cdots + (v^*)^n\right) = \frac{P}{1+g}\,a_{\overline{n}|}^{i^*}.$$

□

## Annuities with Payments in Arithmetic Progression

We next study annuities whose payments increase or decrease by a constant amount each period.

**Example 45** Consider an annuity with $n$ annual payments at annual effective rate $i$. The first payment is \$1, and each subsequent payment is \$1 more than the previous payment. Find the present and accumulated values in each of the following cases:

(1) the first payment is at the end of year 1. Denote these by $(Ia)\overline{n}|[i]$ and $(Is)\overline{n}|[i]$, respectively.

(2) the first payment is at the beginning of year 1. Denote these by $(I\ddot{a})\overline{n}|[i]$ and $(I\ddot{s})\overline{n}|[i]$, respectively.

**Solution.** For end-of-year payments (annuity-immediate), the present value is

$$(Ia)\overline{n}|[i] = \sum_{k=1}^{n} kv^k = \frac{\ddot{a}^i_{\overline{n}|} - nv^n}{i}.$$

Accumulating to time $n$ gives

$$(Is)\overline{n}|[i] = (1+i)^n(Ia)\overline{n}|[i] = \frac{\ddot{s}^i_{\overline{n}|} - n}{i}.$$

For beginning-of-year payments (annuity-due), the present value is

$$(I\ddot{a})\overline{n}|[i] = (1+i)(Ia)\overline{n}|[i] \qquad \text{and} \qquad (I\ddot{s})\overline{n}|[i] = (1+i)(Is)\overline{n}|[i].$$

□

**Example 46** Consider an annuity with $n$ annual payments at annual effective rate $i$. The first payment is \$$n$, and each subsequent payment is \$1 less than the previous payment. Find the present and accumulated values in each of the following cases:

(1) the first payment is at the end of year 1. Denote these by $(Da)\overline{n}|[i]$ and $(Ds)\overline{n}|[i]$,

respectively.

(2) the first payment is at the beginning of year 1. Denote these by $(D\ddot{a})_{\overline{n}|}[i]$ and $(D\ddot{s})_{\overline{n}|}[i]$, respectively.

**Solution.** For end-of-year payments, the present value is

$$(Da)_{\overline{n}|}[i] = \sum_{k=1}^{n}(n+1-k)v^k = (n+1)a^i_{\overline{n}|} - (Ia)_{\overline{n}|}[i],$$

and the accumulated value is

$$(Ds)_{\overline{n}|}[i] = (n+1)s^i_{\overline{n}|} - (Is)_{\overline{n}|}[i].$$

For beginning-of-year payments, the corresponding values are

$$(D\ddot{a})_{\overline{n}|}[i] = (1+i)(Da)_{\overline{n}|}[i] \qquad \text{and} \qquad (D\ddot{s})_{\overline{n}|}[i] = (1+i)(Ds)_{\overline{n}|}[i].$$

□

**Example 47 (SOA Exam FM Question)** A perpetuity costs 77.1 and pays annually at the end of each year. It pays 1 at the end of year 2, 2 at the end of year 3, and in general $n$ at the end of year $(n+1)$. After year $(n+1)$, payments remain level at $n$. If the annual effective interest rate is 10.5%, find $n$.

**Solution.** Let $i = 0.105$ and $v = (1+i)^{-1}$. Then

$$77.1 = \left(v^2 + 2v^3 + \cdots + nv^{n+1}\right) + n\left(v^{n+2} + v^{n+3} + \cdots\right).$$

So

$$77.1 = v\,(Ia)_{\overline{n}|}[0.105] + \frac{nv^{n+1}}{0.105}.$$

Checking nearby integers gives

$$n = 18 \Rightarrow PV \approx 75.668, \qquad n = 19 \Rightarrow PV \approx 77.097, \qquad \text{and} \qquad n = 20 \Rightarrow PV \approx 78.390.$$

Therefore, the required integer is

$$n = 19.$$

$\square$

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### Lecture 17 — Friday, October 18, 2013

## Continuously Paying Annuities

As payment frequency increases, $m$-thly annuity values converge to their counterparts with continuous payments. For fixed $n$ and $i$,

$$\lim_{m\to\infty} a_{\overline{n}|}^{(m)} = \lim_{m\to\infty} \ddot{a}_{\overline{n}|}^{(m)} = \frac{1-v^n}{\delta} \qquad \text{and} \qquad \delta = \log(1+i),$$

where $v = (1+i)^{-1}$ is the yearly discount factor.

For $n = 10$ and $i = 5\%$, the convergence is illustrated below.

| $m$ | Payment each $1/m$ year | $a_{\overline{10}\vert}^{(m)}$ | $\ddot{a}_{\overline{10}\vert}^{(m)}$ |
|---|---|---|---|
| 1 | 1 | 7.721735 | 8.107822 |
| 2 | 0.5 | 7.817079 | 8.010123 |
| 12 | $\frac{1}{12}$ | 7.897133 | 7.929306 |
| 52 | $\frac{1}{52}$ | 7.909497 | 7.916922 |
| 365 | $\frac{1}{365}$ | 7.912680 | 7.913737 |

For continuously paying annuities at annual effective rate $i \neq 0$, we write

$$\bar{a}_{\overline{n}|}^{i} = \frac{1-v^n}{\delta}, \qquad \bar{s}_{\overline{n}|}^{i} = \frac{(1+i)^n - 1}{\delta}, \qquad \text{and} \qquad \delta = \log(1+i).$$

If $\delta$ is given directly, then

$$\bar{a}_{\overline{n}|}^{i} = \frac{1-e^{-\delta n}}{\delta} \qquad \text{and} \qquad \bar{s}_{\overline{n}|}^{i} = \frac{e^{\delta n}-1}{\delta}.$$

At $i = 0$ (equivalently, $\delta = 0$), both values are $n$, as follows either directly or by continuity.

**Alternative proof.** From the $m$-thly formulas,

$$a_{\overline{n}|}^{(m)} = \frac{1 - v^n}{i^{(m)}} \qquad \text{and} \qquad \ddot{a}_{\overline{n}|}^{(m)} = \frac{1 - v^n}{d^{(m)}}.$$

As $m \to \infty$, we have $i^{(m)} \to \delta$ and $d^{(m)} \to \delta$, so

$$\lim_{m\to\infty} a_{\overline{n}|}^{(m)} = \lim_{m\to\infty} \ddot{a}_{\overline{n}|}^{(m)} = \frac{1 - v^n}{\delta} = \bar{a}_{\overline{n}|}^{i}.$$

Accumulating to time $n$ gives

$$\bar{s}_{\overline{n}|}^{i} = (1+i)^n \bar{a}_{\overline{n}|}^{i} = \frac{(1+i)^n - 1}{\delta}.$$

$\square$

More generally, for a general accumulation function $a(t)$ and continuous payment rate $f(t)$ over $[0, n]$,

$$PV = \int_0^n \frac{f(t)}{a(t)}\,\mathrm{d}t.$$

**Example 48** Find the present value of an annuity that pays continuously at annual rate $t$ at time $t$ for the next $n$ years. The annual effective interest rate is $i$.

**Solution.** Let $\delta = \log(1+i)$, so $v^t = e^{-\delta t}$. Then

$$PV = \int_0^n t v^t\,\mathrm{d}t = \int_0^n t e^{-\delta t}\,\mathrm{d}t.$$

Integration by parts gives

$$PV = \frac{1 - e^{-\delta n}(1 + \delta n)}{\delta^2} = \frac{1 - v^n(1 + \delta n)}{\delta^2}.$$

This formula applies when $i \neq 0$; when $i = 0$, the present value is $\int_0^n t\,\mathrm{d}t = n^2/2$.

$\square$

**Example 49** Consider a perpetuity that pays continuously at annual rate $(1+k)^t$ at time $t$. The annual effective interest rate is $i$, where $0 < k < i$. Find the present value.

**Solution.** The present value is

$$PV = \int_0^\infty (1+k)^t (1+i)^{-t}\, \mathrm{d}t = \int_0^\infty e^{-(\log(1+i)-\log(1+k))t}\, \mathrm{d}t.$$

Since $0 < k < i$, the exponent coefficient is positive, so the integral converges and

$$PV = \frac{1}{\log(1+i) - \log(1+k)} = \frac{1}{\log\left(\dfrac{1+i}{1+k}\right)}.$$

$\square$

**Example 50** Find the present value of a 5-year continuously paying annuity if the payment rate at time $t$ is $t^3$ and the force of interest is $\delta_t = 0.001t$ for $0 \le t \le 5$.

**Solution.** Since

$$\delta_t = \frac{a'(t)}{a(t)},$$

we have

$$a(t) = \exp\left(\int_0^t 0.001u\, \mathrm{d}u\right) = e^{0.0005t^2}.$$

Therefore,

$$PV = \int_0^5 \frac{t^3}{a(t)}\, \mathrm{d}t = \int_0^5 t^3 e^{-0.0005t^2}\, \mathrm{d}t.$$

Use $u = t^2$, so $\mathrm{d}u = 2t\, \mathrm{d}t$ and $t^3\, \mathrm{d}t = \dfrac{u}{2}\, \mathrm{d}u$.

$$\begin{aligned} PV &= \frac{1}{2}\int_0^{25} u e^{-0.0005u}\, \mathrm{d}u \\ &= \frac{1}{2}\left[-\left(\frac{u}{0.0005} + \frac{1}{0.0005^2}\right) e^{-0.0005u}\right]_0^{25} \approx 154.95. \end{aligned}$$

$\square$

### Lecture 18 — Monday, October 21, 2013

# 5. Repayment of Debts: The Amortization Method

Two common methods for repaying a loan are the **amortization method** and the **sinking fund method**. Under the amortization method, the borrower makes a sequence of periodic payments, where each payment is split into an interest portion and a portion that repays principal. Under the sinking fund method, the borrower pays interest to the lender and simultaneously deposits into a separate fund that accumulates to the principal at maturity.

