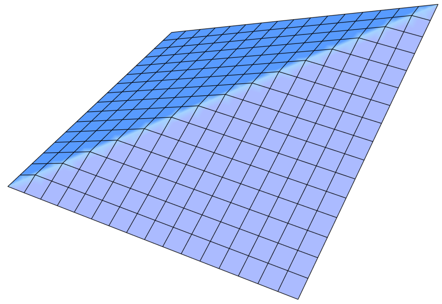

ACTSC 446: Mathematical Models for Finance

Prof. Bin Li • Winter 2015 • University of Waterloo
MW 8:30–9:50AM in DC 1350

Transcribed, LaTeXed, and edited by Rob Zimmerman

Note: These are personal lecture notes, not official course materials. They may contain errors (which by default you should attribute to me, Rob Zimmerman, rather than the course instructor). The permanent home of these — and all of my other lecture notes — is `https://rob-zimmerman.github.io/coursenotes`.

Contents

Lecture 1 — Monday, January 05, 2015

1. Overview of Derivatives and Basic Option Strategies

We start by introducing the basic language of derivatives, explaining the main reasons these contracts are used, and surveying the most common derivative products studied below.

A **derivative** is a financial instrument whose price is determined by the price of one or more underlying assets. An **underlying asset** may be a single asset, a basket of assets, or even another derivative, but there must be an independent way to determine its current value.

Common examples of underlying assets include stock indices such as the S&P 500 and the Dow Jones Industrial Average, interest rate benchmarks such as 10-year Canadian Treasury bonds and LIBOR, foreign exchange rates such as the U.S. dollar, the Chinese yuan, and the euro, commodities such as oil, natural gas, gold, corn, and cattle, and other quantities such as credit or real estate. In all of these examples, the future value of the underlying asset is random.

Example 1.1 Suppose that you want to buy one ounce of gold at the end of the year, when the current price is about $1200 per ounce. You consider entering a contract with a gold seller under which you receive $100 from the seller if the price of one ounce of gold at the end of the year exceeds $1200, while you pay $100 to the seller otherwise. This contract is a derivative because its payoff depends on the future price of gold. From the buyer's perspective, the contract reduces the loss associated with a high future gold price, while from the seller's perspective it reduces the loss associated with a low future gold price. It's win-win!

Derivative Markets and Uses

Derivatives are commonly used for insurance, for speculation (i.e., gambling on the movement of the underlying assets), for reducing transaction costs, and for regulatory arbitrage.

Derivative contracts are traded in two main settings. **Over-the-counter (OTC) derivatives** form a significantly larger market. They are privately traded, offer less price transparency, have lower liquidity, involve higher credit risk, and typically come with lower fees and taxes together with greater contractual freedom. **Exchange-traded derivatives (ETDs)** form a smaller market and are publicly traded in standardized settings.

The same market can be viewed from several perspectives. An **end user** employs derivatives to manage risk, to speculate, to reduce costs, or to avoid regulation. A **market maker** buys derivatives from customers who wish to sell and sells derivatives to customers who wish to buy, earning the **bid-ask spread** in the process. **Economic observers** (or regulators) study derivative markets and attempt to make sense of their overall behaviour. These roles are analogous to the relationship among customers, car dealers, and the Ministry of Transportation.

These perspectives lead to two central issues. The first is the valuation of derivatives. For exchange-traded derivatives, we typically observe a market price. For over-the-counter derivatives, we seek a no-arbitrage price (essentially a fair price), either in discrete time, as in binomial models, or in continuous time, as in the Black–Scholes model. The second issue is the use of derivatives, especially for hedging and for speculation.

Lecture 2 — Wednesday, January 07, 2015

Common Derivative Contracts

The four most common derivative contracts are forwards, futures, options, and swaps.

A **forward** is a nonstandardized contract between two parties to buy or to sell an asset at a specified future time at a price agreed upon today. The **underlying asset** is the asset on which the forward is based, the **expiration date** is the time at which the asset is delivered, and the **forward price** is the price the buyer will pay at the expiration date.

If S_t denotes the spot price of the underlying asset at time $t \geq 0$, if T is the expiration date, and if K is the forward price, then taking the **long side** means agreeing to buy and taking the **short side** means agreeing to sell. The payoff functions at time T are

$$\text{payoff to the long forward} = S_T - K$$

and

$$\text{payoff to the short forward} = K - S_T.$$

Example 2.1 Consider a six-month S&R index forward contract with forward price $K = \$1020$. The payoffs at maturity are given by the following table.

Index price after 6 months	Payoff to the long	Payoff to the short
900	-120	120
950	-70	70
1000	-20	20
1020	0	0
1050	30	-30
1100	80	-80

Graphically, the payoff from the long forward in Figure 1 is a straight line with slope 1 as a function of S_T, and it crosses the horizontal axis at $S_T = K$. The payoff from the short forward is its reflection through the horizontal axis.

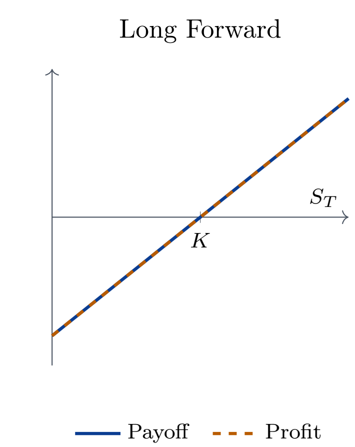

FIGURE 1

A **futures** contract is a standardized contract between two parties to buy or to sell a specified asset of standardized quantity and quality at a specified future time and at a given fixed price.

Both forwards and futures allow parties to lock in terms today for a transaction at a future date. A futures contract is standardized, exchange traded, backed by a clearinghouse, and marked to market daily; many positions are closed before maturity. A forward is an over-the-counter agreement whose delivery price is fixed at inception and whose gains or losses are usually settled at maturity, so it generally carries more counterparty credit risk. Both instruments can be used for hedging or speculation, and either may be closed or offset before delivery if the contract and market permit it.

An **option** is a contract that gives the buyer the right, but not the obligation, to buy or to sell an underlying asset at a specified strike price on or before expiration. The **underlying asset** is the asset on which the option is based, the **expiration date** is the latest date on which the option may be exercised, **exercise** means carrying out the transaction specified by the option, the **strike price** is the price paid for the asset if the option is exercised, the **exercise style** describes when exercise is permitted, and the **payoff type** describes the kind of option.

There are three standard exercise styles. A **European-style** option can only be exercised at maturity. An **American-style** option can be exercised at any time on or before maturity. A **Bermudan-style** option can only be exercised on specified dates on or before maturity.

There are also several payoff types. A **call option** gives the buyer the right to buy the underlying asset, while a **put option** gives the buyer the right to sell it. An **exotic option** contains more complex financial features, whereas a **vanilla option** is a nonexotic option, usually meaning a plain call or put.

Forwards, futures, and options are all derivatives, and both forwards and options have an expiration date and a strike price. However, the payoff type for a forward is unique while options admit many payoff structures. A forward imposes an obligation, whereas an option grants a right. At inception, a forward contract usually has value zero, while an option has a positive price.

Option Payoffs and Profits

Let us focus on European options. Let T denote the expiration date, let S_T denote the underlying asset price at maturity, and let K denote the strike price. Writing $\max(x, 0) = (x)_+$ and using FV to denote future value, the payoffs and profits from long calls and long puts at time T are

$$\text{long call payoff} = (S_T - K)_+ \qquad \text{and} \qquad \text{long call profit} = (S_T - K)_+ - \text{FV(price of the call)},$$

and

$$\text{long put payoff} = (K - S_T)_+ \qquad \text{and} \qquad \text{long put profit} = (K - S_T)_+ - \text{FV(price of the put)}.$$

For a call option, the contract **is in the money** when $S(0) > K$, **at the money** when $S(0) = K$, and **out of the money** when $S(0) < K$. For a put option, the contract is **in the money** when $S(0) < K$, **at the money** when $S(0) = K$, and **out of the money** when $S(0) > K$.

For each option position, it is useful to draw both the payoff and the profit. In the panels of Figure 2, the blue curve represents payoff and the orange curve represents profit.

Lecture 3 — Monday, January 12, 2015

Example 3.1 Consider a call option on the S&R index with six months to expiration and strike price $K = \$1000$. Suppose that the risk-free rate over six months is 2%, and that the call premium is \$93.81. If the index after six months is \$1100, then

$$\text{payoff} = (1100 - 1000)_+ = 100$$

and

$$\text{profit} = (1100 - 1000)_+ - 93.81 \cdot 1.02 = 100 - 95.68 = 4.32.$$

If instead the index after six months is \$900, then

$$\text{payoff} = (900 - 1000)_+ = 0$$

and

$$\text{profit} = (900 - 1000)_+ - 93.81 \cdot 1.02 = 0 - 95.68 = -95.68.$$

Example 3.2 Consider a put option on the S&R index with six months to expiration and strike price $K = \$1000$. Suppose that the risk-free rate over six months is 2%, and that the put premium is \$74.20. If the index after six months is \$1100, then

$$\text{payoff} = (1000 - 1100)_+ = 0$$

and

$$\text{profit} = (1000 - 1100)_+ - 74.20 \cdot 1.02 = 0 - 75.68 = -75.68.$$

If instead the index after six months is \$900, then

$$\text{payoff} = (1000 - 900)_+ = 100$$

and

$$\text{profit} = (1000 - 900)_+ - 74.20 \cdot 1.02 = 100 - 75.68 = 24.32.$$

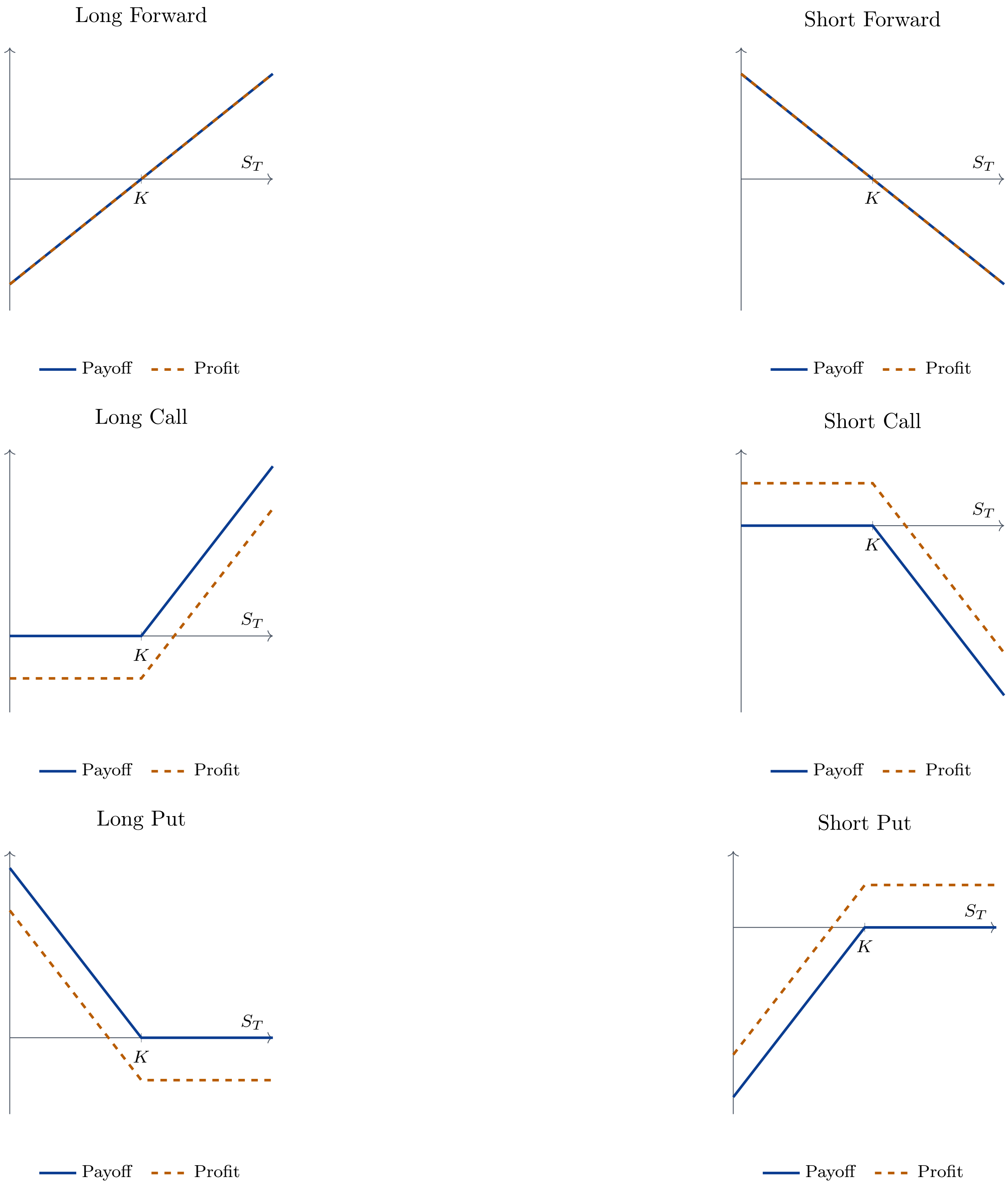
Long Forward
S_T
K
Payoff
Profit
Short Forward
S_T
K
Payoff
Profit
Long Call
S_T
K
Payoff
Profit
Short Call
S_T
K
Payoff
Profit
Long Put
S_T
K
Payoff
Profit
Short Put
S_T
K
Payoff
Profit

Figure 2

The basic payoff and profit formulas are as follows:

$$\begin{aligned}
\text{long forward payoff} &= S_T - K \\
\text{short forward payoff} &= K - S_T \\
\text{long call payoff} &= (S_T - K)_+ \\
\text{long call profit} &= (S_T - K)_+ - \text{FV(premium)} \\
\text{short call payoff} &= -(S_T - K)_+ \\
\text{short call profit} &= \text{FV(premium)} - (S_T - K)_+ \\
\text{long put payoff} &= (K - S_T)_+ \\
\text{long put profit} &= (K - S_T)_+ - \text{FV(premium)} \\
\text{short put payoff} &= -(K - S_T)_+ \\
\text{short put profit} &= \text{FV(premium)} - (K - S_T)_+
\end{aligned}$$

The payoff diagrams show that the profit graph for a long option is obtained by shifting the corresponding payoff graph downward by the future value of the premium, while the profit graph for a short option is obtained by shifting the corresponding payoff graph upward by that amount.

Position	Maximum Loss	Maximum Profit
Long forward	$-K$	Unlimited
Long call	$-$FV(premium)	Unlimited
Short put	$-K +$ FV(premium)	FV(premium)

These three profit profiles are compared on the same set of axes in Figure 3.

Position	Maximum Loss	Maximum Profit
Short forward	Unlimited	K
Short call	Unlimited	FV(premium)
Long put	$-$FV(premium)	$K -$ FV(premium)

We now study basic option strategies used either for insurance or for speculation. The main theme is that different portfolios can produce the same payoff, so an option position can often be reinterpreted as a stock-and-bond position and vice versa.