For now we focus on amortization. Let $L$ be the original loan amount, repaid by $n$ level end-of-period payments of amount $R$ at effective rate $i$ per period. Then the equation of value is

$$L = R\, a^{i}_{\overline{n}|}.$$

## Outstanding Balance (Outstanding Principal)

Let $B_k$ denote the outstanding balance immediately after the $k$-th payment. There are two standard formulas.

**Retrospective method.** Accumulate the original loan and subtract the accumulated value of the first $k$ payments:

$$B_k = L(1+i)^k - R\, s^{i}_{\overline{k}|}.$$

**Prospective method.** Discount the remaining $n-k$ payments:

$$B_k = R\, a^{i}_{\overline{n-k}|}.$$

If $R$ is exact, both methods agree. If $R$ is rounded up and the final payment is reduced, the retrospective method is usually the safer direct calculation, unless the prospective approach is adjusted for the irregular final payment.

**Example 51** Olivia borrows \$10,000 for 10 years at an annual effective rate of 10%, repaid by level annual payments at year-end.

(1) Determine the level payment $R$.

(2) Determine the outstanding balance just after the 5th payment using both the prospective and retrospective methods.

(3) Suppose $R$ is rounded up to the nearest 10 cents.

   (1) Determine the concluding payment.

   (2) Determine the outstanding balance just after the 5th payment using both methods.

**Solution.** For (1), from

$$10000 = R\, a_{\overline{10}|}^{0.10},$$

we get

$$R = \frac{10000}{a_{\overline{10}|}^{0.10}} \approx 1627.4539.$$

For (2), after five payments,

$$B_5 = R\, a_{\overline{5}|}^{0.10} \approx 6169.33$$

(prospective), and

$$B_5 = 10000(1.1)^5 - R\, s_{\overline{5}|}^{0.10} \approx 6169.33$$

(retrospective). For (1), rounding up to the nearest 10 cents gives $R = 1627.5$. The concluding payment is the amount due at time 10 after nine payments of 1627.5:

$$C = \left(10000(1.1)^9 - 1627.5\, s_{\overline{9}|}^{0.10}\right)(1.1) \approx 1626.77.$$

For (2), the retrospective balance at time 5 is

$$B_5 = 10000(1.1)^5 - 1627.5\, s_{\overline{5}|}^{0.10} \approx 6169.05.$$

Since the final payment is irregular, the prospective method must be adjusted as follows.

$$B_5 = 1627.5\, a_{\overline{5}|}^{0.10} + \frac{C - 1627.5}{(1.1)^5} \approx 6169.05.$$

□

**Example 52** A loan of \$400,000 is to be repaid over 20 years by monthly payments (annuity-immediate) at nominal rate $i^{(12)} = 6\%$. After the 36th payment, the bank raises the rate to 6.5%. Find the outstanding balance immediately after the 36th payment.

**Solution.** For the original terms, the monthly effective rate is

$$j = \frac{0.06}{12} = 0.005,$$

with 240 total payments. The level payment is

$$R = \frac{400000}{a_{\overline{240}|}^{0.005}} \approx 2865.7242.$$

After 36 payments, 204 payments remain, so

$$B_{36} = R\, a_{\overline{204}|}^{0.005} \approx 365945.78.$$

The rate change affects future payments, but not the balance immediately after the 36th payment.

□

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### Lecture 19 — Wednesday, October 23, 2013

## Amortization Schedules

Let $I_k$ and $P_k$ be the interest and principal portions of the $k$-th payment. Since $R = I_k + P_k$ and interest is charged on $B_{k-1}$,

$$I_k = iB_{k-1}, \qquad P_k = R - I_k, \qquad \text{and} \qquad B_k = B_{k-1} - P_k.$$

Using the prospective formula $B_{k-1} = R\, a_{\overline{n-k+1}|}^{i}$, we also have

$$I_k = R\left(1 - v^{n-k+1}\right) \qquad \text{and} \qquad P_k = Rv^{n-k+1}.$$

These quantities allow us to construct an **amortization schedule**.

| Period ($k$) | Payment | Interest Payment, $I_k$ | Principal Payment, $P_k$ | Outstanding Balance, $B_k$ |
|---|---|---|---|---|
| 0 | – | – | – | $L = R \cdot a^{i}_{\overline{n}\rceil}$ |
| 1 | $R$ | $R \cdot (1 - v^{n})$ | $R \cdot v^{n}$ | $R \cdot a^{i}_{\overline{n-1}\rceil}$ |
| 2 | $R$ | $R \cdot (1 - v^{n-1})$ | $R \cdot v^{n-1}$ | $R \cdot a^{i}_{\overline{n-2}\rceil}$ |
| 3 | $R$ | $R \cdot (1 - v^{n-2})$ | $R \cdot v^{n-2}$ | $R \cdot a^{i}_{\overline{n-3}\rceil}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $k$ | $R$ | $R \cdot (1 - v^{n-k+1})$ | $R \cdot v^{n-k+1}$ | $R \cdot a^{i}_{\overline{n-k}\rceil}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $n-1$ | $R$ | $R \cdot (1 - v^{2})$ | $R \cdot v^{2}$ | $R \cdot a^{i}_{\overline{1}\rceil}$ |
| $n$ | $R$ | $R \cdot (1 - v)$ | $R \cdot v$ | 0 |
| Totals | $n \cdot R$ | $n \cdot R - L$ | $L$ | |

Useful totals over the full amortization period are

$$\sum_{k=1}^{n} I_k = nR - L \qquad \text{and} \qquad \sum_{k=1}^{n} P_k = L.$$

**Example 53** A loan of \$900 is amortized by 6 equal annual payments at $i^{(1)} = 10\%$. Determine the yearly payment and construct the amortization schedule.

**Solution.** The level payment is

$$R = \frac{900}{a^{0.10}_{\overline{6}\rceil}} \approx 206.6466.$$

Using $I_k = 0.10B_{k-1}$, $P_k = R - I_k$, and $B_k = B_{k-1} - P_k$, we obtain the schedule below. The calculations use the unrounded payment and balances internally; each displayed entry is then rounded independently to cents, so a displayed row can differ by one cent from the sum of its displayed components.

| Period ($k$) | Payment | Interest Payment, $I_k$ | Principal Payment, $P_k$ | Outstanding Balance, $B_k$ |
|---|---|---|---|---|
| 0 | – | – | – | 900.00 |
| 1 | 206.65 | 90.00 | 116.65 | 783.35 |
| 2 | 206.65 | 78.34 | 128.31 | 655.04 |
| 3 | 206.65 | 65.50 | 141.14 | 513.90 |
| 4 | 206.65 | 51.39 | 155.26 | 358.64 |
| 5 | 206.65 | 35.86 | 170.78 | 187.86 |
| 6 | 206.65 | 18.79 | 187.86 | 0.00 |

□

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### Lecture 20 — Friday, October 25, 2013

## Refinancing a Loan

If market rates fall during the repayment period, a borrower may refinance. To assess whether refinancing is worthwhile, compare the old and new payment structures while including any prepayment penalties.

**Example 54** A \$150,000 mortgage is amortized by monthly payments over 25 years. The interest rate is $i^{(2)} = 9.5\%$ for the first 5 years.

(1) Rounding up to the nearest cent, calculate the regular monthly payment and the reduced final payment if the 9.5% rate continued for the full 25 years.

(2) Find the outstanding balance after (i) 2 years and (ii) 5 years.

(3) During the first 5 years, early principal repayment incurs a penalty of 6 months of interest. After 2 years, rates fall to $i^{(2)} = 8\%$ for a 3-year mortgage. Find the new monthly payment if the loan is refinanced so that the outstanding balance at the end of year 5 is unchanged. Should the borrower refinance? Ignore all costs other than this penalty.

**Solution.** Let
$$j=\left(1+\frac{0.095}{2}\right)^{1/6}-1\approx 0.00776438.$$
Over 25 years there are 300 monthly payments.

For (1), the exact level payment is

$$R=\frac{150000}{a^{j}_{\overline{300}|}}\approx 1291.5414,$$

so the regular payment is $R=1291.55$ (rounded up to the nearest cent). With this rounded payment, the final payment is reduced. At month 300, the concluding payment is

$$C=\left(150000(1+j)^{299}-1291.55\,s^{j}_{\overline{299}|}\right)(1+j)\approx 1281.44.$$

For (2), outstanding balances are

$$B_{24}=150000(1+j)^{24}-1291.55\,s^{j}_{\overline{24}|}\approx 146666.52,$$

$$B_{60}=150000(1+j)^{60}-1291.55\,s^{j}_{\overline{60}|}\approx 140349.12.$$

For (3), the penalty at month 24 is six months of interest on $B_{24}$.

$$\text{Penalty}=6jB_{24}\approx 6832.65.$$

Let
$$j'=\left(1+\frac{0.08}{2}\right)^{1/6}-1\approx 0.00655820,$$
and let $X$ be the refinanced monthly payment for the next 36 months. Requiring the same balance $B_{60}$ at the end of year 5 gives

$$B_{24}+\text{Penalty}=X\,a^{j'}_{\overline{36}|}+B_{60}(1+j')^{-36}.$$

Hence,
$$X\approx 1331.72.$$

Under the old terms, 36 payments total

$$36(1291.55)=46495.80.$$

Under refinancing, 36 payments total approximately

$$36(1331.72) = 47942.02.$$

Since the refinanced payment stream is higher (and this already includes only the stated penalty cost), refinancing is not beneficial here.