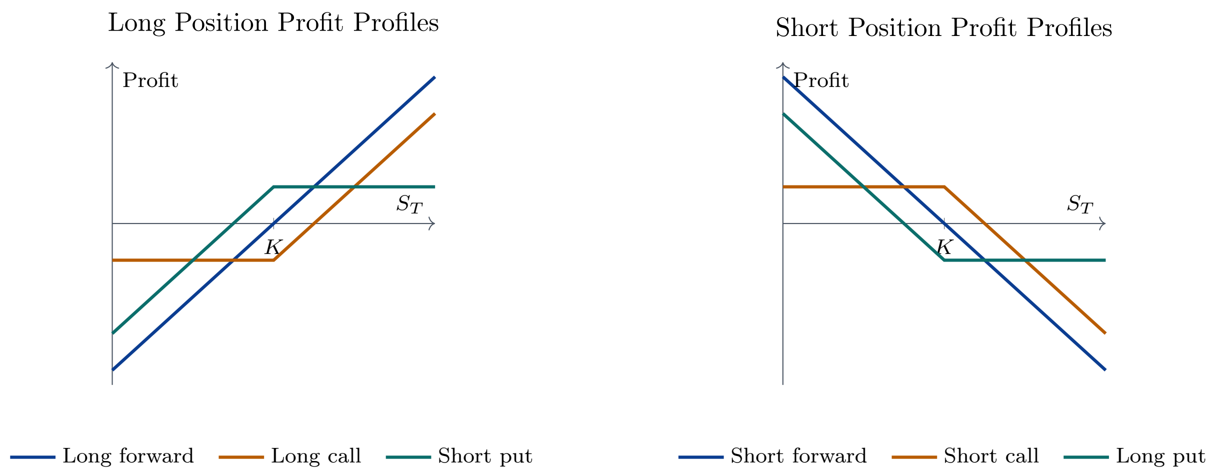

FIGURE 3

Insurance Strategies

The first group of portfolios is designed to insure a position in the underlying asset. The four strategies introduced here are the **floor**, **covered call writing**, the **cap**, and **covered put writing**. Their payoffs are summarized in the following table.

Strategy		Payoff at time T
Floor	Long stock and long put	$S_T + (K - S_T)_+$
Covered call writing	Long stock and short call	$S_T - (S_T - K)_+$
Cap	Short stock and long call	$-S_T + (S_T - K)_+$
Covered put writing	Short stock and short put	$-S_T - (K - S_T)_+$

These formulas can be simplified algebraically. For the floor,

$$S_T + (K - S_T)_+ = K + (S_T - K)_+ = \begin{cases} S_T & \text{if } S_T \geq K \\ K & \text{if } S_T < K \end{cases}$$

so the floor is equivalent to a bond plus a call. For covered call writing,

$$S_T - (S_T - K)_+ = K - (K - S_T)_+ = \begin{cases} K & \text{if } S_T \geq K \\ S_T & \text{if } S_T < K \end{cases}$$

so it is equivalent to a bond minus a put. For the cap,

$$-S_T + (S_T - K)_+ = -K + (K - S_T)_+ = \begin{cases} -K & \text{if } S_T \geq K \\ -S_T & \text{if } S_T < K \end{cases}$$

so it is equivalent to a negative bond plus a put. Finally, for covered put writing,

$$-S_T - (K - S_T)_+ = -K - (S_T - K)_+ = \begin{cases} -S_T & \text{if } S_T \geq K \\ -K & \text{if } S_T < K \end{cases}$$

so it is equivalent to a negative bond minus a call.

Figure 4 reproduces the schematic payoff and profit shapes for these four insurance strategies. The profit graphs are vertical shifts of the corresponding payoff graphs, reflecting the initial cost of the stock position together with the option premium.

Note that different positions can have the same payoff because of the identity

$$(S_T - K)_+ - (K - S_T)_+ = S_T - K.$$

In brief, this can be written as

$$\text{call} - \text{put} = \text{stock} - \text{bond},$$

so we can use a call and a put to create a **synthetic forward**.

The equivalent positions are summarized as follows.

Position	Equivalent to	Name
Index + put	zero-coupon bond + call	insured asset (floor)
Index − call	zero-coupon bond − put	covered call (stock + short call)
−index + call	−zero-coupon bond + put	insured short (cap)
−index − put	−zero-coupon bond − call	covered written put

Speculation Strategies

The second group of strategies uses options to speculate on the future movement of the underlying asset. The main idea is that we can build option portfolios that bet on upward movement, downward movement, high volatility, or low volatility, often at a lower cost than taking a direct position in the underlying asset.

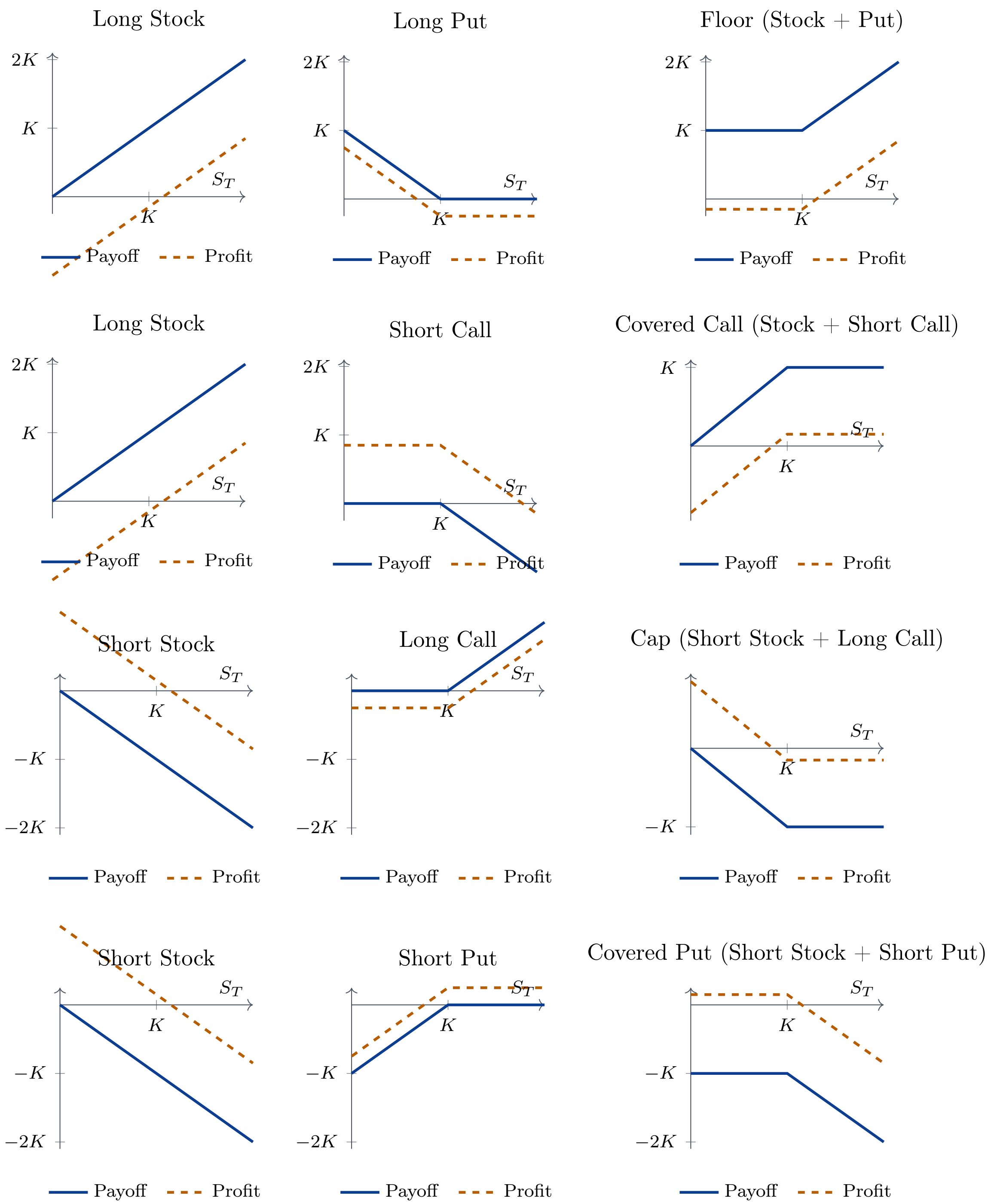
Long Stock
Long Put
Floor (Stock + Put)
Long Stock
Short Call
Covered Call (Stock + Short Call)
Short Stock
Long Call
Cap (Short Stock + Long Call)
Short Stock
Short Put
Covered Put (Short Stock + Short Put)
$2K$
K
$-K$
$-2K$
S_T
Payoff
Profit

FIGURE 4

A **spread** is a position consisting only of calls or only of puts. A **bull spread** is formed by going long one call and short another call with a higher strike, thereby betting on upward movement. A **bear spread** is formed by going long one call and short another call with a lower strike, thereby betting on downward movement. A **box spread** consists of a long synthetic forward together with a short synthetic forward at a different strike, and it can be used to reduce taxes. A **ratio spread** goes long m options and short n options at a different strike in order to reduce the cost of the portfolio. The same kinds of spreads can also be built from puts rather than calls.

The profiles of the bull, bear, and box spreads are represented schematically in Figure 5.

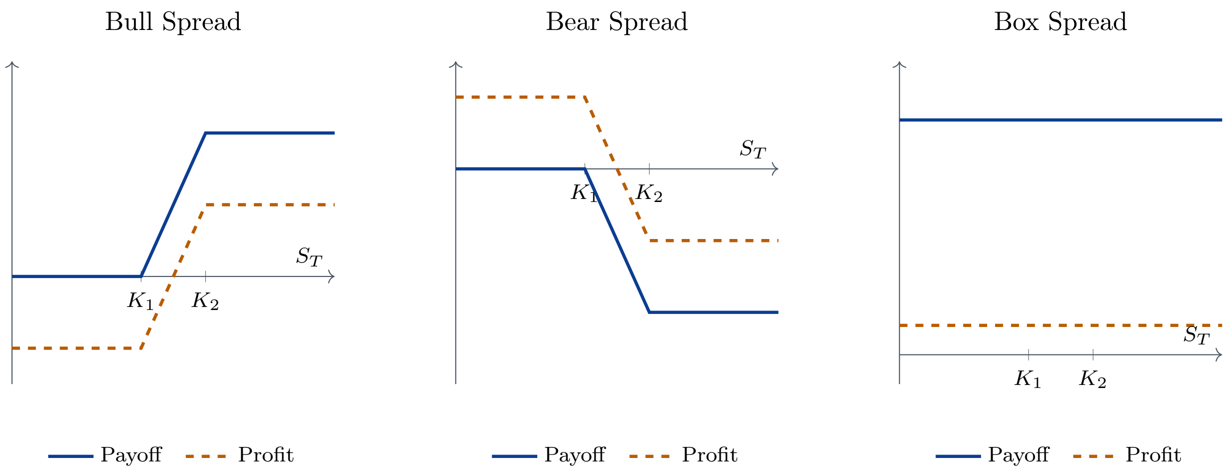

FIGURE 5

A **collar** consists of a long put together with a short call at a higher strike. The **collar width** is the difference between the two strikes.

For example, a collar may use a short 45-strike call together with a long 40-strike put. When the collar is combined with a long position in the underlying asset, it limits both positive and negative returns.

A **straddle** consists of a long at-the-money call and a long at-the-money put with the same strike. A **strangle** consists of a long out-of-the-money call together with a long out-of-the-money put. A **butterfly** is formed by shorting a straddle and going long a strangle.

The intended uses are qualitative. A long straddle is a bet on high volatility. A long strangle is also a bet on high volatility, but it is cheaper because the strikes are out of the money. A long butterfly is a bet on low volatility.

A butterfly can be constructed by overlaying a written straddle with an out-of-the-money call and an out-of-the-money put. The resulting payoff and profit have the schematic tent shape shown in Figure 6.

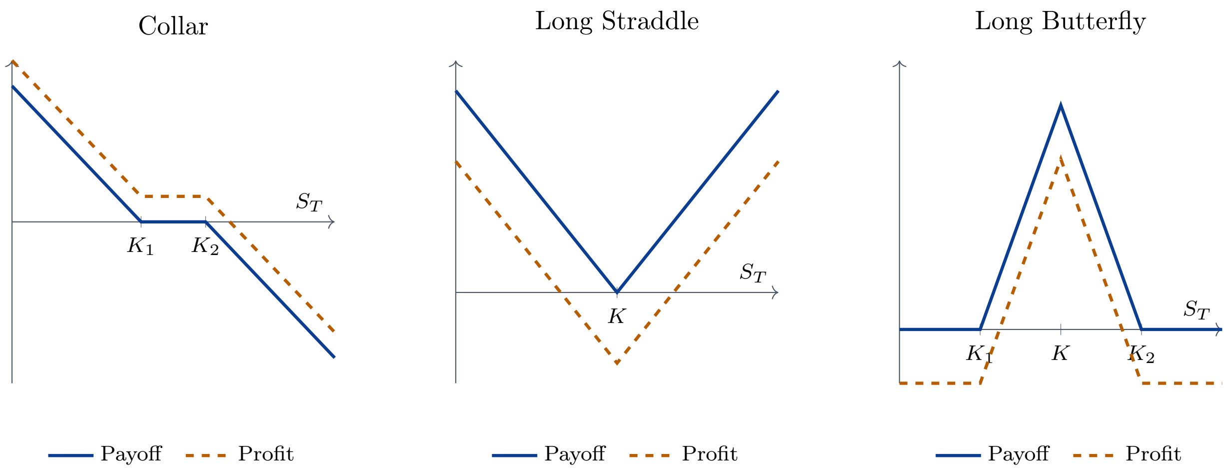

FIGURE 6

Lecture 4 — Wednesday, January 14, 2015

2. Option Price Relationships

Put–Call Parity

Let $\{S_t\}_{0\leq t\leq T}$ denote a stock price process, let $C(S_0, K, T)$ be the premium of a European call on the stock with strike K and maturity T, let $P(S_0, K, T)$ be the premium of a European put on the stock with the same strike and maturity, and let r be the continuously compounded interest rate. The key payoff identity is

$$(S_T - K)_+ - (K - S_T)_+ = S_T - K,$$

which is an example of the **law of one price**: two portfolios with the same payoff at time T must have the same price at time 0. From it, we obtain the following fundamental identity:

Throughout this section, we assume an arbitrage-free frictionless market, a deterministic interest rate, and the ability to trade the stock, options, and risk-free asset without short-sale restrictions. Any discrete dividends used below are also assumed known in advance.

Theorem 4.1 (Put–call parity) For European options,

$$C(S_0, K, T) - P(S_0, K, T) = \mathrm{PV}(S_T) - e^{-rT}K.$$

The quantity $\mathrm{PV}(S_T)$ is the **prepaid forward price**. The following formulas are each justified by the law of one price:

$$\mathrm{PV}(S_T) = S_0 \quad \text{if there is no dividend,}$$

$$\mathrm{PV}(S_T) = S_0 - \mathrm{PV}(\mathrm{Div}) \quad \text{for known discrete dividends, and}$$

$$\mathrm{PV}(S_T) = e^{-\delta T}S_0 \quad \text{for a continuous dividend yield } \delta.$$

We thus have

$$C(S_0, K, T) - P(S_0, K, T) = S_0 - e^{-rT}K$$

in the absence of dividends,

$$C(S_0, K, T) - P(S_0, K, T) = S_0 - \mathrm{PV}(\mathrm{Div}) - e^{-rT}K$$

for discrete dividends, and

$$C(S_0, K, T) - P(S_0, K, T) = e^{-\delta T}S_0 - e^{-rT}K$$

under a continuous dividend yield δ.

Example 4.2 Suppose that $S_0 = 40$, that the continuously compounded interest rate is 8%, and that the options expire in three months. A 40-strike European call sells for 0.74. We seek the no-arbitrage premium of the corresponding 40-strike European put.

If the stock pays a dividend of 5 just before expiration and a dividend of 7 at the end of the second month, then

$$\mathrm{PV}(\mathrm{Div}) = 5e^{-0.08/4} + 7e^{-0.08/6} = 11.808.$$

Hence put–call parity gives

$$\begin{aligned} P(S_0, K, T) &= C(S_0, K, T) - S_0 + \mathrm{PV}(\mathrm{Div}) + e^{-rT}K \\ &= 0.74 - 40 + 11.808 + e^{-0.08/4} \cdot 40 = 11.756. \end{aligned}$$

If instead the stock pays a continuous dividend yield of 6%, then

$$P(S_0, K, T) = C(S_0, K, T) - e^{-0.06/4} S_0 + e^{-0.08/4} K = 0.74 - e^{-0.06/4} \cdot 40 + e^{-0.08/4} \cdot 40 = 0.543.$$

Lecture 5 — Monday, January 19, 2015

Generalized and Currency Parity

Consider two assets with price processes $\{S_t\}_{0 \le t \le T}$ and $\{Q_t\}_{0 \le t \le T}$. Let $C(S_0, Q_0, T)$ denote the premium of the right to receive one share of asset 1 by giving up one share of asset 2 at time T, which will be exercised only if $S_T > Q_T$. Similarly, let $P(S_0, Q_0, T)$ denote the premium of the right to receive one share of asset 2 by giving up one share of asset 1 at time T, which will be exercised only if $Q_T > S_T$.

At maturity, the two exchange option payoffs satisfy

$$(S_T - Q_T)_+ - (Q_T - S_T)_+ = S_T - Q_T.$$

Applying the law of one price again gives the **generalized parity relation**

$$C(S_0, Q_0, T) - P(S_0, Q_0, T) = \mathrm{PV}(S_T) - \mathrm{PV}(Q_T).$$

The same parity idea applies to **currency options** (or **foreign exchange options**). Let the domestic currency be dollars with interest rate $r_\$$, let the foreign currency be euros with interest rate $r_€$, and let the current exchange rate be €1 = $\$x_0$. If $C_\$(x_0, K, T)$ is the premium of the right to buy €1 for $\$K$ at time T, and $P_\$(x_0, K, T)$ is the premium of the right to sell €1 for $\$K$ at time T, then

$$\mathrm{PV}(€1) = €e^{-r_€ T} = \$x_0 e^{-r_€ T} \qquad \text{and} \qquad \mathrm{PV}(\$K) = \$K e^{-r_\$ T}.$$

Therefore the parity relation (in dollar units) is

$$C_\$(x_0, K, T) - P_\$(x_0, K, T) = x_0 e^{-r_€ T} - K e^{-r_\$ T}.$$

We also have the identity (in dollar units)

$$C_{\$}(x_0, K, T) = x_0 K \, P_{€}\left(\frac{1}{x_0}, \frac{1}{K}, T\right),$$

which matches a dollar-denominated call on euros with an appropriately rescaled euro-denominated put.

Bounds, Monotonicity, and Convexity

We now compare option prices in three directions: varying the strike, varying the exercise style, and varying the maturity.

Premium as a function of strike

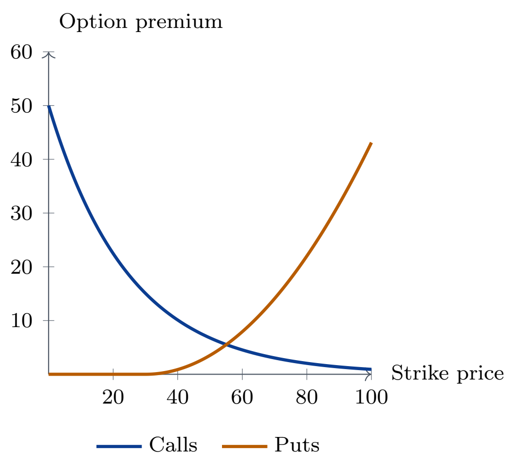

Figure 7

Figure 7 emphasizes that call prices decrease as the strike increases, while put prices increase as the strike increases. It also suggests that both price curves are convex in the strike.

Proposition 5.1 (Monotonicity) If $K_1 < K_2$, then

$$C(S_0, K_1, T) \geq C(S_0, K_2, T) \qquad \text{and} \qquad P(S_0, K_1, T) \leq P(S_0, K_2, T).$$

Proof. Since $K_1 < K_2$, we have

$$(S_T - K_1)_+ \geq (S_T - K_2)_+.$$

By no-arbitrage payoff dominance, this implies $C(S_0, K_1, T) \geq C(S_0, K_2, T)$. Similarly,

$$(K_1 - S_T)_+ \leq (K_2 - S_T)_+,$$

so $P(S_0, K_1, T) \leq P(S_0, K_2, T)$. □

Proposition 5.2 (Lipschitzness) If $K_1 < K_2$, then

$$C(S_0, K_1, T) - C(S_0, K_2, T) \leq e^{-rT}(K_2 - K_1)$$

and

$$P(S_0, K_2, T) - P(S_0, K_1, T) \leq e^{-rT}(K_2 - K_1).$$

Proof. For calls,

$$(S_T - K_1)_+ - (S_T - K_2)_+ \leq K_2 - K_1.$$

No-arbitrage payoff dominance, applied after discounting the deterministic upper bound to time 0, gives the required inequality for calls. The put inequality is proved in exactly the same way from

$$(K_2 - S_T)_+ - (K_1 - S_T)_+ \leq K_2 - K_1.$$

□

Definition 5.3 A function $f : \mathbb{R} \to \mathbb{R}$ is **convex** if for every $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

Proposition 5.4 $f : \mathbb{R} \to \mathbb{R}$ is convex if and only if, for every $x < y < z$,

$$\frac{f(x) - f(y)}{x - y} \leq \frac{f(y) - f(z)}{y - z}.$$

Proof. *Forward implication.* Assume that f is convex and fix $x < y < z$. Write

$$y = \lambda x + (1 - \lambda)z \qquad \text{and} \qquad \lambda := \frac{z - y}{z - x} \in (0, 1).$$

By convexity,

$$f(y) \leq \lambda f(x) + (1 - \lambda)f(z).$$

Rearranging this inequality gives

$$(z-y)\big(f(y)-f(x)\big) \le (y-x)\big(f(z)-f(y)\big),$$

and dividing by $(z-y)(y-x)>0$ yields

$$\frac{f(y)-f(x)}{y-x} \le \frac{f(z)-f(y)}{z-y}.$$

Reverse implication. Assume the slope inequality holds for every $x<y<z$. For fixed $x<z$ and $\lambda \in (0,1)$, set

$$y = \lambda x + (1-\lambda)z.$$

Then $x<y<z$, so

$$\frac{f(y)-f(x)}{y-x} \le \frac{f(z)-f(y)}{z-y}.$$

Let $a := y-x>0$ and $b := z-y>0$. The inequality is

$$b\big(f(y)-f(x)\big) \le a\big(f(z)-f(y)\big),$$

which is equivalent to

$$(a+b)f(y) \le bf(x) + af(z).$$

Since $a+b = z-x$, $b/(a+b) = \lambda$, and $a/(a+b) = 1-\lambda$, we obtain

$$f(y) \le \lambda f(x) + (1-\lambda)f(z),$$

so f is convex. $\square$

FIGURE 8

The two panels in Figure 8 compare the chord definition of convexity with its equivalent condition on secant slopes.

Proposition 5.5 Both $C(S_0, K, T)$ and $P(S_0, K, T)$ are convex functions of the strike K.

Proof. For $0 \le \lambda \le 1$ and strikes $K_1, K_2 > 0$,

$$\big(S_T - \lambda K_1 - (1-\lambda)K_2\big)_+ = \big(\lambda(S_T - K_1) + (1-\lambda)(S_T - K_2)\big)_+$$

and therefore, using the basic inequality $(x+y)_+ \le x_+ + y_+$,

$$\begin{aligned}\big(S_T - \lambda K_1 - (1-\lambda)K_2\big)_+ &\le (\lambda(S_T - K_1))_+ + ((1-\lambda)(S_T - K_2))_+ \\ &= \lambda(S_T - K_1)_+ + (1-\lambda)(S_T - K_2)_+.\end{aligned}$$

Discounting to time 0 gives

$$C(S_0, \lambda K_1 + (1-\lambda)K_2, T) \le \lambda C(S_0, K_1, T) + (1-\lambda)C(S_0, K_2, T),$$

so C is convex. The same argument works for P. □

Corollary 5.6 If $K_1 < K_2 < K_3$, then

$$\frac{C(S_0, K_1, T) - C(S_0, K_2, T)}{K_1 - K_2} \le \frac{C(S_0, K_2, T) - C(S_0, K_3, T)}{K_2 - K_3}$$

and

$$\frac{P(S_0, K_1, T) - P(S_0, K_2, T)}{K_1 - K_2} \le \frac{P(S_0, K_2, T) - P(S_0, K_3, T)}{K_2 - K_3}.$$

Proof. This follows immediately from Propositions 5.4 and 5.5. □

Example 5.7 Consider three call options on the same stock index, all expiring in one year, with simple interest rate $r = 20\%$:

Strike K	50	59	65
Premium $C(S_0, K, T)$	14	8.9	5

Does there exist any arbitrage?

Solution. Monotonicity holds because $14 > 8.9 > 5$. The Lipschitz bounds also hold:

$$C(S_0, 50, T) - C(S_0, 59, T) = 5.1 < \frac{59 - 50}{1.2} = 7.5$$

and

$$C(S_0, 59, T) - C(S_0, 65, T) = 3.9 < \frac{65 - 59}{1.2} = 5.$$

However, convexity fails because the required slope inequality is reversed:

$$\frac{C(S_0, 50, T) - C(S_0, 59, T)}{50 - 59} = -\frac{14 - 8.9}{9} = -0.567$$

while

$$\frac{C(S_0, 59, T) - C(S_0, 65, T)}{59 - 65} = -\frac{8.9 - 5}{6} = -0.65.$$

Thus $C(S_0, 59, T)$ is overpriced! Indeed,

$$C(S_0, 59, T) = C(S_0, 0.4 \cdot 50 + 0.6 \cdot 65, T) > 0.4C(S_0, 50, T) + 0.6C(S_0, 65, T),$$

or equivalently,

$$10C(S_0, 59, T) > 4C(S_0, 50, T) + 6C(S_0, 65, T).$$

An arbitrage strategy is therefore to long four 50-strike calls and six 65-strike calls, while shorting ten 59-strike calls. The time-0 profit is

$$10C(S_0, 59, T) - 4C(S_0, 50, T) - 6C(S_0, 65, T) = 3.$$

□

The corresponding payoff at expiration is

$$\text{Payoff} = \begin{cases} 0, & S_T \le 50, \\ 4(S_T - 50), & 50 < S_T \le 59, \\ 4(S_T - 50) - 10(S_T - 59) = 390 - 6S_T, & 59 < S_T \le 65, \\ 4(S_T - 50) - 10(S_T - 59) + 6(S_T - 65) = 0, & 65 < S_T. \end{cases}$$

Its nonnegative tent shape is plotted in Figure 9.

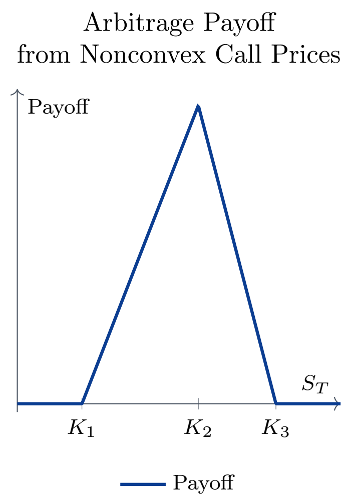

Figure 9

Lecture 6 — Wednesday, January 21, 2015

Exercise Styles

Let $C_E(S_0, K, T)$ and $P_E(S_0, K, T)$ denote the premia for a European call and put (respectively), and let $C_A(S_0, K, T)$ and $P_A(S_0, K, T)$ denote the corresponding premia for American options.

Proposition 6.1 We have the following upper bounds:

$$S_0 \geq C_A(S_0, K, T) \geq C_E(S_0, K, T)$$

and

$$K \geq P_A(S_0, K, T) \geq P_E(S_0, K, T).$$

Proof. Since an American option includes all exercise opportunities of the corresponding European option,

$$C_A(S_0, K, T) \geq C_E(S_0, K, T) \qquad \text{and} \qquad P_A(S_0, K, T) \geq P_E(S_0, K, T).$$

Also, call payoff is bounded by the stock value, $(S_\tau - K)^+ \leq S_\tau$, so no-arbitrage gives $C_A(S_0, K, T) \leq S_0$. For puts, $(K - S_\tau)^+ \leq K$, hence $P_A(S_0, K, T) \leq K$. Combining gives the stated inequalities. □

Proposition 6.2 We also have the following lower bounds:

$$C_A(S_0, K, T) \geq C_E(S_0, K, T) \geq \big(\mathrm{PV}(S_T) - \mathrm{PV}(K)\big)_+$$

and

$$P_A(S_0, K, T) \geq P_E(S_0, K, T) \geq \big(\mathrm{PV}(K) - \mathrm{PV}(S_T)\big)_+.$$

Proof. Put–call parity gives

$$C_E(S_0, K, T) = P_E(S_0, K, T) + \mathrm{PV}(S_T) - \mathrm{PV}(K) \geq \mathrm{PV}(S_T) - \mathrm{PV}(K),$$

since $P_E(S_0, K, T) \geq 0$. Together with $C_E(S_0, K, T) \geq 0$, this yields

$$C_E(S_0, K, T) \geq \big(\mathrm{PV}(S_T) - \mathrm{PV}(K)\big)_+.$$

The put bound is analogous. □

Proposition 6.3 If the stock pays no dividend and $r \geq 0$, then it is never strictly optimal to exercise an American call early. Consequently,

$$C_A(S_0, K, T) = C_E(S_0, K, T).$$

Proof. At any time $t < T$, put–call parity gives

$$C_E(S_t, K, T - t) = P_E(S_t, K, T - t) + S_t - e^{-r(T-t)}K \geq S_t - K.$$

Immediate exercise produces $(S_t - K)_+$, whereas the value of holding is at least the corresponding European value. The displayed inequality, together with $C_E \geq 0$, shows that holding is always at least as valuable as exercising. Thus early exercise is never strictly optimal, and the American and European call values agree:

$$C_A(S_0, K, T) = C_E(S_0, K, T).$$

□

When dividends are present, the situation changes.

Proposition 6.4 Early exercise of an American call at time $t < T$ is not rational if

$$P_E(S_t, K, T-t) + K - \mathrm{PV}_{t,T}(K) > \mathrm{PV}_{t,T}(\mathrm{Div}).$$

Proof. By put–call parity, at any early time $t < T$, we have

$$\begin{aligned} C_A(S_t, K, T-t) &\geq C_E(S_t, K, T-t) \\ &= P_E(S_t, K, T-t) + S_t - \mathrm{PV}_{t,T}(\mathrm{Div}) - \mathrm{PV}_{t,T}(K) \\ &= S_t - K + P_E(S_t, K, T-t) + K - \mathrm{PV}_{t,T}(\mathrm{Div}) - \mathrm{PV}_{t,T}(K) \\ &> S_t - K. \end{aligned}$$

□

This does *not* imply that we should exercise the American call when

$$\mathrm{PV}_{t,T}(\mathrm{Div}) > P_E(S_t, K, T-t) + K - \mathrm{PV}_{t,T}(K).$$

A sufficiently large dividend merely creates an incentive to exercise early in order to acquire the stock before the dividend is paid.