$\square$

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**Lecture 21 — Monday, October 28, 2013**

# 6. Bonds

## Bond Pricing Basics

A **bond** is a loan from the investor (buyer/lender) to the issuer (borrower). Over the bond term, the investor receives periodic payments called **coupons** and, at the **maturity date** (or **redemption date**), receives the **redemption value**.

At issue (time 0), the investor pays the bond price $P$. At each coupon date, the investor receives a coupon payment, and at maturity receives the final coupon together with the redemption value.

For terminology, the **face value** or **par value** $F$ is the reference amount used to compute coupons, while the **redemption value** $C$ is the lump sum repaid at maturity. If $C = F$, the bond is said to be *redeemed at par*. If redemption is quoted as a percentage of par, convert directly; for example, "redeemable at 102" means $C = 1.02F$. The **coupon rate per payment period** $r$ determines the coupon amount $Fr$. In practice, coupon rates are often quoted as nominal rates, commonly convertible semiannually, so they should first be converted to the effective rate per coupon period. The **time to maturity** $n$ is the number of coupon periods remaining, and the **yield** (or **yield to maturity**) $i$ is the expected return per payment period if the bond is held to maturity and coupons are reinvested at rate $i$.

Let

$$F = \text{face value}, \qquad C = \text{redemption value}, \qquad \text{and} \qquad r = \text{coupon rate per payment period},$$

$$i = \text{yield rate per payment period} \qquad \text{and} \qquad n = \text{number of remaining coupon periods.}$$

Then each coupon is $Fr$, and the bond price at a coupon date is

$$P = Fr\, a_{\overline{n}|}^{i} + Cv^n \qquad \text{and} \qquad v = \frac{1}{1+i}.$$

**Example 55** Consider a bond with face value \$1000, semiannual coupons at $i^{(2)} = 5\%$, 10 years to maturity, yield $i^{(2)} = 4\%$, and redemption at par. Find the purchase price.

**Solution.** Per half-year,

$$r = \frac{0.05}{2} = 0.025, \qquad i = \frac{0.04}{2} = 0.02, \qquad n = 20, \qquad \text{and} \qquad F = C = 1000.$$

So

$$P = 1000(0.025)\, a_{\overline{20}|}^{0.02} + 1000(1.02)^{-20} \approx 1081.76.$$

□

**Example 56** A \$10,000 par value 10-year bond with 8% annual coupons is bought on November 1, 2010 to yield an annual effective rate of 8%.

(1) Find the purchase price on November 1, 2010.

(2) Find the purchase price on November 1, 2012 if the buyer at that date requires yield 9%.

**Solution.** For (1), coupon and yield are both 8%, so the bond is priced at par.

$$P_{2010} = 10000.$$

For (2), on November 1, 2012, there are 8 annual coupons remaining. Hence

$$P_{2012} = 800\, a_{\overline{8}|}^{0.09} + 10000(1.09)^{-8} \approx 9446.52.$$

$\square$

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### Lecture 22 — Wednesday, October 30, 2013

## Premium and Discount

An equivalent form for pricing the bond is

$$P = C + (Fr - Ci)\, a_{\overline{n}|}^{i}.$$

A bond is priced

$$\text{at a } \textbf{premium} \text{ if } P > C \quad (\text{premium} = P - C),$$

$$\text{at a } \textbf{discount} \text{ if } P < C \quad (\text{discount} = C - P).$$

From this formula, we obtain the following relation.

$$P > C \iff Fr > Ci \qquad \text{and} \qquad P < C \iff Fr < Ci.$$

On the investor's side, if $C$ is viewed as principal and $i$ as the desired interest rate, then the "fair" periodic interest would be $Ci$, whereas the actual coupon is $Fr$. Therefore, if $Fr = Ci$, the bond should sell for $P = C$. If $Fr < Ci$, the coupon stream is too low relative to the target yield, so the bond sells at a discount and $P < C$. If $Fr > Ci$, the coupon stream is attractive relative to the target yield, so the bond sells at a premium and $P > C$.

If a bond is redeemable at par, $F = C$, so

$$P = F + F(r - i)\, a_{\overline{n}|}^{i}.$$

Hence, when redemption is at par, we have

$$r = i \Rightarrow P = F, \qquad r > i \Rightarrow P > F, \qquad \text{and} \qquad r < i \Rightarrow P < F.$$

This form is also useful in accounting/tax discussions via **book value**: the PV of remaining bond cash flows discounted at the *original* purchase yield. For a premium bond, book value decreases from $P$ to $C$ over time; this is called **premium amortization** or a **write-down**. For a discount bond, book value increases from $P$ to $C$ over time; this is called **discount accumulation** or a

**write-up**.

The amortization schedule is as follows.

| Period ($k$) | Coupon (Payment) | Interest on Book Value, $I_k$ (Interest Payment) | Book Value Adjustment, $P_k$ (Write Down or Principal Payment) | Book Value, $B_k$ (Outstanding Balance) |
|---|---|---|---|---|
| 0 | | | | $P = C + (Fr - Ci)\, a^i_{\overline{n}\rvert}$ |
| 1 | $Fr$ | $Fr - (Fr - Ci)v^n$ | $(Fr - Ci) \cdot v^n$ | $C + (Fr - Ci)\, a^i_{\overline{n-1}\rvert}$ |
| 2 | $Fr$ | $Fr - (Fr - Ci)v^{n-1}$ | $(Fr - Ci) \cdot v^{n-1}$ | $C + (Fr - Ci)\, a^i_{\overline{n-2}\rvert}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $k$ | $Fr$ | $Fr - (Fr - Ci)v^{n-k+1}$ | $(Fr - Ci) \cdot v^{n-k+1}$ | $C + (Fr - Ci)\, a^i_{\overline{n-k}\rvert}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $n-1$ | $Fr$ | $Fr - (Fr - Ci)v^2$ | $(Fr - Ci) \cdot v^2$ | $C + (Fr - Ci)\, a^i_{\overline{1}\rvert}$ |
| $n$ | $Fr$ | $Fr - (Fr - Ci)v$ | $(Fr - Ci) \cdot v$ | $C$ |
| Totals | $n \cdot Fr$ | $n \cdot Fr - (Fr - Ci)\, a^i_{\overline{n}\rvert}$ | $\underbrace{(Fr - Ci)\, a^i_{\overline{n}\rvert}}_{=\text{Amount of the premium}}$ | |

## Lecture 23 — Friday, November 01, 2013

**Example 57** A 15-year bond with annual coupons of \$80 is bought at a premium to yield $i^{(1)} = 4\%$. If the premium write-down in the 4th payment is \$5, determine the purchase price and the write-down in the 12th payment.

**Solution.** Let

$$D = Fr - Ci.$$

For annual coupons, $n = 15$, $Fr = 80$, and $i = 0.04$. From the amortization schedule formula,

$$\text{write-down in period } k = Dv^{n-k+1}.$$

Given period 4 write-down is 5,

$$5 = Dv^{12} \Longrightarrow D = 5(1.04)^{12} \approx 8.0052.$$

Then

$$Ci = Fr - D = 80 - 8.0052 = 71.9948 \Longrightarrow C = \frac{71.9948}{0.04} \approx 1799.87.$$

Hence the purchase price is

$$P = C + D\, a_{\overline{15}|}^{0.04} \approx 1888.88.$$

The period 12 write-down is

$$Dv^{15-12+1} = Dv^4 \approx 6.84.$$

□

**Example 58** A $1000 par value 5-year bond with 3% annual coupons is purchased to yield $i^{(1)} = 4\%$. Construct the bond amortization schedule.

**Solution.** Here $F = C = 1000$, coupon $Fr = 30$, and $i = 0.04$. The price is

$$P = 30\, a_{\overline{5}|}^{0.04} + 1000(1.04)^{-5} \approx 955.481777.$$

Since the bond is purchased at a discount, the book value adjustment is negative each year; its magnitude is the annual write-up.

| Period ($k$) | Coupon (Payment) | Interest on Book Value, $I_k$ (Interest Payment) | Book Value Adjustment, $P_k$ (Write Down or Principal Payment) | Book Value, $B_k$ (Outstanding Balance) |
|---|---|---|---|---|
| 0 | | | | 955.481777 |
| 1 | 30.000000 | 38.219271 | −8.219271 | 963.701048 |
| 2 | 30.000000 | 38.548042 | −8.548042 | 972.249090 |
| 3 | 30.000000 | 38.889964 | −8.889964 | 981.139053 |
| 4 | 30.000000 | 39.245562 | −9.245562 | 990.384615 |
| 5 | 30.000000 | 39.615385 | −9.615385 | 1000.000000 |
| Totals | 150.000000 | 194.518223 | −44.518223 | |

□

**Example 59** Continuing from Example 58, determine the write-up in the third year.

**Solution.** In year 3, the book value adjustment is $-8.889964$. Therefore, the write-up magnitude is

$$8.889964 \approx 8.89.$$

$\square$

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### Lecture 24 — Monday, November 04, 2013

## Callable Bonds

For a **callable bond**, the issuer may redeem before maturity on specified call dates. The call provision is valuable to the issuer, so (all else equal) a callable bond sells at a lower price (higher yield) than an otherwise equivalent non-callable bond.