Proposition 6.5 Similarly, early exercise of an American put at time $t < T$ is not rational if

$$\mathrm{PV}_{t,T}(\mathrm{Div}) > K - \mathrm{PV}_{t,T}(K) - C_E(S_t, K, T-t).$$

Proof. Again by put–call parity, at any early time $t < T$, we have

$$\begin{aligned} P_A(S_t, K, T-t) &\geq P_E(S_t, K, T-t) \\ &= C_E(S_t, K, T-t) - S_t + \mathrm{PV}_{t,T}(\mathrm{Div}) + \mathrm{PV}_{t,T}(K) \\ &= K - S_t + C_E(S_t, K, T-t) - K + \mathrm{PV}_{t,T}(\mathrm{Div}) + \mathrm{PV}_{t,T}(K) \\ &> K - S_t. \end{aligned}$$

□

A sufficiently large dividend therefore discourages early exercise of the American put.

Maturity Effects

Suppose that

$$0 < t < T_1 < T_2.$$

Then American options satisfy the monotonicity relations

$$C_A(S_t, K, T_1 - t) \leq C_A(S_t, K, T_2 - t)$$

and

$$P_A(S_t, K, T_1 - t) \leq P_A(S_t, K, T_2 - t),$$

since the longer-dated option offers all of the exercise opportunities of the shorter-dated option and more.

For European calls on a non-dividend-paying stock when $r \geq 0$, the same monotonicity holds:

$$C_E(S_t, K, T_1 - t) \leq C_E(S_t, K, T_2 - t).$$

However, with dividends, the ordering of European calls with different maturities is not determined in general. Likewise, the ordering of European puts with different maturities need not be fixed without additional assumptions.

Tutorial 1 — Friday, January 23, 2015

3. Binomial Pricing Models

The binomial model fits inside the broader pricing pipeline in Figure 10. We first choose a model for the underlying asset, then select a pricing approach, and finally compute the option price.

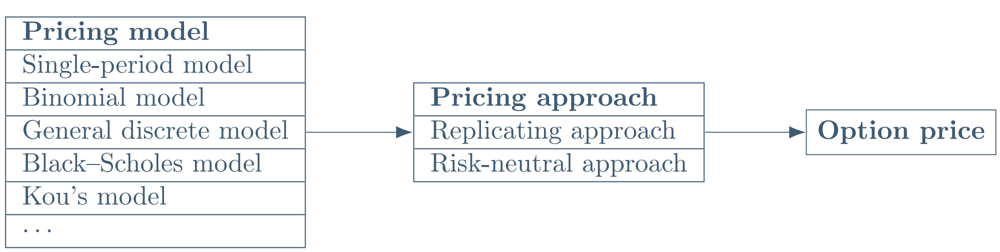

FIGURE 10

The Single-Period Binomial Model

In a **single-period binomial model**, the stock price over the next time step can move to only two states, called the up state and the down state, with fixed ratios

$$0 < d < u.$$

Here d is the **down factor** and u is the **up factor**. If h is the time to expiration, if r is the continuously compounded risk-free rate, if δ is the continuously compounded dividend rate, and if $S(0)$ is the current stock price, then

$$S(h,u) = uS(0) \qquad \text{and} \qquad S(h,d) = dS(0).$$

For an option with payoffs $X(h,u)$ and $X(h,d)$ at time h, the pricing problem is to determine $X(0)$.

The stock tree and the corresponding option payoff tree have the form shown in Figure 11.

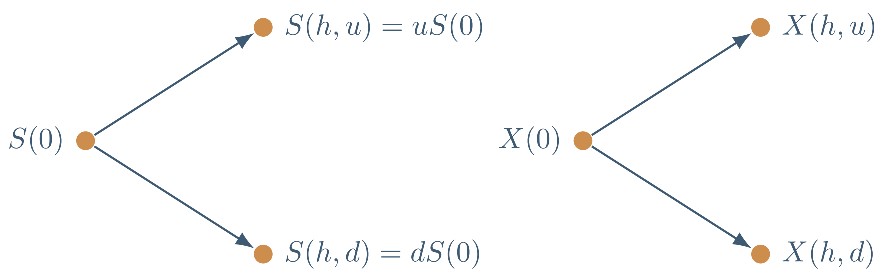

Figure 11

Definition 1.1 Let B be the amount invested in a bond and let Δ be the number of stock shares held. The pair (B, Δ) is called a **replicating portfolio** for the option if

$$e^{rh}B + \Delta e^{\delta h}S(h,u) = X(h,u)$$

and

$$e^{rh}B + \Delta e^{\delta h}S(h,d) = X(h,d).$$

Solving these two equations gives

$$B = e^{-rh}\frac{uX(h,d) - dX(h,u)}{u-d}$$

and

$$\Delta = e^{-\delta h}\frac{X(h,u)-X(h,d)}{uS(0)-dS(0)} = e^{-\delta h}\frac{X(h,u)-X(h,d)}{(u-d)S(0)}.$$

Proposition 1.2 (Replicating portfolio pricing formula) If a replicating portfolio exists, then the option price is

$$X(0) = B + \Delta S(0) = e^{-rh}\left(\frac{e^{(r-\delta)h}-d}{u-d}X(h,u) \;+\; \frac{u-e^{(r-\delta)h}}{u-d}X(h,d)\right).$$

Proof. By the law of one price, any portfolio with the same payoff at time h must have the same price at time 0. Since the replicating portfolio satisfies

$$e^{rh}B + \Delta e^{\delta h}S(h,\omega) = X(h,\omega), \quad \omega \in \{u,d\},$$

the option price must equal the current cost of the portfolio, namely

$$X(0) = B + \Delta S(0).$$

Substituting the formulas for B and Δ yields the stated expression. $\square$

Example 1.3 Consider a single-period binomial model with

$$S(0) = 41, \qquad u = \frac{60}{41}, \qquad \text{and} \qquad d = \frac{30}{41},$$

and

$$h = 1, \qquad r = 8\%, \qquad \text{and} \qquad \delta = 0\%.$$

Determine the price of a one-year European call with strike 40.

Solution. The two stock prices at maturity are

$$S(1,u) = 60 \qquad \text{and} \qquad S(1,d) = 30,$$

so the call payoffs are

$$X(1,u) = (60-40)_+ = 20 \qquad \text{and} \qquad X(1,d) = (30-40)_+ = 0.$$

The replicating equations are

$$e^{0.08}B + 60\Delta = 20 \qquad \text{and} \qquad e^{0.08}B + 30\Delta = 0.$$

Solving gives

$$B = -18.462 \qquad \text{and} \qquad \Delta = 0.667.$$

Therefore the call price is

$$C(40) = B + \Delta S(0) = 8.871.$$

$\square$

Lecture 7 — Monday, January 26, 2015

Market Completeness, Arbitrage, and Risk-Neutral Measures

Definition 7.1 A binomial stock market model is called **complete** if any European option has a replicating portfolio (B, Δ).

Proposition 7.2 A single-period binomial model is complete as long as $d < u$.

Proof. The replicating portfolio is determined by a 2×2 linear system. Since $d < u$, the two future stock prices are distinct, so the determinant is nonzero and the system has a unique solution. $\square$

Definition 7.3 A portfolio (B, Δ) is an **arbitrage portfolio** if its initial cost is zero, if its future payoff is nonnegative in every state, and if its future payoff is strictly positive in at least one state. The binomial model is **arbitrage-free** if no such portfolio exists.

Proposition 7.4 The single-period binomial model is arbitrage-free if and only if

$$d < e^{(r-\delta)h} < u.$$

Proof. If $e^{(r-\delta)h} \leq d < u$, then the stock dominates the bond in both states, so borrowing from the bank and buying the stock produces an arbitrage. If $d < u \leq e^{(r-\delta)h}$, then shorting the stock and investing the proceeds in the bond produces an arbitrage.

Reverse implication. Suppose

$$d < e^{(r-\delta)h} < u,$$

and set

$$q_u = \frac{e^{(r-\delta)h} - d}{u - d} \qquad \text{and} \qquad q_d = \frac{u - e^{(r-\delta)h}}{u - d}.$$

Both weights are positive and sum to one. The discounted risk-neutral value of any zero-cost portfolio is therefore zero. An arbitrage would instead have a nonnegative payoff in both states and a positive payoff in at least one state, giving it a strictly positive risk-neutral value. This contradiction proves that the model is arbitrage-free. □

Definition 7.5 A pair (q_u, q_d) is called a **risk-neutral measure** if the following hold:

1. $q_u e^{\delta h} S(h, u) + q_d e^{\delta h} S(h, d) = e^{rh} S(0)$.
2. $q_u + q_d = 1$.
3. $q_u > 0 \qquad \text{and} \qquad q_d > 0$.

Combining Conditions (1) and (2) gives

$$q_u = \frac{e^{(r-\delta)h} - d}{u - d} \qquad \text{and} \qquad q_d = \frac{u - e^{(r-\delta)h}}{u - d}.$$

The positivity conditions $q_u > 0$ and $q_d > 0$ in Condition (3) hold exactly when

$$d < e^{(r-\delta)h} < u,$$

so the existence of a risk-neutral measure is equivalent to the absence of arbitrage.

Theorem 7.6 (Risk-neutral pricing formula) If $\mathbb{Q}$ is the risk-neutral measure on $\{u, d\}$ with $\mathbb{Q}(\{u\}) = q_u$ and $\mathbb{Q}(\{d\}) = q_d$, then

$$X(0) = e^{-rh} \mathbb{E}^{\mathbb{Q}} \left[X(h) \right].$$

Proof. From Proposition 1.2,

$$\begin{aligned} X(0) &= e^{-rh}\left(\frac{e^{(r-\delta)h}-d}{u-d}X(h,u) \;+\; \frac{u-e^{(r-\delta)h}}{u-d}X(h,d)\right) \\ &= e^{-rh}\big(q_u X(h,u) + q_d X(h,d)\big) \\ &= e^{-rh}\mathbb{E}^{\mathbb{Q}}\left[X(h)\right]. \end{aligned}$$

□

Example 7.7 Consider the same model as in Example 1.3, with

$$S(0) = 41, \qquad u = \frac{60}{41}, \qquad \text{and} \qquad d = \frac{30}{41},$$

and

$$h = 1, \qquad r = 8\%, \qquad \text{and} \qquad \delta = 0\%.$$

Determine the price of a one-year European call with strike 40.

Solution. Since $u = 60/41$ and $d = 30/41$, the risk-neutral probabilities are

$$q_u = \frac{e^{0.08} - 30/41}{(60/41) - (30/41)} = 0.48 \qquad \text{and} \qquad q_d = \frac{60/41 - e^{0.08}}{(60/41) - (30/41)} = 0.52.$$

Therefore

$$C(40) = e^{-0.08}\big(q_u \cdot 20 + q_d \cdot 0\big) = 8.871,$$

which agrees with the price found by replication. □

Risk-neutral pricing can be interpreted by contrasting a **risk-averse** investor, who prefers receiving \$1000 for sure, with a **risk-neutral** investor, who is indifferent between receiving \$1000 for sure and receiving \$2000 or \$0 with equal probability.

Remark 7.8 Under the risk-neutral measure, every traded asset's total return has the risk-free expected growth factor. For a stock paying continuous dividend yield δ, this means that its expected price appreciation is $r - \delta$, while price appreciation together with reinvested dividends has expected growth rate r. The risk-neutral measure is an artificial pricing measure; its probabilities need not describe real-world frequencies.

The single-period analysis extends next to multi-period binomial models and binomial trees.

Lecture 8 — Wednesday, January 28, 2015

Multi-Period and American Options

The **multi-period binomial model** assumes N trading times

$$\{0, h, 2h, \ldots, Nh\} \qquad \text{and} \qquad T = Nh,$$

and with up and down factors satisfying the no-arbitrage condition

$$d < e^{(r-\delta)h} < u.$$

At time nh, a state of nature is described by a word $\omega \in \{u, d\}^n$, or equivalently by a product $u^m d^{n-m}$ with $0 \leq m \leq n$. The stock price in that state is

$$S(nh, \omega) = S(nh, u^m d^{n-m}) = u^m d^{n-m} S(0).$$

The first two periods of the recombining stock tree are shown in Figure 12.

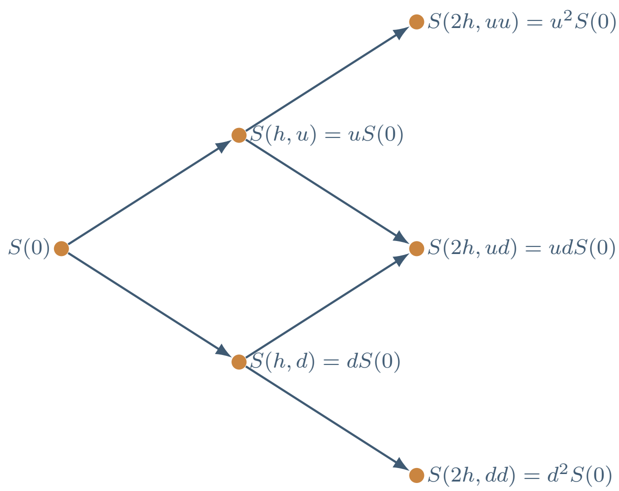

Figure 12

At maturity, the American and European values coincide. Thus, for every state at maturity ω,

$$X_{\text{call}}(T, \omega) = \big(S(T, \omega) - K\big)_+$$

and

$$X_{\text{put}}(T,\omega) = \big(K - S(T,\omega)\big)_+.$$

At an earlier node (nh,ω), an American option value is the larger of the exercise value and the value of holding.

If q_u and q_d are the single-period risk-neutral probabilities, then the value of holding at node (nh,ω) is

$$e^{-rh}\big(q_u X((n+1)h,\omega u) + q_d X((n+1)h,\omega d)\big).$$

Therefore the American call satisfies the recursion

$$X_{\text{call}}(nh,\omega) = \max\Big\{S(nh,\omega) - K,\ e^{-rh}\big(q_u X_{\text{call}}((n+1)h,\omega u) + q_d X_{\text{call}}((n+1)h,\omega d)\big)\Big\},$$

and the American put similarly satisfies

$$X_{\text{put}}(nh,\omega) = \max\Big\{K - S(nh,\omega),\ e^{-rh}\big(q_u X_{\text{put}}((n+1)h,\omega u) + q_d X_{\text{put}}((n+1)h,\omega d)\big)\Big\}.$$

The one-step branch used in this backward recursion is shown in Figure 13.

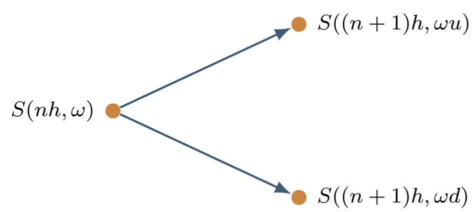

FIGURE 13

The binomial pricing procedure for an American vanilla option is as follows.

Step 1. First construct the N-period binomial tree for the stock price

$$S(nh,\omega) = S(nh, u^m d^{n-m}) = u^m d^{n-m} S(0), \quad \text{for } 0 \le m \le n \le N.$$

Step 2. Determine the value of the American option at maturity $X(T) = X(Nh)$ in all states.

Step 3. Compute the single-period risk-neutral probabilities

$$q_u = \frac{e^{(r-\delta)h} - d}{u-d} \qquad \text{and} \qquad q_d = \frac{u - e^{(r-\delta)h}}{u-d} = 1 - q_u.$$

Step 4. Move backwards through the tree one time period at a time. At each time, use the risk-neutral pricing formula to compare immediate exercise with the discounted value of holding.

The first three steps are the same as in European pricing. The difference lies entirely in the final step. For a European option, we discount the continuation values directly back to time 0 to obtain $X(0)$. For an American option, we must also decide at every node whether early exercise is better than continuing to hold the option.

Example 8.1 Let

$$S(0) = K = 40, \qquad r = 8\%, \qquad \sigma = 30\%, \qquad \delta = 0, \qquad N = 2, \qquad \text{and} \qquad T = 0.5.$$

Calculate the premium of European and American puts using a forward tree with $h = 0.25$.

Solution. Then $h = T/N = 0.25$, and the forward tree parameterization gives

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = 1.1853 \qquad \text{and} \qquad d = e^{(r-\delta)h-\sigma\sqrt{h}} = 0.8781.$$

The corresponding stock prices are

$$S(h, u) = 47.412 \qquad \text{and} \qquad S(h, d) = 35.124,$$

and

$$S(2h, uu) = 56.198, \qquad S(2h, ud) = 41.632, \qquad \text{and} \qquad S(2h, dd) = 30.842.$$

The single-period risk-neutral probabilities are

$$q_u = \frac{e^{0.08/4} - 0.8781}{1.1853 - 0.8781} = 0.4626 \qquad \text{and} \qquad q_d = 0.5374.$$

At maturity, the put payoffs are 0, 0, and 9.158. Working backward gives the European price

$$P_E(0) = e^{-0.08/2}\big(q_d^2 \cdot 9.158\big) = 2.541.$$

For the American put, the down node at time h has value of holding 4.824 but intrinsic value

$$40 - 35.124 = 4.876,$$

so early exercise is optimal there. Hence the American value is

$$P_A(0) = 2.569,$$

which is slightly larger than the European value. $\square$

The node labels in Figure 14 pair each stock price with its American put value; the down node at time h is the node where early exercise is optimal.

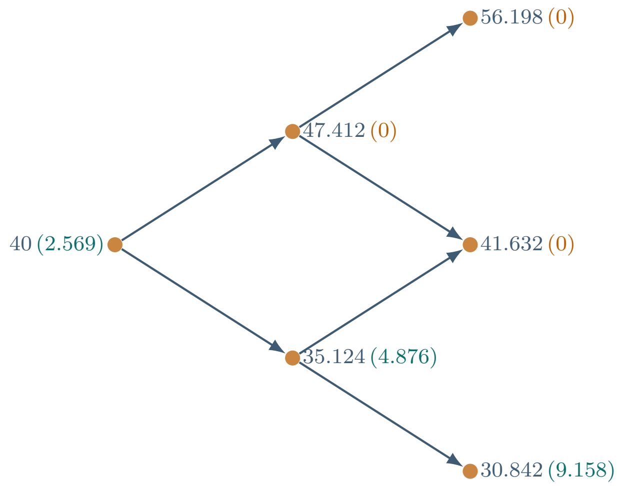

FIGURE 14

Example 8.2 Consider a three-period at-the-money American put with one-year time steps, with

$$r = 10\%, \qquad \delta = 6.5\%, \qquad S(0) = K = 300, \qquad u = 1.25, \qquad \text{and} \qquad d = 0.7.$$

Determine the price of the put option.

Solution. The risk-neutral probabilities are

$$q_u = \frac{e^{0.1-0.065} - 0.7}{1.25 - 0.7} = 0.61022 \qquad \text{and} \qquad q_d = 0.38978.$$

Backward induction gives

$$P_A(0) = 39.7263.$$

Figure 15 records the stock price at each node together with the American put value. At the nodes (h, d) and $(2h, dd)$, early exercise is optimal because the intrinsic values 90 and 153 dominate the value of holdings there.

$\square$

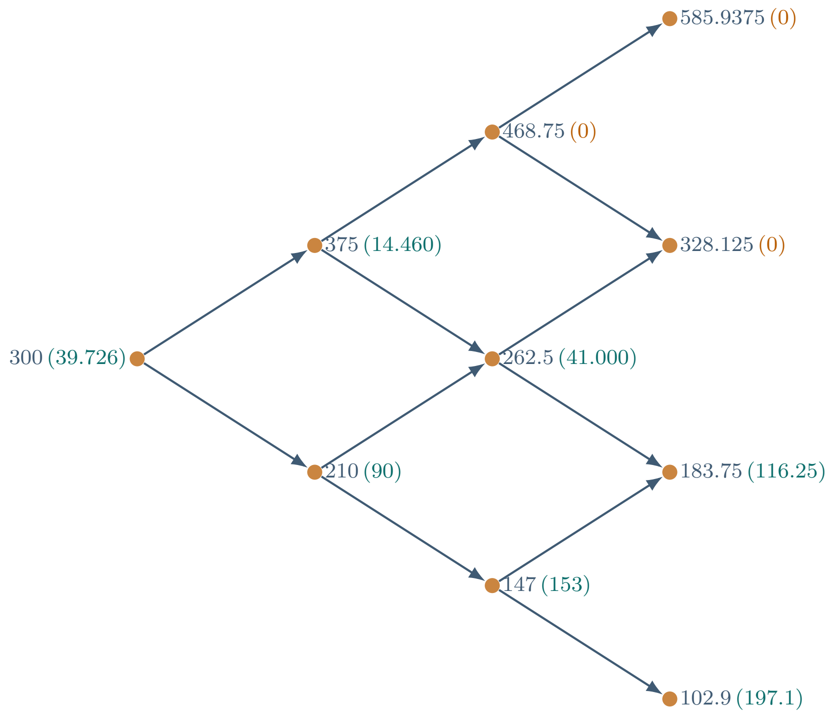

Figure 15

Lecture 9 — Monday, February 02, 2015

Exotic Options

An **exotic option** is an option whose payoff is more complicated than that of a vanilla call or put. In many cases the payoff depends on the entire path of the stock price rather than only on the price at maturity $S(T)$.

Standard examples include Asian options, lookback options, all-or-nothing options, barrier options, gap options, and compound options. The formulas below should be distinguished carefully because the strike, the averaging rule, and the path condition all matter.

An **Asian option** depends on the average underlying price over $[0, T]$. There are many notions of averaging that can be used; for example, for monitoring dates $h, 2h, \ldots, Nh = T$, the **arithmetic**

average price is

$$A(T) = \frac{1}{N}\sum_{n=1}^{N} S(nh),$$

and the **geometric average price** is

$$G(T) = \left(\prod_{n=1}^{N} S(nh)\right)^{1/N}.$$

Using $A(T)$ for concreteness, the four basic Asian payoffs are

$$\begin{aligned}
\text{Payoff of call with fixed strike} &= \big(A(T) - K\big)_+ \\
\text{Payoff of put with fixed strike} &= \big(K - A(T)\big)_+ \\
\text{Payoff of call with floating strike} &= \big(S(T) - A(T)\big)_+ \\
\text{Payoff of put with floating strike} &= \big(A(T) - S(T)\big)_+
\end{aligned}$$

A **lookback option** depends on the running maximum or running minimum of the stock price:

$$\begin{aligned}
\text{Payoff of call with fixed strike} &= \left(\max_{0\le t\le T} S(t) - K\right)_+ \\
\text{Payoff of put with fixed strike} &= \left(K - \min_{0\le t\le T} S(t)\right)_+ \\
\text{Payoff of call with floating strike} &= S(T) - \min_{0\le t\le T} S(t) \\
\text{Payoff of put with floating strike} &= \max_{0\le t\le T} S(t) - S(T)
\end{aligned}$$

The tree indexed by paths in Figure 16 previews why recombination can fail for an Asian payoff. It uses $S(0) = 100$, up and down factors 1.2 and 0.8, and a fixed-strike Asian call with strike 100, where the average is taken over times 1 and 2. The paths ud and du finish at the same stock price but produce averages 108 and 88, and hence payoffs 8 and 0.

A **barrier option** has a payoff that is activated or extinguished when the underlying price breaches a threshold L. A **binary option** (or **all-or-nothing option**) instead pays a fixed amount or delivers an asset when a specified event occurs. For **knock-in options**,

$$\begin{aligned}
\text{Payoff of up-and-in call} &= (S(T) - K)_+ \mathbb{1}_{\{\max_{0\le t\le T} S(t) > L\}} \\
\text{Payoff of down-and-in call} &= (S(T) - K)_+ \mathbb{1}_{\{\min_{0\le t\le T} S(t) \le L\}}
\end{aligned}$$

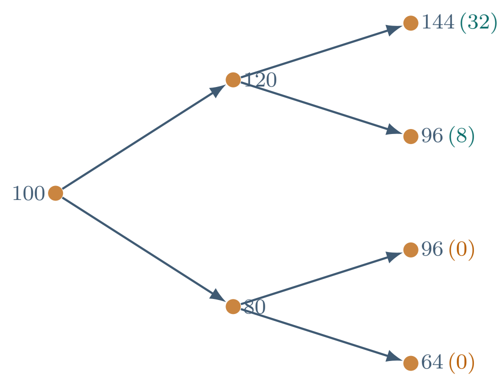

FIGURE 16

and for **knock-out options**,

$$\text{Payoff of up-and-out call} = (S(T)-K)_+ \mathbb{1}_{\{\max_{0\le t\le T} S(t)\le L\}}$$
$$\text{Payoff of down-and-out call} = (S(T)-K)_+ \mathbb{1}_{\{\min_{0\le t\le T} S(t)> L\}}$$

Puts are defined analogously. The barrier decomposition identities are

$$\text{up-and-in call} + \text{up-and-out call} = \text{regular call}$$

and

$$\text{down-and-in call} + \text{down-and-out call} = \text{regular call},$$

because the two barrier events partition the sample paths:

$$(S(T)-K)_+ = (S(T)-K)_+ \left(\mathbb{1}_{\{\max_{0\le t\le T} S(t)\le L\}} + \mathbb{1}_{\{\max_{0\le t\le T} S(t)> L\}}\right)$$

and

$$(S(T)-K)_+ = (S(T)-K)_+ \left(\mathbb{1}_{\{\min_{0\le t\le T} S(t)\le L\}} + \mathbb{1}_{\{\min_{0\le t\le T} S(t)> L\}}\right).$$

Example 9.1 Path dependence can already be seen on the two-step tree with first-step stock prices 120 and 80, and stock prices at maturity

$$144, \qquad 96, \qquad 96, \qquad \text{and} \qquad 64.$$

The paths ud and du both end at 96, but they produce different path statistics.

The same tree also illustrates indicator payoffs and barrier payoffs. If monitoring occurs only at times h and $2h$, then the all-or-nothing payoff

$$\mathbb{1}_{\{\max\{S(h),S(2h)\}>105\}}$$

equals 1 on the paths uu and ud, but equals 0 on the paths du and dd. Likewise, the barrier payoff

$$(S(T)-90)_+\mathbb{1}_{\{\max\{S(h),S(2h)\}>100\}}$$

takes the values 54, 6, 0, and 0 on the paths uu, ud, du, and dd, respectively.

The corresponding pathwise indicator values are displayed in Figure 17.

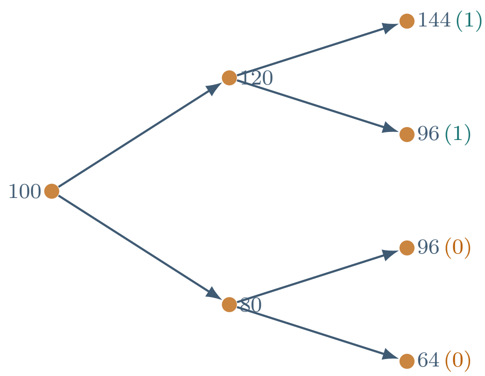

FIGURE 17

A **gap option** has a discontinuous payoff. For a European call, the payoff takes the form $(S(T)-K)\mathbb{1}_{S(T)\geq K_t}$, where K_t is the **trigger price**. For a European put, the payoff takes the form $(K-S(T))\mathbb{1}_{S(T)<K_t}$.

Example 9.2 A representative example is

$$(S(T)-90)\mathbb{1}_{\{S(T)>100\}}.$$

Algebraically,

$$(S(T)-90)\mathbb{1}_{\{S(T)>100\}}=\big(S(T)-100+10\big)\mathbb{1}_{\{S(T)>100\}}=(S(T)-100)_++10\mathbb{1}_{\{S(T)>100\}},$$

so this gap option can be decomposed into a standard call plus ten units of an all-or-nothing payoff.

A **compound option** is an option on another option. If the underlying option has strike K and maturity T_2, and if the compound option has strike x and maturity $T_1<T_2$, then exercising the

compound option at time T_1 means buying or selling the underlying option for the price x.

If $C(S_{T_1}, K, T_2 - T_1)$ denotes the time-T_1 value of the underlying call, then the two payoffs are

$$\text{Payoff of CallOnCall (at time } T_1) = \big(C(S_{T_1}, K, T_2 - T_1) - x\big)_+$$
$$\text{Payoff of PutOnCall (at time } T_1) = \big(x - C(S_{T_1}, K, T_2 - T_1)\big)_+$$

The corresponding put–call parity is

$$\begin{aligned} &\text{CallOnCall}(S_0, K, x, T_1, T_2) - \text{PutOnCall}(S_0, K, x, T_1, T_2) \\ &= \text{PV}_{0,T_1}\big(C(S_{T_1}, K, T_2 - T_1)\big) - \text{PV}_{0,T_1}(x) \\ &= C(S_0, K, T_2) - xe^{-rT_1}. \end{aligned}$$

Path-Dependent Pricing with Binomial Trees

A **combined** binomial tree merges the paths ud and du because they lead to the same stock price at time $2h$. An **uncombined** tree keeps those paths separate. At time nh, a combined tree has $n+1$ nodes, whereas an uncombined tree has 2^n nodes. For example, the set of states at time $2h$ for an uncombined tree are $\{uu, ud, du, dd\}$, rather than $\{u^2, ud, d^2\}$ as for a combined tree.

The reason for using an uncombined tree is now clear. A path-dependent payoff distinguishes between ud and du, even though both paths may end at the same stock price at maturity. Thus, pricing an exotic option requires the full path information, as Figure 18 emphasizes.

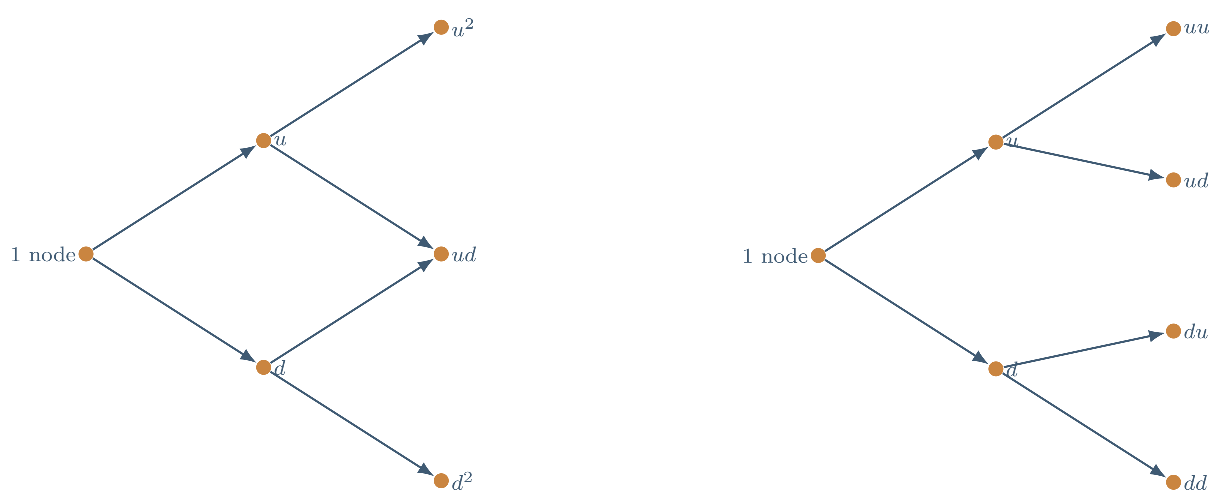

FIGURE 18

For a European exotic option, the pricing procedure is as follows:

Step 1. First construct the N-period uncombined tree for the stock price.