Calls are most likely when market rates fall: the issuer can redeem and refinance at lower rates. Since the issuer controls call timing, the investor must price using the redemption date or dates least favourable to the investor so the required yield is still guaranteed.

For a **bond callable at par**, the following rules apply:

$$\text{premium case } (r > i) : \text{the earliest call date is adverse to the investor,}$$

$$\text{discount case } (r < i) : \text{the latest redemption date is adverse to the investor.}$$

If call/redemption values are not all at par, this shortcut may fail; compute all relevant present values at the target yield and use the minimum price that guarantees the investor the required yield.

**Example 60** A \$1000 bond paying 8% annual coupons is redeemable at par in 10 years, but callable at par in 6 years. Determine the price that guarantees each of the following yields:

(1) 6%

(2) 10%

**Solution.** For (1), $r = 8\% > 6\%$, so this is the premium case; use earliest call date ($n = 6$).

$$P = 80\, a_{\overline{6}|}^{0.06} + 1000(1.06)^{-6} \approx 1098.35.$$

For (2), $r = 8\% < 10\%$, so this is the discount case; use latest redemption date ($n = 10$).

$$P = 80\, a_{\overline{10}|}^{0.10} + 1000(1.10)^{-10} \approx 877.11.$$

$\square$

**Example 61 (SOA Exam Question)** Sue purchases a 10-year bond with par value $X$ and semiannual coupons at nominal annual rate of 4% convertible semiannually. The price is \$1021.50. The bond can be called at par on any coupon date starting at the end of year 5. The price guarantees Sue at least $i^{(2)} = 6\%$. Find $X$.

**Solution.** Let the yield per half-year be

$$i = \frac{0.06}{2} = 0.03.$$

Coupons are $0.02X$ each half-year. Since $0.02 < 0.03$, this is the discount case for callable-at-par pricing, so the adverse date is the latest one (20 half-years).

$$1021.50 = 0.02X\, a_{\overline{20}|}^{0.03} + X(1.03)^{-20}.$$

Hence,

$$X = \frac{1021.50}{0.02\, a_{\overline{20}|}^{0.03} + (1.03)^{-20}} \approx 1200.03.$$

$\square$

**Example 62** A \$5000 callable bond pays semiannual coupons at 9.5% nominal and is redeemable at par in 20 years. It may be called at the end of year 15 for \$5250. Determine the price to yield at least $i^{(2)} = 8.5\%$.

**Solution.** Per half-year,

$$i = \frac{0.085}{2} = 0.0425 \qquad \text{and} \qquad \text{coupon} = 5000\left(\frac{0.095}{2}\right) = 237.5.$$

Because the call price is not par, compute all relevant present values at yield $i$ and take the lower price.

$$P_{15} = 237.5\, a_{\overline{30}|}^{0.0425} + 5250(1.0425)^{-30} \approx 5491.20,$$

$$P_{20} = 237.5\, a_{\overline{40}|}^{0.0425} + 5000(1.0425)^{-40} \approx 5476.93.$$

Therefore,

$$P = \min\{P_{15}, P_{20}\} = 5476.93.$$

□

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### Lecture 25 — Wednesday, November 06, 2013

## Pricing a Bond Between Coupon Dates

If a bond is sold before maturity, its market price depends on the yield prevailing at that time. Suppose just after a coupon date the bond value is

$$P_0 = Fr\, a_{\overline{n}|}^{i} + Cv^n.$$

If $d$ days have elapsed since that coupon date and $\tau$ is the number of days in one coupon period, the **full** (**dirty**) price is

$$P_{\text{dirty}} = (1+i)^{d/\tau} P_0.$$

The quoted price is called the **clean** (**flat**) price.

$$P_{\text{clean}} = P_{\text{dirty}} - Fr\left(\frac{d}{\tau}\right),$$

which subtracts the accrued coupon from the dirty price.

The dirty price is the actual transaction price. The quoted market price is usually the clean price, which removes the buildup of accrued interest and is smoother over time.

**Example 63** Consider a 2-year bond issued on August 1, 2005 with face value \$100 and 4% semiannual coupons. Assume yield $i^{(2)} = 5\%$.

(1) Calculate the price on August 1, 2005.

(2) Calculate the full (dirty) price on October 2, 2005.

**Solution.** For (1), per half-year,

$$i = \frac{0.05}{2} = 0.025, \qquad Fr = 100\left(\frac{0.04}{2}\right) = 2, \qquad \text{and} \qquad n = 4,$$

so

$$P_0 = 2\, a_{\overline{4|}}^{0.025} + 100(1.025)^{-4} \approx 98.12.$$

For (2), from August 1 to October 2 there are $d = 62$ days. The coupon period from August 1 to February 1 has $\tau = 184$ days, so

$$P_{\text{dirty}} = (1.025)^{62/184} P_0 \approx 98.94.$$

$\square$

**Example 64** Calculate the quoted (clean) price on October 2, 2005 for the bond in the previous example.

**Solution.** Using

$$P_{\text{clean}} = P_{\text{dirty}} - Fr\left(\frac{d}{\tau}\right),$$

with $P_{\text{dirty}} \approx 98.9388$, $Fr = 2$, and $d/\tau = 62/184$,

$$P_{\text{clean}} \approx 98.9388 - 2\left(\frac{62}{184}\right) \approx 98.26.$$

$\square$

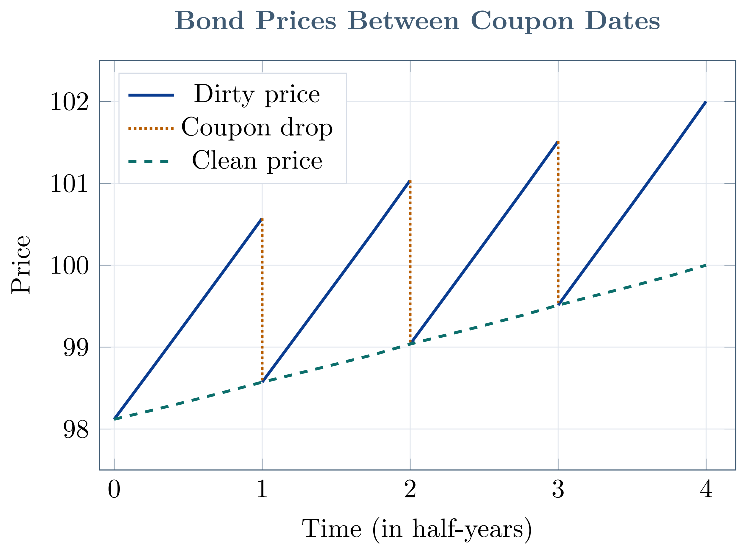


FIGURE 10

In Figure 10, the solid blue curve illustrates the behaviour of the dirty price over time. Discontinuities occur at each coupon date: just before the payment, the dirty price has accrued a full coupon period's worth of interest at rate $i$; just after the payment, the price drops by $Fr$ (the coupon amount), as indicated by the dotted orange lines. The dashed teal curve illustrates the behaviour of the clean price over time. Because the accrued coupon interest is subtracted out, the clean price evolves smoothly across coupon dates. This smoothness is precisely why the clean price is the quoted convention: if the clean price were ever to move abruptly, it would indicate a change in market conditions—such as a shift in market interest rates or a downgrade of the issuer's credit rating—rather than the routine effect of coupon accrual.

Just after a coupon date, the **book value** is the PV of remaining cash flows discounted at the *original* yield (at issue). The **selling price** is the PV of those same remaining cash flows discounted at the *current market* yield, so the two need not coincide.

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## Lecture 26 — Friday, November 08, 2013

## Zero Coupon Bonds

A **zero coupon bond** (also called a **discount bond** or **deep-discount bond**) has no periodic coupons ($r = 0$). Therefore its price is simply the present value of the redemption payment.

Compared with coupon bonds, zero-coupon bonds are often useful when targeting a specific future date/amount and when avoiding coupon reinvestment uncertainty.

**Example 65** Find the price of a \$1000 zero coupon bond with term 5 years if the yield rate is $i^{(2)} = 8\%$.

**Solution.** The half-year yield is

$$i = \frac{0.08}{2} = 0.04,$$

and there are 10 half-year periods in 5 years. Hence

$$P = 1000(1.04)^{-10} \approx 675.56.$$

□

## Yield Rates and the Method of Averages

Given price, coupon, redemption value, and term, we typically find the exact yield numerically. For a quick estimate from $P$, coupon amount $Fr$, redemption value $C$, and term $n$, we can use the method of averages:

$$\hat{i} = \frac{\text{average interest payment}}{\text{average amount invested}},$$

where

$$\text{average interest payment} = \frac{nFr + C - P}{n} \qquad \text{and} \qquad \text{average amount invested} = \frac{P + C}{2}.$$

**Example 66** Using the method of averages, find the approximate yield on a 10-year \$1000 bond with 5% annual coupons priced at \$1020.

**Solution.** Here

$$n = 10, \qquad Fr = 1000(0.05) = 50, \qquad C = 1000, \qquad \text{and} \qquad P = 1020.$$

So

$$\text{average interest payment} = \frac{10(50) + 1000 - 1020}{10} = 48,$$

$$\text{average amount invested} = \frac{1020 + 1000}{2} = 1010.$$

Therefore,

$$\hat{i} = \frac{48}{1010} \approx 0.04752 = 4.75\%.$$

This value is a quick approximation and can be refined by interpolation if a more accurate yield is needed.

□

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**Lecture 27 — Monday, November 11, 2013**

# 7. General Cash Flows and Portfolios (Rates of Return)

The term **present value** is used when all cash flows are discounted to a focal date, while **future value** is used when all cash flows are accumulated to a focal date. More generally, an investment can be valued at any intermediate date by accumulating earlier cash flows forward and discounting later cash flows backward.

We now study three return measures.

yield rate (internal rate of return), dollar-weighted rate, and time-weighted rate.

## Yield Rate (Internal Rate of Return)

The **yield rate**, or **internal rate of return (IRR)**, is the effective rate that equates the present value of inflows and outflows. It is often interpreted as the investment's expected return under reinvestment at the same rate.

It measures how attractive an investment is. As a decision metric, higher yield is preferred by investors (lenders), while lower yield is preferred by borrowers.

**Example 67** Consider a 4-year investment project in which \$10,000 is invested at time 0, an additional \$5,000 is invested at time 1, and returns of \$8,000, \$7,000, and \$3,000 are received at times 2, 3, and 4, respectively. Show that the IRR is 7.971893%.

**Solution.** Let $i$ be the IRR and $v = (1+i)^{-1}$. The equation of value at time 0 is

$$10000 + 5000v = 8000v^2 + 7000v^3 + 3000v^4.$$

Equivalently,

$$-10000 - 5000v + 8000v^2 + 7000v^3 + 3000v^4 = 0,$$

and plugging in

$$v = (1 + 0.07971893)^{-1}$$

confirms that $v$ is an (approximate) root of the quartic polynomial above.

□

Note that a yield may fail to exist or may be non-unique for some cash flow patterns. A common uniqueness condition is that net cash flows change sign only once (which follows from Descartes' rule of signs).

The IRR also assumes reinvestment at that same rate. If realized reinvestment rates differ, the actual return will differ from the stated yield.

**Example 68** In the previous example, suppose the net cash flows received at the ends of years $2, 3, 4$ are reinvested at 9%. Find the actual effective annual return over the 4 years.

**Solution.** First accumulate the received cash flows to time 4 at 9%:

$$FV_4 = 8000(1.09)^2 + 7000(1.09) + 3000 = 20134.8.$$

Let $j$ be the actual annual return over the full project horizon. Then

$$10000(1+j)^4 + 5000(1+j)^3 = 20134.8.$$

Solving gives

$$j \approx 0.083393 = 8.34\%.$$

□

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### Lecture 28 — Wednesday, November 13, 2013

## Dollar-Weighted Rate of Return

Let $A$ and $B$ be the fund balances at times 0 and 1, respectively. At each time $t$, let $c_t$ be a contribution and $w_t$ a withdrawal, and write

$$c = \sum_t c_t \qquad \text{and} \qquad w = \sum_t w_t$$

for the total contributions and withdrawals. Then

$$I = B - A - c + w$$

is the interest earned during the year. How can we determine the rate of return on the portfolio over the year? In general, this can be challenging. It is much easier if we assume simple interest for any fractional part of the year:

**Proposition** Using the simple interest approximation $(1+i)^t \approx 1 + i \cdot t$, the rate of return is

$$i_{\text{DW}} = \frac{I}{A + \sum_t c_t(1-t) - \sum_t w_t(1-t)}.$$

This is called the **dollar-weighted rate of return**.

**Proof.** Take the focal date at $t = 1$. Under the simple interest approximation, a contribution $c_t$ made at time $t$ accumulates to approximately

$$c_t\big(1 + i(1-t)\big),$$

while a withdrawal $w_t$ made at time $t$ reduces the year-end fund by approximately

$$w_t\big(1 + i(1-t)\big).$$

Therefore,

$$B = A(1+i) + \sum_t c_t\big(1 + i(1-t)\big) - \sum_t w_t\big(1 + i(1-t)\big).$$

Expanding and grouping the terms involving $i$ gives

$$B = A + c - w + i\left(A + \sum_t c_t(1-t) - \sum_t w_t(1-t)\right).$$

Since

$$I = B - A - c + w,$$

we obtain

$$I = i\left(A + \sum_t c_t(1-t) - \sum_t w_t(1-t)\right),$$

and hence

$$i = \frac{I}{A + \sum_t c_t(1-t) - \sum_t w_t(1-t)}.$$

□

This is also called the **money-weighted return**. It is sensitive to the size and timing of contributions/withdrawals.

**Example 69 (SOA Exam FM Question)** An association has fund balance 75 on January 1 and 60 on December 31. At each month-end during the year, it deposits 10. Withdrawals are 5 on February 28, 25 on June 30, 80 on October 15, and 35 on October 31. Calculate the dollar-weighted rate of return for the year.

**Solution.** The exact IRR from a financial calculator is

$$i \approx 10.97797525\%.$$

For the dollar-weighted approximation, use month fractions of the year and take October 15 as 9.5/12 of a year.

$$I = B - A - c + w = 60 - 75 - 120 + 145 = 10.$$

Also,

$$\sum_t c_t(1-t) = 10\left(\frac{11+10+\cdots+1+0}{12}\right) = 55,$$

$$\sum_t w_t(1-t) = 5\left(\frac{10}{12}\right) + 25\left(\frac{6}{12}\right) + 80\left(\frac{2.5}{12}\right) + 35\left(\frac{2}{12}\right) = 39.1667.$$

Hence

$$i_{\text{DW}} = \frac{10}{75+55-39.1667} \approx 0.11009 = 11.01\%.$$

□

One drawback of the dollar-weighted rate of return is that it is highly sensitive to the timing and amount of contributions and withdrawals. To see this, consider a fund with balance \$1000 at the beginning of the year, \$500 after 6 months, and \$1000 at year-end.

If no contributions or withdrawals are made, then the fund simply falls to \$500 midway through the year and recovers to \$1000 by year-end. In this case,

$$I = 1000 - 1000 = 0 \qquad \text{and} \qquad i_{\text{DW}} = 0\%.$$

This correctly reflects that the overall annual return is 0%.

Now suppose instead that the investor contributes \$500 after 6 months. Then the year-end balance would be \$2000. In that case,

$$I = 2000 - 1000 - 500 = 500,$$

and since the contribution is made halfway through the year,

$$i_{\text{DW}} = \frac{500}{1000 + 500(1 - 0.5)} = \frac{500}{1250} = 40\%.$$

On the other hand, if the investor withdraws \$500 after 6 months, then the year-end balance would be \$0. In that case,

$$I = 0 - 1000 + 500 = -500,$$

so

$$i_{\text{DW}} = \frac{-500}{1000 - 500(1 - 0.5)} = \frac{-500}{750} = -66.67\%.$$

Neither 40% nor −66.67% is a good measure of the fund's underlying performance, since the investment path itself still corresponds to a 0% return before the effect of external cash flows. The point is that the dollar-weighted rate gives more influence to periods when more money is actually invested. That is why it is called *dollar-weighted*. If an investor contributes more money just before a period of strong performance, the measured return increases; if the investor contributes more just before a period of weak performance, the measured return decreases. Thus, the dollar-weighted return reflects not only fund performance, but also how lucky or unlucky the investor was in timing contributions and withdrawals.

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### Lecture 29 — Friday, November 15, 2013

## Time-Weighted Rate of Return

Suppose external cash flows divide the year into $n+1$ subperiods. Let $y_k$ be the return in subperiod $k$ computed from balances immediately before cash flows. Then the **time-weighted rate of interest/return** is

$$1 + i_{\text{TW}} = \prod_{k=1}^{n+1} (1 + y_k).$$

Compared with dollar-weighted return, time-weighted return is designed to remove the effects of investor cash flow timing and better reflect underlying fund performance.

**Example 70** An investor buys \$2000 of stock today. After 2 months the value is \$2200, and she buys another \$1000 of stock. After another 6 months the value is \$2700, and she sells \$800 of stock. Four months later, the stock value is \$2100. Calculate her time-weighted and dollar-weighted rates of return.

**Solution.** *Time-weighted return.* The subperiod returns are

$$y_1 = \frac{2200}{2000} - 1 = 0.10,$$

$$y_2 = \frac{2700}{2200+1000} - 1 = \frac{2700}{3200} - 1 = -0.15625,$$

$$y_3 = \frac{2100}{2700-800} - 1 = \frac{2100}{1900} - 1 = 0.105263.$$

So

$$i_{\mathrm{TW}} = (1+y_1)(1+y_2)(1+y_3) - 1 \approx 0.02582 = 2.58\%.$$

*Dollar-weighted return.* Here

$$I = 2100 - 2000 - 1000 + 800 = -100,$$

and

$$A + \sum_t c_t(1-t) - \sum_t w_t(1-t) = 2000 + 1000\left(1 - \frac{2}{12}\right) - 800\left(1 - \frac{8}{12}\right) = 2566.6667.$$

Therefore,

$$i_{\mathrm{DW}} = \frac{-100}{2566.6667} \approx -0.03896 = -3.90\%.$$

□

## Stocks

We close with preferred and common stock valuation via present value of dividends (dividend discount modeling).