Step 2. Determine the exotic option payoffs at maturity $X(T) = X(Nh)$ in all 2^N states.

Step 3. Compute the single-period risk-neutral probabilities

$$q_u = \frac{e^{(r-\delta)h} - d}{u - d} \qquad \text{and} \qquad q_d = \frac{u - e^{(r-\delta)h}}{u - d} = 1 - q_u.$$

Step 4. Determine the European option price by directly applying risk-neutral pricing:

$$X(0) = e^{-rT} \sum_{\omega \in \Omega_T} \mathbb{Q}(\{\omega\}) X(T, \omega),$$

where Ω_T is the set of all states of nature at maturity, which has 2^N elements because every up-down word must be kept distinct.

For an American exotic option, the first three steps are the same, but the fourth uses backward recursion:

Step 4'. Determine the American option price $X(0)$ using the following recursion: At each path $\omega \in \{u, d\}^n$,

$$X(nh, \omega) = \max\Big\{\text{exercise payoff at } (nh, \omega), \\ e^{-rh}\big(q_u X((n+1)h, \omega u) + q_d X((n+1)h, \omega d)\big)\Big\}.$$

This is exactly the American option recursion from before, except that the nodes are now indexed by paths rather than being determined solely by the current stock price.

The four cases can be organized as follows. European vanilla options use a combined tree and direct pricing. American vanilla options use a combined tree and recursive pricing. European exotic options use an uncombined tree and direct pricing. American exotic options use an uncombined tree and recursive pricing.

Option class	Tree structure	Pricing method
European vanilla	Combined	Direct risk-neutral valuation
American vanilla	Combined	Backward recursion with exercise test
European exotic	Uncombined	Direct risk-neutral valuation
American exotic	Uncombined	Backward recursion with exercise test

Lecture 10 — Wednesday, February 04, 2015

Example 10.1 Suppose that

$$S(0) = 100 = K, \qquad r = 8\%, \qquad \delta = 0\%, \qquad u = 1.2,$$
$$d = 0.8, \qquad T = 2, \qquad \text{and} \qquad h = 1.$$

Find the time-0 price of a European Asian call with maturity payoff

$$C_E(2) = \left(\frac{S(1)+S(2)}{2} - 100\right)_+ .$$

Solution. The one-step risk-neutral probabilities are

$$q_u = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{0.08} - 0.8}{1.2 - 0.8} \approx 0.708218 \qquad \text{and} \qquad q_d = 1 - q_u = 0.2918.$$

Since $T = 2$ and $h = 1$, there are two periods and four paths to maturity:

Path	$S(1)$	$S(2)$	$C_E(2) = \left(\frac{S(1)+S(2)}{2} - 100\right)_+$
uu	120	144	32
ud	120	96	8
du	80	96	0
dd	80	64	0

Hence direct risk-neutral valuation gives

$$C_E(0) = e^{-rT}\left(32\, q_u^2 + 8\, q_u q_d\right) = e^{-0.08\cdot 2}\left(32\, q_u^2 + 8\, q_u q_d\right) \approx 15.0859.$$

$\square$

The tree indexed by paths for this Asian call example is shown in Figure 19. The two nodes with $S(2) = 96$ must be kept distinct because their payoffs are different.

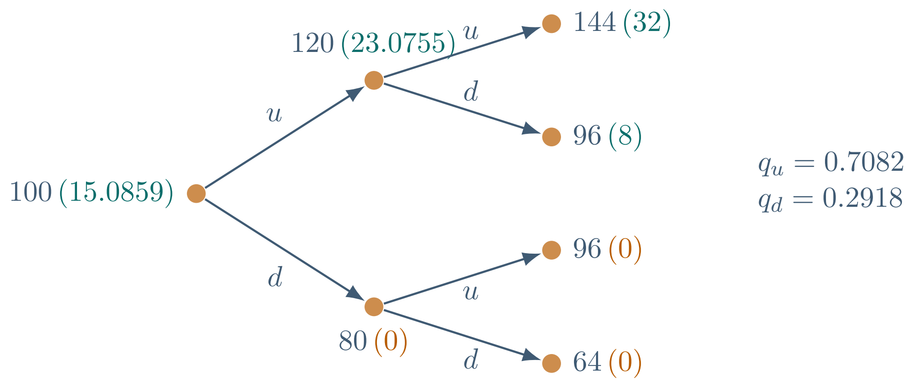

FIGURE 19

Example 10.2 Suppose that

$$S(0) = 100 = K, \qquad r = 8\%, \qquad \delta = 0\%, \qquad u = 1.2,$$
$$d = 0.8, \qquad T = 2, \qquad \text{and} \qquad h = 1.$$

Determine the time-0 price of an American Asian call with exercise payoff at time $n \in \{1, 2\}$ given by

$$C_A(n) = \left(\frac{1}{n} \sum_{m=1}^{n} S(m) - 100 \right)_+ .$$

Solution. The one-period risk-neutral probabilities are

$$q_u = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{0.08} - 0.8}{1.2 - 0.8} \approx 0.708218 \qquad \text{and} \qquad q_d = 1 - q_u = 0.2918.$$

At maturity $n = 2$, the pathwise payoffs are

Path	$C_A(2) = \left(\frac{S(1)+S(2)}{2} - 100 \right)_+$
uu	32
ud	8
du	0
dd	0

At time $n = 1$, for the up node $S(1) = 120$, the immediate exercise value is

$$C_A^{\text{ex}}(1, u) = (120 - 100)_+ = 20,$$

and the continuation value is

$$C_A^{\text{cont}}(1, u) = e^{-0.08}(32q_u + 8q_d) \approx 23.0755,$$

so

$$C_A(1, u) = \max\{20, 23.0755\} = 23.0755.$$

For the down node $S(1) = 80$, both immediate and continuation values are 0, hence

$$C_A(1, d) = 0.$$

Finally,

$$C_A(0) = e^{-0.08}\big(q_u C_A(1, u) + q_d C_A(1, d)\big) = e^{-0.08}\big(q_u \cdot 23.0755 + q_d \cdot 0\big) \approx 15.0859.$$

□

Figure 20 records the same pathwise states and the American values obtained by comparing exercise and continuation at time 1.

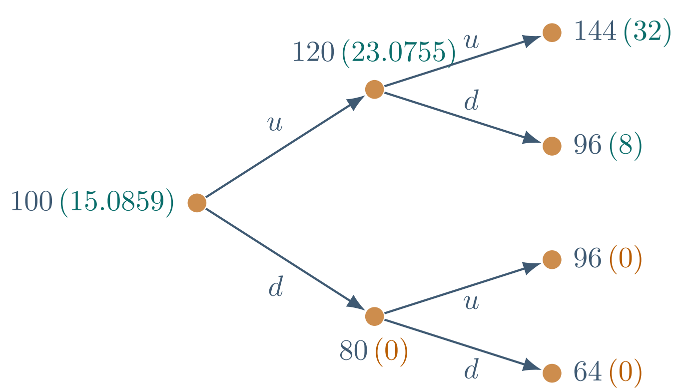

FIGURE 20

Lecture 11 — Monday, February 09, 2015

4. General Discrete-Period Asset Pricing

This lecture generalizes the single-period binomial model to an arbitrary finite state space. The main objective is to express arbitrage, completeness, and pricing in linear-algebraic terms and then use that formulation to state the first and second fundamental theorems of asset pricing in the single-period setting.

Single-Period Finite-State Models

For the single-period general model, we impose two simplifications. First, the interest rate is a simple interest rate with positive gross return $1 + rT$. Second, all risky assets are assumed to be non-dividend-paying. These assumptions keep the notation cleaner and isolate the linear-algebraic issues.

The usual single-period binomial model can be viewed as a special case of a general **finite-state model**. In both cases there are only two trading dates, namely time 0 and time T, and there are n traded assets, one of which is risk-free. The difference is that the binomial model has two states $\Omega = \{u, d\}$, whereas the general model has

$$\Omega = \{\omega_1, \dots, \omega_m\}.$$

In the single-period binomial model, Asset 1 is the risk-free asset with $S_1(0) = 1$ and $S_1(T) = 1 + rT$, and the $n - 1$ risky assets have initial price $S_i(0)$ and time T price

$$S_i(T, u) = uS_i(0) \qquad \text{and} \qquad S_i(T, d) = dS_i(0), \quad \text{for all } i = 2, \dots, n,$$

subject to the no-arbitrage condition

$$0 < d < 1 + rT < u.$$

In the **general single-period model**, there are two trading dates 0 and T, there are n traded assets, and there are m states of nature at time T. Asset 1 is the risk-free asset, with

$$S_1(0) = 1 \qquad \text{and} \qquad S_1(T, \omega_1) = \cdots = S_1(T, \omega_m) = 1 + rT,$$

where r is the simple interest rate.

The time-0 price vector is written as

$$\boldsymbol{S}(0) = \begin{pmatrix} S_1(0) \\ S_2(0) \\ \vdots \\ S_n(0) \end{pmatrix} \in \mathbb{R}^n,$$

and the matrix of time-T asset prices across all states is

$$\mathbf{S}(T, \Omega) = \begin{pmatrix} S_1(T, \omega_1) & \cdots & S_1(T, \omega_m) \\ S_2(T, \omega_1) & \cdots & S_2(T, \omega_m) \\ \vdots & \ddots & \vdots \\ S_n(T, \omega_1) & \cdots & S_n(T, \omega_m) \end{pmatrix} \in \mathbb{R}^{n \times m}.$$

Arbitrage, Replication, and Completeness

A **portfolio** (or **trading strategy**) is a vector

$$\boldsymbol{\pi} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_n \end{pmatrix} \in \mathbb{R}^n,$$

where π_i denotes the number of shares held in asset i. The portfolio value at time 0 is

$$V^{\boldsymbol{\pi}}(0) = \boldsymbol{S}(0)^\top \boldsymbol{\pi} = \sum_{i=1}^n \pi_i S_i(0),$$

and the vector of portfolio values at time T is

$$\boldsymbol{V}^{\boldsymbol{\pi}}(T, \Omega) = \begin{pmatrix} V^{\boldsymbol{\pi}}(T, \omega_1) \\ \vdots \\ V^{\boldsymbol{\pi}}(T, \omega_m) \end{pmatrix} = \mathbf{S}(T, \Omega)^\top \boldsymbol{\pi} = \begin{pmatrix} \displaystyle\sum_{i=1}^n \pi_i S_i(T, \omega_1) \\ \vdots \\ \displaystyle\sum_{i=1}^n \pi_i S_i(T, \omega_m) \end{pmatrix} \in \mathbb{R}^m.$$

Thus the payoff at maturity of a portfolio is obtained by multiplying the $m \times n$ matrix $\mathbf{S}(T,\Omega)^\top$ by the holdings vector $\boldsymbol{\pi}$.

Definition 11.1 A portfolio $\boldsymbol{\pi} \in \mathbb{R}^n$ is an **arbitrage** if all three of the following conditions hold:

1. $V^{\boldsymbol{\pi}}(0) = 0$.
2. $V^{\boldsymbol{\pi}}(T,\omega_i) \geq 0$ for every state $\omega_i \in \Omega$.
3. $V^{\boldsymbol{\pi}}(T,\omega_{i_0}) > 0$ for at least one state $\omega_{i_0} \in \Omega$.

This is the exact analogue of the earlier binomial definition: the portfolio has zero initial cost, never loses money, and makes a strictly positive gain in at least one state.

The next concept is replication, which records when a portfolio matches a target payoff exactly.

A European **contingent claim** is described by its payoff vector at maturity

$$\boldsymbol{X}(T,\Omega) = \begin{pmatrix} X(T,\omega_1) \\ \vdots \\ X(T,\omega_m) \end{pmatrix} \in \mathbb{R}^m.$$

A trading strategy $\boldsymbol{\pi}$ is a **replicating portfolio** for this claim if

$$\boldsymbol{V}^{\boldsymbol{\pi}}(T,\Omega) = \mathbf{S}(T,\Omega)^\top \boldsymbol{\pi} = \boldsymbol{X}(T,\Omega).$$

If the linear system above has a solution, then the law of one price implies that the derivative price at time 0 must be

$$X(0) = \boldsymbol{S}(0)^\top \boldsymbol{\pi}.$$

A single-period market is called **complete** if every contingent claim has a replicating portfolio.

Proposition 11.2 The general single-period market is complete if and only if

$$\operatorname{rank}\big(\mathbf{S}(T,\Omega)^\top\big) = m.$$

Proof. The equation

$$\mathbf{S}(T,\Omega)^\top \boldsymbol{\pi} = \boldsymbol{X}(T,\Omega)$$

must be solvable for every right-hand side $\boldsymbol{X}(T,\Omega) \in \mathbb{R}^m$. This happens exactly when the column space of $\mathbf{S}(T,\Omega)^\top$ is all of $\mathbb{R}^m$, which is equivalent to saying that the matrix has rank m. $\square$

In particular, completeness requires $m \le n$. If there are more states than traded assets, then we cannot usually span every possible payoff vector.

Lecture 12 — Wednesday, February 11, 2015

Risk-Neutral Measures

For each asset i, we may view

$$\boldsymbol{S}_i(T,\Omega) = \begin{pmatrix} S_i(T,\omega_1) \\ \vdots \\ S_i(T,\omega_m) \end{pmatrix} \in \mathbb{R}^m$$

as describing a random variable $S_i(T)$ defined on the state space $\Omega = \{\omega_1, \ldots, \omega_m\}$. If $\mathbb{Q}$ is a probability measure on Ω, its expectation is

$$\mathbb{E}^{\mathbb{Q}}[S_i(T)] := \sum_{j=1}^{m} S_i(T,\omega_j) \cdot \mathbb{Q}(\{\omega_j\}).$$

Definition 12.1 A probability measure $\mathbb{Q}$ on Ω is called a **risk-neutral measure** if its probability vector

$$\boldsymbol{q}(\Omega) := \begin{pmatrix} q(\omega_1) \\ \vdots \\ q(\omega_m) \end{pmatrix} = \begin{pmatrix} \mathbb{Q}(\{\omega_1\}) \\ \vdots \\ \mathbb{Q}(\{\omega_m\}) \end{pmatrix}$$

satisfies

$$\mathbf{S}(T,\Omega)\boldsymbol{q}(\Omega) = (1+rT)\boldsymbol{S}(0)$$

and

$$q(\omega_i) > 0 \quad \text{for each } \omega_i \in \Omega.$$

The matrix equation says exactly that under $\mathbb{Q}$, the expected value at maturity of every asset equals its risk-free growth:

$$\mathbb{E}^{\mathbb{Q}}\left[S_i(T)\right] = (1+rT)S_i(0), \quad i = 1, \ldots, n,$$

where

$$\mathbb{E}^{\mathbb{Q}}\left[S_i(T)\right] = \sum_{j=1}^{m} S_i(T, \omega_j) q(\omega_j).$$

Equivalently, every asset has expected gross return $1 + rT$ over the period under the risk-neutral measure.