**Bondholders** are lenders to the firm, whereas **stockholders** are owners of equity. Bondholders generally have priority over stockholders if the firm defaults. Unlike bonds, stocks pay dividends rather than coupons, have no fixed maturity date, and failure to pay dividends does not automatically constitute default.

**Preferred stock** typically has fixed dividends; **common stock** dividends are variable and linked to firm performance/growth.

For preferred stock with constant dividend $D$ each period and effective rate $i$ per period,

$$P = \frac{D}{i}.$$

For common stock with first dividend $D$ one period from now, growth rate $g$ per period, and required yield $i$ per period, assume $-1 < g < i$ so that the dividend stream is well defined and its present value converges. Then

$$P = \frac{D}{i - g}.$$

**Example 71** Calculate the price of a share of preferred stock paying quarterly dividends of \$3 if the effective quarterly interest rate is 4%.

**Solution.** This is a perpetuity-immediate with $D = 3$ and $i = 0.04$, so

$$P = \frac{3}{0.04} = 75.$$

□

**Example 72** A stock pays quarterly dividends with first dividend \$3 due in 3 months. The company is expected to grow at $i^{(4)} = 8\%$. Calculate the share price if the investor's effective quarterly yield is 2.5%.

**Solution.** The quarterly dividend growth rate is

$$g = \frac{0.08}{4} = 0.02, \qquad i = 0.025, \qquad \text{and} \qquad D = 3.$$

Hence

$$P = \frac{D}{i-g} = \frac{3}{0.025 - 0.02} = 600.$$

$\square$

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**Lecture 30 — Monday, November 18, 2013**

# 8. Term Structure of Interest Rates

## Yield Curves

The **term structure of interest rates** describes how yields vary with time to maturity. A graph of this relationship is called a **yield curve**.

At issue, bond terms (coupon, redemption, maturity) are set and market pricing determines the purchase price; the yield is then implied by those quantities. In practice, one often studies government curves (e.g., treasury bills/notes/bonds) because of liquid pricing. Any yield curve should compare securities with similar risk. Common shapes include the upward-sloping (normal) curve in Figure 11, the inverted curve in Figure 12, and the flat curve in Figure 13.

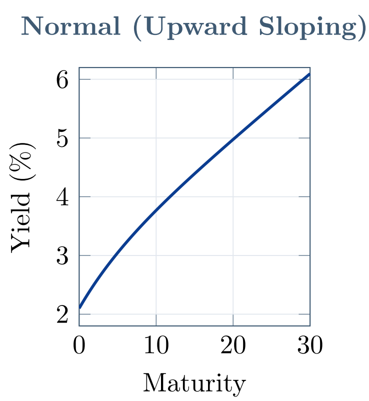


FIGURE 11

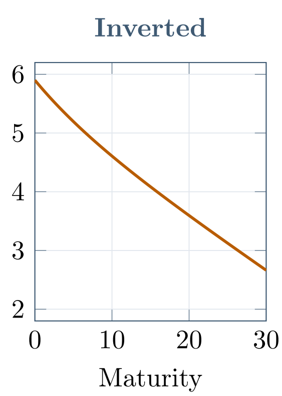


FIGURE 12

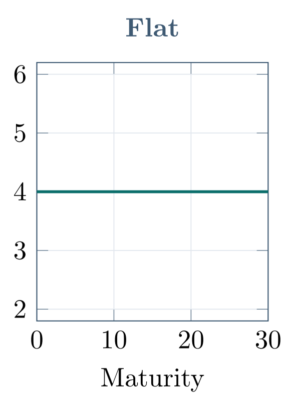


FIGURE 13

The key point for pricing is that, in general, one rate does not fit all maturities.

## Lecture 31 — Wednesday, November 20, 2013

### Spot Rates

Let $P(0,t)$ be the price at time 0 of a $t$-year zero coupon bond with face value 1. The $t$-year **spot rate** $s_t$ satisfies

$$P(0,t) = (1+s_t)^{-t}.$$

Equivalently, $s_t$ is the effective annual borrowing/lending rate locked in over the next $t$ years.

For a general fixed income cash flow stream $A_t$ at times $t = 1, 2, \ldots, n$, pricing by spot rates is

$$P = \sum_{t=1}^{n} A_t(1+s_t)^{-t}.$$

This can be viewed as decomposing the cash flow stream into zero-coupon components, each discounted at the spot rate for its maturity.

**Example 73** You are given four zero coupon bonds, each with face value \$100.

| Bond | Maturity (years) | Price |
|---|---|---|
| 1 | 1 | 96.618 |
| 2 | 2 | 92.456 |
| 3 | 3 | 87.630 |
| 4 | 4 | 82.270 |

Compute annual effective spot rates $s_1, s_2, s_3, s_4$.

**Solution.** Using $P(0,t) = 100(1+s_t)^{-t}$:

$$s_1 = \frac{100}{96.618} - 1 \approx 0.035004 = 3.50\%,$$

$$s_2 = \left(\frac{100}{92.456}\right)^{1/2} - 1 \approx 0.039998 = 4.00\%,$$

$$s_3 = \left(\frac{100}{87.630}\right)^{1/3} - 1 \approx 0.044999 = 4.50\%,$$

$$s_4 = \left(\frac{100}{82.270}\right)^{1/4} - 1 \approx 0.050001 = 5.00\%.$$

□

**Example 74** Determine the price of a 4-year \$1000 bond with 5% annual coupons using the spot rates from Example 73.

**Solution.** Cash flows are \$50 at years $1, 2, 3$ and \$1050 at year 4. Thus

$$P = \frac{50}{1+s_1} + \frac{50}{(1+s_2)^2} + \frac{50}{(1+s_3)^3} + \frac{1050}{(1+s_4)^4}.$$

Using the discount factors from Example 73,

$$P = 50\left(\frac{96.618}{100}\right) + 50\left(\frac{92.456}{100}\right) + 50\left(\frac{87.63}{100}\right) + 1050\left(\frac{82.27}{100}\right) \approx 1002.19.$$

□

Spot rates and yield rates play different roles. A spot rate is the return from a single payment (or zero-coupon investment) over a specific horizon $t$. A yield rate is the expected rate of return on an investment that may have multiple cash flows over the investment period.

**Example 75** You are given the following bonds (face value \$1000, annual coupons).

| Maturity (years) | Coupon rate | Yield to maturity |
|---|---|---|
| 1 | 0% | 3% |
| 2 | 5% | 4% |
| 3 | 4% | 5% |

Compute implied annual effective spot rates $s_1, s_2, s_3$.

**Solution.** We bootstrap from shortest maturity to longest.

From the 1-year zero coupon bond,

$$P_1 = 1000(1.03)^{-1} = 970.8738 \Longrightarrow s_1 = 3\%.$$

For the 2-year 5% coupon bond,

$$P_2 = \frac{50}{1.04} + \frac{1050}{(1.04)^2} = 1018.8609.$$

Also,

$$P_2 = \frac{50}{1+s_1} + \frac{1050}{(1+s_2)^2}.$$

Substitute $s_1 = 0.03$ and solve for the next spot rate.

$$s_2 \approx 0.040250 = 4.03\%.$$

For the 3-year 4% coupon bond,

$$P_3 = \frac{40}{1.05} + \frac{40}{(1.05)^2} + \frac{1040}{(1.05)^3} = 972.7675,$$

and

$$P_3 = \frac{40}{1+s_1} + \frac{40}{(1+s_2)^2} + \frac{1040}{(1+s_3)^3}.$$

Substitute $s_1, s_2$ and solve for the next spot rate.

$$s_3 \approx 0.050555 = 5.06\%.$$

$\square$

### Lecture 32 — Friday, November 22, 2013

## Forward Rates

With the accumulation function written as

$$a(t) = (1 + s_t)^t,$$

the **forward rate** $f_{[t,t+k]}$ is the effective annual interest rate over $[t, t + k]$.

$$(1 + f_{[t,t+k]})^k = \frac{a(t + k)}{a(t)} = \frac{(1 + s_{t+k})^{t+k}}{(1 + s_t)^t}.$$

Equivalently,

$$(1 + s_{t+k})^{t+k} = (1 + s_t)^t \cdot (1 + f_{[t,t+k]})^k.$$

For one-year forwards,

$$1 + f_{[k,k+1]} = \frac{(1 + s_{k+1})^{k+1}}{(1 + s_k)^k}.$$

The forward rate $f_{[t,t+k]}$ can be interpreted as the rate agreed upon today for borrowing or lending over the future interval $[t, t + k]$. Equivalently, it is the return implied today on a $t$-year deferred $k$-year zero-coupon position. It is a rate implied by the market, not a guaranteed forecast of the future spot rate.

Forward rates are useful for locking in future reinvestment/borrowing rates and managing rate uncertainty.

**Example 76** Using the term structure from Example 73, compute the implied $k$-year deferred 1-year forward rates for $k = 0, 1, 2, 3$.

**Solution.**

$$f_{[0,1]} = s_1 \approx 3.5004\%,$$

$$f_{[1,2]} = \frac{(1 + s_2)^2}{1 + s_1} - 1 \approx 4.5016\%,$$

$$f_{[2,3]} = \frac{(1+s_3)^3}{(1+s_2)^2} - 1 \approx 5.5072\%,$$

$$f_{[3,4]} = \frac{(1+s_4)^4}{(1+s_3)^3} - 1 \approx 6.5151\%.$$

□

**Example 77** Assume the term structure from Example 73. A firm's project produces a cash flow at the end of year 2. The firm wants to reinvest that cash flow at the end of year 2 for one year at a rate of at least 5%. What should the firm do?

**Solution.** From Example 76,

$$f_{[2,3]} \approx 5.5072\% > 5\%.$$

So the firm should lock in this rate today, for example by entering a forward agreement (equivalently, a 2-year deferred 1-year lending position). This guarantees at least 5% for reinvestment over year 3.

□

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**Lecture 33 — Monday, November 25, 2013**

# 9. Asset-Liability Management (ALM)

**Asset-liability management (ALM)** aims to structure assets so that liability cash flows can be met as they come due. Perfect cash flow matching is ideal but not always feasible, so we develop measures of interest rate risk and immunization techniques.

**Example 78** Assume annual effective interest $i = 6\%$. You borrow $L = \$747258.17$ today and must repay \$1,000,000 in 5 years. You may invest in 3-, 5-, and/or 7-year zero coupon bonds. Compare the following portfolios.

(1) Portfolio A: invest $L$ in the 5-year zero coupon bond.

(2) Portfolio B: invest $L$ in the 3-year zero coupon bond.

(3) Portfolio C: invest $L$ in the 7-year zero coupon bond.

(4) Portfolio D: invest $L/2$ in each of the 3- and 7-year zero coupon bonds.

**Solution.** *Portfolio A.* The 5-year face value is

$$L(1.06)^5 \approx 1000000,$$

so this is an exact match.

*Portfolio B.* The 3-year proceeds are

$$L(1.06)^3 \approx 889996.44,$$

so funds must be reinvested from year 3 to year 5 at the rates then prevailing.

*Portfolio C.* The 7-year face value is

$$L(1.06)^7 \approx 1123600,$$

so the bond must be sold at year 5, creating price risk.

*Portfolio D.* Investing half of $L$ in each bond gives face values

$$444998.22 \quad \text{(3-year)}, \qquad \text{and} \qquad 561800 \quad \text{(7-year)}.$$

If the yield after setup is $y$ and then remains level, the asset value at time 5 is

$$CF_5(y) = 444998.22(1+y)^2 + 561800(1+y)^{-2},$$

which is tabulated below for several values of $y$.

| $y$ | $CF_5(y)$ |
|---|---|
| 6% | 1000000 |
| 5% | 1000179.6985 |
| 7% | 1000176.3396 |
| 4% | 1000725.7552 |
| 9% | 1001558.2054 |

In all listed scenarios, the portfolio covers the \$1,000,000 liability at year 5. Our goal is to learn how to construct such portfolios.

□

## Measuring Interest Rate Risk

**Interest rate risk** is the risk that changes in market yields alter an investment's value. Recall that the price of a bond is

$$P(i) = \sum_{k=1}^{n} Fr(1+i)^{-k} + C(1+i)^{-n}.$$

In general, higher coupon rates reduce price sensitivity, and higher initial yields also reduce price sensitivity. By contrast, the longer the major cash flows are pushed into the future, the greater the sensitivity tends to be. In addition, when initial yields are low, a rate decrease tends to increase the price by more than a rate increase of the same size decreases it; this asymmetry is one manifestation of convexity.

## Macaulay and Modified Duration

For cash flows $A_t$ at times $t = 1, 2, \ldots, n$,

$$P(i) = \sum_{t=1}^{n} A_t(1+i)^{-t}.$$

The quantity of immediate interest is the sensitivity of price to a small instantaneous yield change

$\Delta i$. The exact percentage price change is

$$\frac{P(i+\Delta i)-P(i)}{P(i)}.$$

Using a first-order Taylor expansion,

$$P(i+\Delta i)\approx P(i)+P'(i)\,\Delta i,$$

so

$$\frac{P(i+\Delta i)-P(i)}{P(i)}\approx\frac{P'(i)}{P(i)}\,\Delta i.$$

This motivates the definition of **modified duration**.

$$\bar{v}=-\frac{P'(i)}{P(i)}=\frac{\sum_{t=1}^{n}tA_t(1+i)^{-t-1}}{\sum_{t=1}^{n}A_t(1+i)^{-t}}.$$

Therefore, for a small instantaneous change $\Delta i$,

$$\frac{P(i+\Delta i)-P(i)}{P(i)}\approx-\bar{v}\,\Delta i.$$

So modified duration is the local first-order price sensitivity per unit change in yield. For example, if yields increase by 1% (that is, $\Delta i=0.01$), then the approximate percentage price change is $-0.01\bar{v}$.

Algebraically, modified duration can be written as

$$\bar{v}=\sum_{t=1}^{n}t\left(\frac{A_t(1+i)^{-t-1}}{\sum_{u=1}^{n}A_u(1+i)^{-u}}\right),$$

which resembles a weighted average of times, but these coefficients are not true weights because they sum to $(1+i)^{-1}$ rather than 1.

To obtain a true average time weighted by present values, define the **Macaulay duration** by

$$\bar{d}=(1+i)\bar{v}=\frac{\sum_{t=1}^{n}tA_t(1+i)^{-t}}{\sum_{t=1}^{n}A_t(1+i)^{-t}}.$$

Equivalently,

$$\bar{d}=\sum_{t=1}^{n}t\,w_t,\qquad w_t=\frac{A_t(1+i)^{-t}}{P(i)},\qquad\text{and}\qquad\sum_{t=1}^{n}w_t=1,$$

so Macaulay duration is exactly the average time of the cash flows weighted by their present values.

If we write $\delta = \ln(1+i)$ (force of interest), then

$$P(\delta) = \sum_{t=1}^{n} A_t e^{-\delta t} \qquad \text{and} \qquad \bar{d} = -\frac{P'(\delta)}{P(\delta)}.$$

**Example 79** Calculate the modified and Macaulay durations of a $T$-year zero coupon bond with face value $F$ at annual effective rate $i$.

**Solution.** The price is

$$P(i) = F(1+i)^{-T}.$$

Then

$$P'(i) = -TF(1+i)^{-T-1},$$

so

$$\bar{v} = -\frac{P'(i)}{P(i)} = \frac{T}{1+i} \qquad \text{and} \qquad \bar{d} = (1+i)\bar{v} = T.$$

□

**Example 80** Calculate the modified and Macaulay durations of a perpetuity-immediate paying \$1 each year when $i = 5\%$.

**Solution.** For a perpetuity-immediate,

$$P(i) = \frac{1}{i}.$$

Hence

$$P'(i) = -\frac{1}{i^2} \qquad \text{and} \qquad \bar{v} = -\frac{P'(i)}{P(i)} = \frac{1}{i}.$$

At $i = 0.05$, we have

$$\bar{v} = 20 \qquad \text{and} \qquad \bar{d} = (1.05)(20) = 21 \text{ years.}$$

□

**Example 81** Using the modified duration from Example 80, approximate the new price of the perpetuity-immediate if the annual effective rate changes instantaneously from 5% to 6%.

**Solution.** Here $\Delta i = 0.01$ and $\bar{v} = 20$. So

$$\frac{\Delta P}{P} \approx -\bar{v}\Delta i = -20(0.01) = -0.20.$$

Since $P(0.05) = 20$,

$$P_{\text{new}} \approx 20(1 - 0.20) = 16.$$

For reference, the exact new price is $1/0.06 \approx 16.6667$.

□

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### Lecture 34 — Wednesday, November 27, 2013

## Convexity

Including second-order effects gives the following approximation to the percentage change in price:

$$\frac{P(i + \Delta i) - P(i)}{P(i)} \approx -\bar{v}\,\Delta i + \frac{1}{2}\bar{c}(\Delta i)^2,$$

where the **convexity** is

$$\bar{c} = \frac{P''(i)}{P(i)} = \frac{\sum_{t=1}^{n} t(t+1)A_t(1+i)^{-t-2}}{\sum_{t=1}^{n} A_t(1+i)^{-t}}.$$

**Example 82** Return to Example 81. Compute the convexity and obtain a better approximation to the new price after the rate change from 5% to 6%.

**Solution.** For $P(i) = 1/i$,

$$P''(i) = \frac{2}{i^3} \Longrightarrow \bar{c} = \frac{P''(i)}{P(i)} = \frac{2}{i^2}.$$

At $i = 0.05$,

$$\bar{c} = \frac{2}{0.05^2} = 800.$$

Thus,

$$\frac{\Delta P}{P} \approx -20(0.01) + \frac{1}{2}(800)(0.01)^2 = -0.20 + 0.04 = -0.16,$$

so

$$P_{\text{new}} \approx 20(1 - 0.16) = 16.8,$$

which is closer to the exact 16.6667.

□

---

### Lecture 35 — Friday, November 29, 2013

## Immunization

In the previous sections we quantified interest rate risk and approximated how values change when rates move. **Immunization** addresses the corresponding portfolio-design question: how should we choose assets so that a fixed-income portfolio continues to cover its liabilities after a small instantaneous shock to the yield?

Let the asset and liability cash flow streams be $\{A_t, t > 0\}$ and $\{L_t, t > 0\}$, and define

$$A(i) = \sum_{t>0} A_t(1+i)^{-t}, \qquad L(i) = \sum_{t>0} L_t(1+i)^{-t}, \qquad \text{and} \qquad S(i) = A(i) - L(i).$$

Here $A(i)$ is the present value of assets, $L(i)$ is the present value of liabilities, and $S(i)$ is the present value surplus. Assume $i^*$ is the current annual effective rate. The local objective is to structure assets so that, for small instantaneous changes $\Delta i$ around $i^*$,

$$S(i^* + \Delta i) \geq S(i^*).$$

Graphically, we want $S(i)$ to be locally minimized at the current rate $i^*$, as in Figure 14.