Theorem 12.2 If a risk-neutral measure $\mathbb{Q}$ exists, then the time-0 price of any attainable contingent claim with payoff $\boldsymbol{X}(T, \Omega)$ is

$$X(0) = \frac{1}{1+rT}\mathbb{E}^{\mathbb{Q}}\left[X(T)\right] = \frac{1}{1+rT}\boldsymbol{X}(T,\Omega)^{\top}\boldsymbol{q}(\Omega).$$

Proof. If $\boldsymbol{X}(T, \Omega)$ is attainable, there exists $\boldsymbol{\pi}$ such that

$$\mathbf{S}(T,\Omega)^{\top}\boldsymbol{\pi} = \boldsymbol{X}(T,\Omega).$$

Hence

$$\mathbb{E}^{\mathbb{Q}}\left[X(T)\right] = \boldsymbol{X}(T,\Omega)^{\top}\boldsymbol{q}(\Omega) = \boldsymbol{\pi}^{\top}\mathbf{S}(T,\Omega)\boldsymbol{q}(\Omega).$$

Using the defining property of the risk-neutral measure gives

$$\mathbb{E}^{\mathbb{Q}}\left[X(T)\right] = \boldsymbol{\pi}^{\top}(1+rT)\boldsymbol{S}(0) = (1+rT)\boldsymbol{S}(0)^{\top}\boldsymbol{\pi} = (1+rT)X(0),$$

which proves the formula. $\square$

Definition 12.3 A vector

$$\boldsymbol{\psi}(\Omega) = \begin{pmatrix} \psi(\omega_1) \\ \vdots \\ \psi(\omega_m) \end{pmatrix}$$

is called a **state price vector** if

$$\mathbf{S}(T,\Omega)\boldsymbol{\psi}(\Omega) = \boldsymbol{S}(0)$$

and

$$\psi(\omega_i) > 0 \quad \text{for each } \omega_i \in \Omega.$$

The relation between risk-neutral probabilities and state prices is

$$\psi(\omega_i) = \frac{q(\omega_i)}{1+rT} = \frac{1}{1+rT}\mathbb{E}^{\mathbb{Q}}[e_i(T)] \qquad \text{and} \qquad q(\omega_i) = (1+rT)\psi(\omega_i),$$

where

$$e_i(T, \omega_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}.$$

Thus $\psi(\omega_i)$ is the price assigned by this state-price vector to a claim that pays 1 dollar in state ω_i and 0 in every other state. If that claim is attainable, this is its unique no-arbitrage price; if it is not attainable, different state-price vectors may assign different prices to it.

The risk-neutral pricing formula then becomes

$$X(0) = \frac{1}{1+rT}\boldsymbol{X}(T,\Omega)^\top \boldsymbol{q}(\Omega) = \boldsymbol{X}(T,\Omega)^\top \boldsymbol{\psi}(\Omega),$$

or equivalently,

$$X(0) = \frac{1}{1+rT}\sum_{j=1}^{m} X(T,\omega_j)q(\omega_j) = \sum_{j=1}^{m} X(T,\omega_j)\psi(\omega_j).$$

Fundamental Theorems of Asset Pricing

Theorem 12.4 (First fundamental theorem of asset pricing) The following are equivalent in the general single-period model:

1. The market is arbitrage-free.
2. There exists a risk-neutral measure; that is, there exists a positive solution $\boldsymbol{q}(\Omega)$ to
$$\mathbf{S}(T,\Omega)\boldsymbol{q}(\Omega) = (1+rT)\boldsymbol{S}(0).$$
3. There exists a state price vector; that is, there exists a positive solution $\boldsymbol{\psi}(\Omega)$ to
$$\mathbf{S}(T,\Omega)\boldsymbol{\psi}(\Omega) = \boldsymbol{S}(0).$$

Proof. *Equivalence of* (2) *and* (3). If $\boldsymbol{q}(\Omega) > 0$ satisfies
$$\mathbf{S}(T,\Omega)\boldsymbol{q}(\Omega) = (1+rT)\boldsymbol{S}(0),$$
then with $\boldsymbol{\psi}(\Omega) := (1/(1+rT))\boldsymbol{q}(\Omega)$, we obtain
$$\mathbf{S}(T,\Omega)\boldsymbol{\psi}(\Omega) = \boldsymbol{S}(0),$$
and positivity is preserved. The converse follows by $\boldsymbol{q}(\Omega) = (1+rT)\boldsymbol{\psi}(\Omega)$.

(2) *implies* (1). Assume $\boldsymbol{q}(\Omega) > 0$ solves
$$\mathbf{S}(T,\Omega)\boldsymbol{q}(\Omega) = (1+rT)\boldsymbol{S}(0).$$
If an arbitrage $\boldsymbol{\pi}$ existed, then
$$\boldsymbol{S}(0)^\top\boldsymbol{\pi} = 0 \qquad \text{and} \qquad \mathbf{S}(T,\omega_i)^\top\boldsymbol{\pi} \ge 0 \ \forall i,$$
with strict inequality for at least one state. Therefore,
$$0 = (1+rT)\boldsymbol{S}(0)^\top\boldsymbol{\pi} = \big(\mathbf{S}(T,\Omega)\boldsymbol{q}(\Omega)\big)^\top\boldsymbol{\pi} = \boldsymbol{q}(\Omega)^\top\mathbf{S}(T,\Omega)^\top\boldsymbol{\pi} = \sum_{i=1}^{m} q(\omega_i)\,\mathbf{S}(T,\omega_i)^\top\boldsymbol{\pi}.$$
Every term in the sum is nonnegative, and at least one is strictly positive because $q(\omega_i) > 0$, so the sum is > 0, a contradiction.

(1) *implies* (2). Define the linear subspace

$$\mathcal{L} := \left\{ \mathbf{S}(T,\Omega)^\top \boldsymbol{\pi} - (1+rT)\mathbf{1}\, \boldsymbol{S}(0)^\top \boldsymbol{\pi} \;:\; \boldsymbol{\pi} \in \mathbb{R}^n \right\} \subset \mathbb{R}^m,$$

where $\mathbf{1}$ is the m-vector of ones. The no-arbitrage condition is exactly

$$\mathcal{L} \cap \mathbb{R}^m_+ = \{\mathbf{0}\}.$$

By Farkas' lemma, there exists $\boldsymbol{y} \in \mathbb{R}^m_{++}$ such that

$$\boldsymbol{y}^\top \boldsymbol{z} = 0 \quad \text{for all } \boldsymbol{z} \in \mathcal{L}.$$

Hence for every $\boldsymbol{\pi} \in \mathbb{R}^n$,

$$\boldsymbol{y}^\top \mathbf{S}(T,\Omega)^\top \boldsymbol{\pi} = (1+rT)\boldsymbol{y}^\top \mathbf{1}\, \boldsymbol{S}(0)^\top \boldsymbol{\pi}.$$

Since this holds for all $\boldsymbol{\pi}$, we get

$$\mathbf{S}(T,\Omega)\boldsymbol{y} = (1+rT)(\boldsymbol{y}^\top \mathbf{1})\boldsymbol{S}(0).$$

Set

$$\boldsymbol{q}(\Omega) := \frac{\boldsymbol{y}}{\boldsymbol{y}^\top \mathbf{1}}.$$

Then $\boldsymbol{q}(\Omega) > 0$, $\boldsymbol{q}(\Omega)^\top \mathbf{1} = 1$, and

$$\mathbf{S}(T,\Omega)\boldsymbol{q}(\Omega) = (1+rT)\boldsymbol{S}(0).$$

So a risk-neutral measure exists. □

The risk-neutral measure (and hence the state price vector) guaranteed by the first fundamental theorem may not be unique. Uniqueness is ensured in a complete market:

Theorem 12.5 (Second fundamental theorem of asset pricing) The following are equivalent in the general single-period model:

1. The market is arbitrage-free and complete.
2. There exists a unique risk-neutral measure; that is, there exists a unique and positive solution $\boldsymbol{q}(\Omega)$ to
$$\mathbf{S}(T,\Omega)\boldsymbol{q}(\Omega) = (1+rT)\boldsymbol{S}(0).$$
3. There exists a unique state price vector; that is, there exists a unique and positive

solution $\boldsymbol{\psi}(\Omega)$ to
$$\mathbf{S}(T,\Omega)\boldsymbol{\psi}(\Omega) = \boldsymbol{S}(0).$$

Proof. *Equivalence of* (2) *and* (3). As in the proof of the first fundamental theorem, this equivalence follows from

$$\boldsymbol{q}(\Omega) = (1+rT)\boldsymbol{\psi}(\Omega) \qquad \text{and} \qquad \boldsymbol{\psi}(\Omega) = \frac{1}{1+rT}\boldsymbol{q}(\Omega),$$

which preserves positivity and uniqueness.

(1) *implies* (2). If the market is arbitrage-free, the first fundamental theorem gives the existence of a positive solution to
$$\mathbf{S}(T,\Omega)\boldsymbol{q}(\Omega) = (1+rT)\boldsymbol{S}(0).$$
If the market is also complete, then by the completeness criterion

$$\operatorname{rank}(\mathbf{S}(T,\Omega)^\top) = m,$$

hence $\operatorname{rank}(\mathbf{S}(T,\Omega)) = m$. Therefore the linear system above has a unique solution, so the risk-neutral measure is unique.

(2) *implies* (1). Uniqueness implies existence, so by the first fundamental theorem the market is arbitrage-free. Suppose the market were not complete. Then $\operatorname{rank}(\mathbf{S}(T,\Omega)) < m$, so there is $\boldsymbol{z} \neq \mathbf{0}$ with
$$\mathbf{S}(T,\Omega)\boldsymbol{z} = \mathbf{0}.$$
Let $\boldsymbol{q}$ be the unique positive risk-neutral solution. For $\varepsilon > 0$ small enough, both

$$\boldsymbol{q}^{\pm} := \boldsymbol{q} \pm \varepsilon \boldsymbol{z}$$

remain strictly positive, and

$$\mathbf{S}(T,\Omega)\boldsymbol{q}^{\pm} = \mathbf{S}(T,\Omega)\boldsymbol{q} \pm \varepsilon \mathbf{S}(T,\Omega)\boldsymbol{z} = (1+rT)\boldsymbol{S}(0).$$

The risk-free asset's row in $\mathbf{S}(T,\Omega)\boldsymbol{z} = \mathbf{0}$ also gives $(1+rT)\mathbf{1}^\top\boldsymbol{z} = 0$, so $\mathbf{1}^\top\boldsymbol{q}^{\pm} = 1$. Thus $\boldsymbol{q}^+$ and $\boldsymbol{q}^-$ are two distinct risk-neutral probability vectors, a contradiction. Hence the market is complete. $\square$

Tutorial 2 — Friday, February 13, 2015

Example 2.1 Consider the market with

$$\boldsymbol{S}(0) = \begin{pmatrix} 1 \\ 100 \\ 6 \end{pmatrix} \qquad \text{and} \qquad \mathbf{S}(T,\Omega) = \begin{pmatrix} 1.25 & 1.25 & 1.25 \\ 125 & 135 & 120 \\ 5 & 5 & 10 \end{pmatrix}.$$

Is this market arbitrage-free? Is it complete?

Solution. The state price vector $\boldsymbol{\psi}(\Omega)$ solves

$$\mathbf{S}(T,\Omega) \begin{pmatrix} \psi(\omega_1) \\ \psi(\omega_2) \\ \psi(\omega_3) \end{pmatrix} = \boldsymbol{S}(0).$$

Solving gives

$$\boldsymbol{\psi}(\Omega) = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.4 \end{pmatrix}.$$

This solution is positive and unique, so by the second fundamental theorem of asset pricing, the market is arbitrage-free and complete. □

Example 2.2 Now consider

$$\boldsymbol{S}(0) = \begin{pmatrix} 1 \\ 100 \\ 6 \end{pmatrix} \qquad \text{and} \qquad \mathbf{S}(T,\Omega) = \begin{pmatrix} 1.25 & 1.25 & 1.25 \\ 80 & 100 & 120 \\ 5 & 5 & 10 \end{pmatrix}.$$

Is this market arbitrage-free? If not, how can we arbitrage? Is this market complete?

Solution. The state price vector again solves

$$\mathbf{S}(T,\Omega)\boldsymbol{\psi}(\Omega) = \boldsymbol{S}(0),$$

and the solution is

$$\boldsymbol{\psi}(\Omega) = \begin{pmatrix} -0.6 \\ 1 \\ 0.4 \end{pmatrix}.$$

Since one state price is negative, the market is not arbitrage-free.

An explicit arbitrage can be constructed. The second risky asset has initial price 100, but its payoffs at maturity 80, 100, and 120 are all dominated by placing 100 dollars in the bank, which grows to 125 in every state. Therefore we can short one share of the second asset and deposit the proceeds in the bank. With

$$\boldsymbol{\pi} = (100, -1, 0)^\top,$$

the initial cost is

$$V^{\boldsymbol{\pi}}(0) = 100 \cdot 1 + (-1) \cdot 100 + 0 \cdot 6 = 0,$$

while the payoffs at maturity are

$$V^{\boldsymbol{\pi}}(T, \omega_1) = 125 - 80 = 45, \qquad V^{\boldsymbol{\pi}}(T, \omega_2) = 125 - 100 = 25, \\ \text{and} \qquad V^{\boldsymbol{\pi}}(T, \omega_3) = 125 - 120 = 5.$$

Thus $\boldsymbol{\pi}$ is an arbitrage portfolio.

Even so, the market is still complete. The matrix

$$\mathbf{S}(T, \Omega)^\top = \begin{pmatrix} 1.25 & 80 & 5 \\ 1.25 & 100 & 5 \\ 1.25 & 120 & 10 \end{pmatrix}$$

is invertible, so every contingent claim can still be replicated. This example shows that completeness alone does not rule out arbitrage. □

Example 2.3 Consider a market with only one risky asset:

$$\boldsymbol{S}(0) = \begin{pmatrix} 1 \\ 100 \end{pmatrix} \qquad \text{and} \qquad \mathbf{S}(T, \Omega) = \begin{pmatrix} 1.25 & 1.25 & 1.25 \\ 80 & 100 & 140 \end{pmatrix}.$$

Is this market arbitrage-free? Is it complete?

Solution. The state price vector satisfies

$$\mathbf{S}(T, \Omega)\boldsymbol{\psi}(\Omega) = \boldsymbol{S}(0).$$

This system has infinitely many positive solutions, namely

$$\boldsymbol{\psi}(\Omega) = \begin{pmatrix} x \\ 0.3 - 1.5x \\ 0.5 + 0.5x \end{pmatrix}, \quad 0 < x < 0.2.$$

Hence the market is arbitrage-free by the first fundamental theorem of asset pricing, but it is not complete by the second fundamental theorem of asset pricing because the state price vector is not unique. □

Pricing in Incomplete Markets

In an arbitrage-free but incomplete market, there are multiple risk-neutral measures and multiple state price vectors. Some claims are attainable and some are not. The following distinction is useful.

Proposition 2.4 Suppose the market is arbitrage-free but incomplete.

1. If a contingent claim is attainable, then its price is the same under every risk-neutral measure.
2. If a nonnegative contingent claim is not attainable, then its no-arbitrage prices need not be unique, but we still have the **fair price bounds**

$$\max\left\{0, \frac{1}{1+rT} \inf_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}\left[X(T)\right]\right\} \leq X(0) \leq \frac{1}{1+rT} \sup_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}\left[X(T)\right].$$

Proof. For (1), let the claim be attainable, so there exists a replicating portfolio $\boldsymbol{\pi}$ with

$$\mathbf{S}(T,\Omega)^\top \boldsymbol{\pi} = \boldsymbol{X}(T,\Omega) \qquad \text{and} \qquad X(0) = \boldsymbol{S}(0)^\top \boldsymbol{\pi}.$$

For any risk-neutral measure $\mathbb{Q}$ (equivalently any positive $\boldsymbol{q}(\Omega)$ solving $\mathbf{S}(T,\Omega)\boldsymbol{q}(\Omega) = (1 + rT)\boldsymbol{S}(0)$), the same calculation as in the risk-neutral pricing theorem gives

$$X(0) = \frac{1}{1+rT}\mathbb{E}^{\mathbb{Q}}[X(T)].$$

Hence the price is identical for every risk-neutral measure.