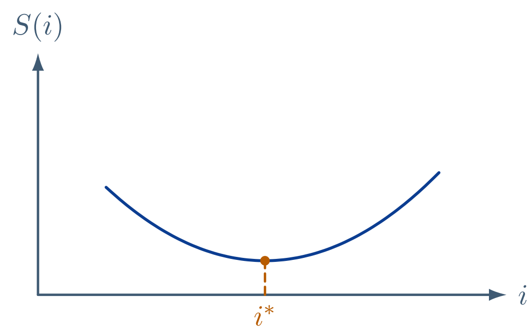


FIGURE 14

This leads to **Redington's immunization conditions:**

$$\text{(I) } S(i^*) = 0, \qquad \text{(II) } S'(i^*) = 0, \qquad \text{and} \qquad \text{(III) } S''(i^*) > 0.$$

Condition (I) matches present values, condition (II) matches durations, and condition (III) is a convexity/dispersion condition.

Condition (I) is equivalent to

$$A(i^*) = L(i^*) \iff \sum_{t>0} A_t(1+i^*)^{-t} = \sum_{t>0} L_t(1+i^*)^{-t}.$$

If condition (I) holds, then condition (II) is equivalent to

$$A'(i^*) = L'(i^*).$$

Since

$$A'(i) = -\sum_{t>0} tA_t(1+i)^{-t-1} \qquad \text{and} \qquad L'(i) = -\sum_{t>0} tL_t(1+i)^{-t-1},$$

we obtain

$$\sum_{t>0} tA_t(1+i^*)^{-t} = \sum_{t>0} tL_t(1+i^*)^{-t}.$$

Dividing by the corresponding present values and using condition (I) gives

$$\bar{d}_A = \bar{d}_L,$$

so the Macaulay durations match. Because $\bar{v} = \bar{d}/(1+i)$, this is equivalent to

$$\bar{v}_A = \bar{v}_L,$$

so modified durations match as well.

If conditions (I) and (II) hold, then condition (III) is equivalent to

$$A''(i^*) > L''(i^*).$$

Using

$$A''(i) = \sum_{t>0} t(t+1)A_t(1+i)^{-t-2} \qquad \text{and} \qquad L''(i) = \sum_{t>0} t(t+1)L_t(1+i)^{-t-2},$$

condition (III) becomes

$$\sum_{t>0} t(t+1)A_t(1+i^*)^{-t-2} > \sum_{t>0} t(t+1)L_t(1+i^*)^{-t-2},$$

which corresponds to asset convexity exceeding liability convexity. Equivalently, using condition (II), this can be written as

$$\sum_{t>0} t^2 A_t(1+i^*)^{-t} > \sum_{t>0} t^2 L_t(1+i^*)^{-t}.$$

Immunization is not a set-and-forget method. Rebalancing is generally required as time passes and rates move, because both present values and durations change. Also, Redington conditions are local in $\Delta i$: they guarantee protection for sufficiently small shocks, not arbitrary large moves. Finally, this setup assumes a flat curve and parallel shifts; in richer models of the term structure, the same core idea remains, but implementation is more general.

**Example 83** A portfolio manager has liabilities \$1000 at time 3 and \$4000 at time 6. The manager may invest in 1- and 8-year zero coupon bonds. Assume annual effective interest 5%.

(1) Determine asset cash flows $A_1$ and $A_8$ so that Redington's first two conditions hold.

(2) Check whether the portfolio is immunized.

**Solution.** Let $v = (1.05)^{-1}$.

For (1), matching PV and first moment gives

$$A_1 v + A_8 v^8 = 1000v^3 + 4000v^6 = 3848.6992,$$

$$1 \cdot A_1 v + 8 \cdot A_8 v^8 = 3 \cdot 1000v^3 + 6 \cdot 4000v^6 = 20500.6823.$$

Solving,

$$A_1 \approx 1543.34 \qquad \text{and} \qquad A_8 \approx 3514.65.$$

For (2), checking the dispersion/convexity criterion gives

$$\sum_{t>0} t^2 L_t (1.05)^{-t} = 115229.5555,$$

$$\sum_{t>0} t^2 A_t (1.05)^{-t} = 153716.5474.$$

Since the asset quantity is larger, condition (III) holds, so the portfolio is immunized (for small instantaneous rate changes).

□

**Example 84** Joe has just turned 60 and will receive a pension of \$5000 yearly, paid at the beginning of each year from age 65. Joe is expected to die just before turning 75. Assume annual effective interest 3%.

(1) Find the present value of the pension payments.

(2) Given that

$$\frac{\mathrm{d}}{\mathrm{d}i} a_{\overline{n}|i} = -\frac{(Ia)_{\overline{n}|i}}{1+i},$$

find the duration of the pension payments.

(3) The pension fund manager wants to construct an asset portfolio satisfying Redington's first two conditions. If the manager can invest in 7-year and 15-year zero-coupon bonds, determine the amount of money to be invested in each bond.

(4) Determine whether the portfolio above is immunized.

(5) Using software, plot the surplus against the interest rate.

**Solution.** For (1),

$$PV = 5000(1.03)^{-4} a_{\overline{10}|0.03} \approx 37894.873683.$$

For (2), if a question does not specify Macaulay or modified duration, we use Macaulay duration by default. Since the present value of pension payments is

$$L(i) = 5000(1+i)^{-4}\, a_{\overline{10}|i},$$

we differentiate using

$$\frac{\mathrm{d}}{\mathrm{d}i} a_{\overline{n}|i} = -\frac{(Ia)_{\overline{n}|i}}{1+i}.$$

Hence

$$\begin{aligned} L'(i) &= 5000(-4)(1+i)^{-5}\, a_{\overline{10}|i} - 5000(1+i)^{-4}(1+i)^{-1}(Ia)_{\overline{10}|i} \\ &= -5000(1+i)^{-5}\left[4a_{\overline{10}|i} + (Ia)_{\overline{10}|i}\right]. \end{aligned}$$

So the Macaulay duration is

$$\begin{aligned} \bar{d}_L &= -(1+0.03)\frac{L'(0.03)}{L(0.03)} \\ &= \frac{5000(1.03)^{-4}\left[4a_{\overline{10}|0.03} + (Ia)_{\overline{10}|0.03}\right]}{37894.873683} \\ &= \frac{350773.814811}{37894.873683} \approx 9.256498 \text{ years.} \end{aligned}$$

For (3), let the invested amounts be $x_7, x_{15}$. Matching PV and duration gives

$$x_7 + x_{15} = 37894.873683 \qquad \text{and} \qquad 7x_7 + 15x_{15} = 350773.814811,$$

so

$$x_7 = 27206.161304 \qquad \text{and} \qquad x_{15} = 10688.712379.$$

For (4), we need to check Redington's third condition, the convexity/dispersion criterion. Software (or a financial calculator) gives

$$\sum_{t>0} t^2 L_t (1.03)^{-t} = 3558195.091636,$$

$$\sum_{t>0} t^2 A_t (1.03)^{-t} = 7^2 x_7 + 15^2 x_{15} = 3738062.189151,$$

so the convexity criterion is satisfied and the portfolio is immunized.

For (5), with corresponding face values

$$A_7 = 33460.146766 \qquad \text{and} \qquad A_{15} = 16652.665612,$$

the surplus function is

$$S(i) = A_7(1+i)^{-7} + A_{15}(1+i)^{-15} - 5000(1+i)^{-4} a_{\overline{10}|i}.$$

Using the annuity-immediate identity

$$a_{\overline{10}|i} = \frac{1-(1+i)^{-10}}{i},$$

we obtain the surplus curve in Figure 15.

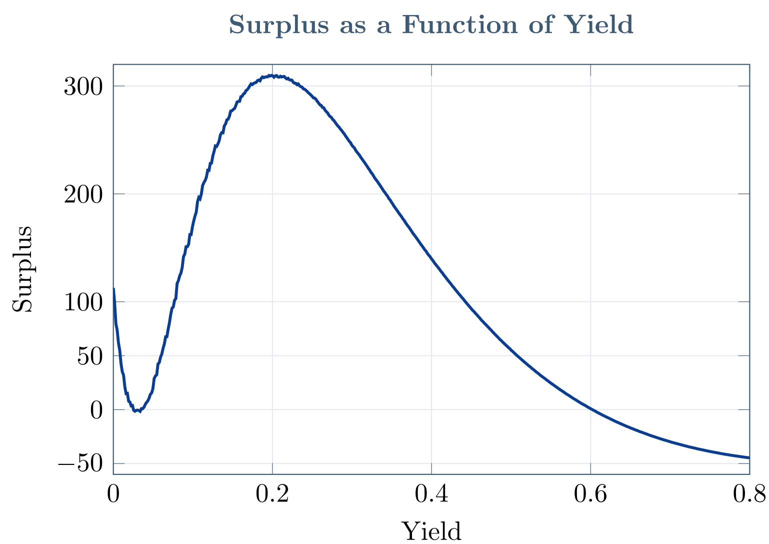


FIGURE 15

The graph in Figure 15 shows that the surplus remains positive over a reasonably wide range of nearby instantaneous yield changes.

□

# Appendix: Lecture and Tutorial Index