For (2), let

$$\Psi := \{\boldsymbol{\psi}(\Omega) > 0 : \; \mathbf{S}(T,\Omega)\boldsymbol{\psi}(\Omega) = \boldsymbol{S}(0)\}$$

be the set of admissible state price vectors, which is nonempty because the market is arbitrage-free. In the enlarged market that includes the claim as an additional asset with price $X(0)$, no-arbitrage is equivalent to existence of some $\boldsymbol{\psi} \in \Psi$ such that

$$X(0) = \boldsymbol{X}(T,\Omega)^{\top}\boldsymbol{\psi}(\Omega).$$

Therefore every arbitrage-free price lies between

$$\inf_{\boldsymbol{\psi}\in\Psi} \boldsymbol{X}(T,\Omega)^{\top}\boldsymbol{\psi}(\Omega) \quad \text{and} \quad \sup_{\boldsymbol{\psi}\in\Psi} \boldsymbol{X}(T,\Omega)^{\top}\boldsymbol{\psi}(\Omega).$$

Using $\boldsymbol{q} = (1+rT)\boldsymbol{\psi}$, this is exactly

$$\frac{1}{1+rT}\inf_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}[X(T)] \le X(0) \le \frac{1}{1+rT}\sup_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}[X(T)].$$

If $X(T) \ge 0$, then $\mathbb{E}^{\mathbb{Q}}[X(T)] \ge 0$ for every $\mathbb{Q}$, and also $X(0) \ge 0$ for any arbitrage-free price (otherwise buying the claim yields nonnegative payoff with strictly positive initial gain). Hence

$$X(0) \ge \max\left\{0, \frac{1}{1+rT}\inf_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}[X(T)]\right\},$$

which is the stated lower bound. □

Different admissible risk-neutral measures can assign different expectations to an unattainable payoff, so the absence of arbitrage determines an interval rather than a single price. Its endpoints need not themselves be arbitrage-free: equivalence requires strictly positive state prices, and an extremum may occur only on the excluded boundary.

Example 2.5 Consider the market with only one risky asset as in Example 2.3:

$$\boldsymbol{S}(0) = \begin{pmatrix} 1 \\ 100 \end{pmatrix} \qquad \text{and} \qquad \mathbf{S}(T,\Omega) = \begin{pmatrix} 1.25 & 1.25 & 1.25 \\ 80 & 100 & 140 \end{pmatrix}.$$

We saw in Example 2.3 that this market is arbitrage-free but incomplete.

1. Which contingent claims are attainable?
2. For the contingent claim with payoff $X(T,\Omega) = (4,2,0)^{\top}$, solve for its price (if it is

attainable) or its price bound (if it is not attainable).

Solution. For (1), we previously solved for the state price vectors

$$\boldsymbol{\psi}(\Omega) = \begin{pmatrix} x \\ 0.3 - 1.5x \\ 0.5 + 0.5x \end{pmatrix}, \quad 0 < x < 0.2.$$

For a generic payoff

$$\boldsymbol{X}(T, \Omega) = \begin{pmatrix} X(T, \omega_1) \\ X(T, \omega_2) \\ X(T, \omega_3) \end{pmatrix},$$

the time-0 price induced by $\boldsymbol{\psi}$ is

$$X(0) = \boldsymbol{X}(T, \Omega)^\top \boldsymbol{\psi}(\Omega).$$

Substituting the parameterized state price vector gives

$$X(0) = 0.3X(T, \omega_2) + 0.5X(T, \omega_3) + x\big(X(T, \omega_1) - 1.5X(T, \omega_2) + 0.5X(T, \omega_3)\big).$$

Therefore the price is independent of x, and hence unique under all risk-neutral measures, if and only if

$$X(T, \omega_1) - 1.5X(T, \omega_2) + 0.5X(T, \omega_3) = 0.$$

This is the attainability condition in this example.

For (2), consider the specific contingent claim

$$\boldsymbol{X}(T, \Omega) = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}.$$

Then

$$4 - 1.5(2) + 0.5(0) = 1 \neq 0,$$

so the claim is not attainable. Its no-arbitrage price is

$$X(0) = \boldsymbol{X}(T, \Omega)^\top \boldsymbol{\psi}(\Omega) = 4x + 2(0.3 - 1.5x) = 0.6 + x.$$

Since $0 < x < 0.2$, the fair price bound is

$$0.6 < X(0) < 0.8.$$

$\square$

Lecture 13 — Monday, February 23, 2015

Example 13.1 Consider the market with

$$\boldsymbol{S}(0) = \begin{pmatrix} 3.5 \\ 1.5 \end{pmatrix} \qquad \text{and} \qquad \mathbf{S}(T, \Omega) = \begin{pmatrix} 8 & 4 \\ 4 & 1 \end{pmatrix}.$$

This and the next example have no designated risk-free asset: the first asset is risky because its payoff varies by state. The state-price versions of the fundamental theorems still apply, but the conversion between state prices and risk-neutral probabilities requires a traded risk-free asset and therefore is not used here.

1. Solve for the state price vector.
2. Is the market arbitrage-free? Is it complete?
3. What is the price of the 5-strike European call on the first asset?

Solution. The state price vector solves

$$\begin{pmatrix} 8 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} \psi(\omega_1) \\ \psi(\omega_2) \end{pmatrix} = \begin{pmatrix} 3.5 \\ 1.5 \end{pmatrix}.$$

Equivalently,

$$\begin{cases} 8\psi(\omega_1) + 4\psi(\omega_2) = 3.5, \\ 4\psi(\omega_1) + \psi(\omega_2) = 1.5. \end{cases}$$

Solving gives

$$\boldsymbol{\psi}(\Omega) = \begin{pmatrix} 0.3125 \\ 0.25 \end{pmatrix}.$$

This state price vector is positive and unique, so the market is arbitrage-free and complete by the first and second fundamental theorems of asset pricing.

If X denotes the 5-strike European call on the first asset, then

$$\boldsymbol{X}(T,\Omega) = \begin{pmatrix}(8-5)_+\\(4-5)_+\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix}.$$

Therefore

$$X(0) = \boldsymbol{X}(T,\Omega)^\top \boldsymbol{\psi}(\Omega) = 3(0.3125) + 0(0.25) = 0.9375.$$

$\square$

Example 13.2 Consider the market with

$$\boldsymbol{S}(0) = \begin{pmatrix}1\\100\end{pmatrix} \qquad \text{and} \qquad \mathbf{S}(T,\Omega) = \begin{pmatrix}1.2 & 0.8 & 1\\80 & 100 & 120\end{pmatrix}.$$

1. Solve for the state price vectors.
2. Is the market arbitrage-free? Is it complete?
3. Determine the no-arbitrage price for the contingent claim with payoff

$$\boldsymbol{X}(T,\Omega) = \begin{pmatrix}0\\1\\5\end{pmatrix}.$$

Solution. The state price vector solves

$$\begin{pmatrix}1.2 & 0.8 & 1\\80 & 100 & 120\end{pmatrix}\begin{pmatrix}\psi(\omega_1)\\\psi(\omega_2)\\\psi(\omega_3)\end{pmatrix} = \begin{pmatrix}1\\100\end{pmatrix}.$$

Writing $s = \psi(\omega_1)$, the solutions are

$$\boldsymbol{\psi}(\Omega) = \begin{pmatrix}s\\16s-5\\5-14s\end{pmatrix}, \quad \frac{5}{16} < s < \frac{5}{14}.$$

Thus there are infinitely many positive state price vectors. The market is arbitrage-free, but incomplete.

For the contingent claim,

$$X(0) = \boldsymbol{X}(T,\Omega)^\top \boldsymbol{\psi}(\Omega) = 16s - 5 + 5(5 - 14s) = 20 - 54s.$$

Since $5/16 < s < 5/14$, this gives the no-arbitrage price bound

$$\frac{5}{7} < X(0) < \frac{25}{8}.$$

□

Example 13.3 Consider a two-period binomial tree with

$$u = 2, \qquad d = 0.5, \qquad T = 1, \qquad S(0) = 100, \qquad r = 5\%, \qquad \text{and} \qquad \delta = 0.$$

Determine the premium of an American lookback call with a floating strike whose exercise payoff at time $t \in \{0, 1/2, 1\}$ is

$$G(t) = S(t) - \min_{0 \le s \le t} S(s).$$

Solution. Here $N = 2$, so $h = T/N = 1/2$. The one-period risk-neutral probabilities are

$$q_u = \frac{e^{0.05/2} - 0.5}{2 - 0.5} \approx 0.350210 \qquad \text{and} \qquad q_d = 1 - q_u = 0.6498.$$

Because this is a lookback option, the uncombined tree must be used. The exercise payoffs at maturity are

(stock value, option value/payoff)

The full tree obtained by backward induction is shown in Figure 21.

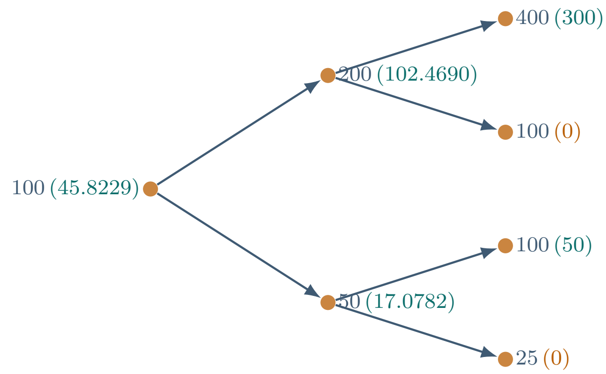

Figure 21

At the up node at time 1/2, the immediate exercise payoff is 100, while the continuation value is

$$e^{-0.05/2}\big(300q_u + 0q_d\big) \approx 102.4690.$$

Therefore

$$X(1/2, u) = \max\{100, 102.4690\} = 102.4690.$$

At the down node at time 1/2, the immediate exercise payoff is 0, while the continuation value is

$$e^{-0.05/2}\big(50q_u + 0q_d\big) \approx 17.0782.$$

Therefore

$$X(1/2, d) = \max\{0, 17.0782\} = 17.0782.$$

Finally,

$$X(0) = e^{-0.05/2}\big(q_u X(1/2, u) + q_d X(1/2, d)\big) \approx 45.8229.$$

Thus the premium of the American lookback call is approximately 45.8229. □

Example 13.4 Consider a stock with current price $S(0) = 1690$, and suppose the continuously compounded risk-free rate is $r = 2\%$. If the stock pays a dividend of 20 at the end of the third month and a dividend of 30 at the end of the ninth month, what is the one-year forward price?

Solution. The dividends are paid at times 1/4 and 3/4, so their future value at time 1 must be subtracted from the accumulated stock price. Hence

$$F_{0,1} = S(0)e^r - \mathrm{FV}(\mathrm{Div}) = 1690e^{0.02} - 20e^{0.02(1-0.25)} - 30e^{0.02(1-0.75)} = 1673.69.$$

$\square$

Example 13.5 Suppose the continuously compounded interest rate is r. A stock pays a dividend D at time $t_0 \in (0, T)$, but the observed European call and put with maturity T and strike K satisfy

$$C(S_0, K, T) - P(S_0, K, T) = S_0 - Ke^{-rT}.$$

Construct an arbitrage portfolio such that $V(0) = 0$ and $V(T) > 0$.

Solution. The observed parity relation omits the present value of the dividend. Consider the portfolio that is short one call, long one put, long one share of the stock, and borrows Ke^{-rT} from the bank. Its values are

	Value at time 0	Value at time T
Short one call	$-C(S_0, K, T)$	$-(S_T - K)_+$
Long one put	$P(S_0, K, T)$	$(K - S_T)_+$
Long one share of stock	S_0	$S_T + De^{r(T-t_0)}$
Borrow from the bank	$-Ke^{-rT}$	$-K$
Total	0	$De^{r(T-t_0)}$

Indeed, the initial value is

$$-C(S_0, K, T) + P(S_0, K, T) + S_0 - Ke^{-rT} = 0$$

by the observed parity relation, and the value at maturity is

$$-(S_T - K)_+ + (K - S_T)_+ + S_T + De^{r(T-t_0)} - K = De^{r(T-t_0)} > 0.$$

Thus this is an arbitrage portfolio. $\square$

Example 13.6 Consider three call options on the same stock index, all expiring in one year:

Strike K	50	59	65
$C(K)$	14	8.9	5

Construct an arbitrage portfolio such that $V(0) < 0$ and $V(T) \geq 0$.

Solution. The call prices violate convexity as a function of the strike:

$$\frac{C(50) - C(59)}{50 - 59} = \frac{14 - 8.9}{-9} > \frac{C(59) - C(65)}{59 - 65} = \frac{8.9 - 5}{-6}.$$

Equivalently,

$$C(59) = C(0.4 \cdot 50 + 0.6 \cdot 65) > 0.4C(50) + 0.6C(65),$$

so the 59-strike call is overpriced.

Take the portfolio that buys four 50-strike calls, buys six 65-strike calls, and shorts ten 59-strike calls. Its initial value is

$$V(0) = 4(14) + 6(5) - 10(8.9) = -3 < 0.$$

Its value at maturity is

$$V(T) = 4(S_T - 50)_+ + 6(S_T - 65)_+ - 10(S_T - 59)_+.$$

Indeed,

$$V(T) = \begin{cases} 0, & S_T \leq 50, \\ 4(S_T - 50), & 50 < S_T \leq 59, \\ 390 - 6S_T, & 59 < S_T \leq 65, \\ 0, & S_T > 65, \end{cases}$$

and this is nonnegative in every case. Therefore the portfolio is an arbitrage portfolio. $\square$

Lecture 14 — Monday, March 02, 2015

Multi-Period Finite-State Models

This lecture extends the single-period framework to a general discrete-time model with multiple trading dates. The main new ideas are that asset prices must be adapted to the information available at each time, trading strategies must be predictable, and replication must be carried out by self-financing portfolios over time.

Consider a model with T time periods, n assets, and m states of nature. The trading dates are

$$I = \{0, 1, \ldots, T\}.$$

The sample space is

$$\Omega = \{\omega_1, \ldots, \omega_m\},$$

and each state at maturity can be viewed as a history

$$\omega = \xi_1 \xi_2 \cdots \xi_T.$$

For example, in a three-period binomial model,

$$\Omega = \{uuu, uud, udu, udd, duu, dud, ddu, ddd\}.$$

Example 14.1 Consider a two-step model in which the first-step outcomes are a_1, a_2, a_3, and the second-step outcomes are indexed by $b_1, \ldots, b_6$. The states at maturity are

$$\omega_1 = a_1 b_1, \qquad \omega_2 = a_1 b_2, \qquad \omega_3 = a_1 b_3,$$
$$\omega_4 = a_2 b_4, \qquad \omega_5 = a_3 b_5, \qquad \text{and} \qquad \omega_6 = a_3 b_6.$$

Suppose the asset price process for asset i is

$$S_i(0) = 100, \qquad S_i(1, a_1) = 120, \qquad S_i(1, a_2) = 100, \qquad \text{and} \qquad S_i(1, a_3) = 80,$$

and at time 2,

$$S_i(2, \omega_1) = 140, \qquad S_i(2, \omega_2) = 110, \qquad \text{and} \qquad S_i(2, \omega_3) = 90,$$

$$S_i(2, \omega_4) = 110, \qquad S_i(2, \omega_5) = 90, \qquad \text{and} \qquad S_i(2, \omega_6) = 70.$$

The corresponding finite-state price tree is shown in Figure 22.

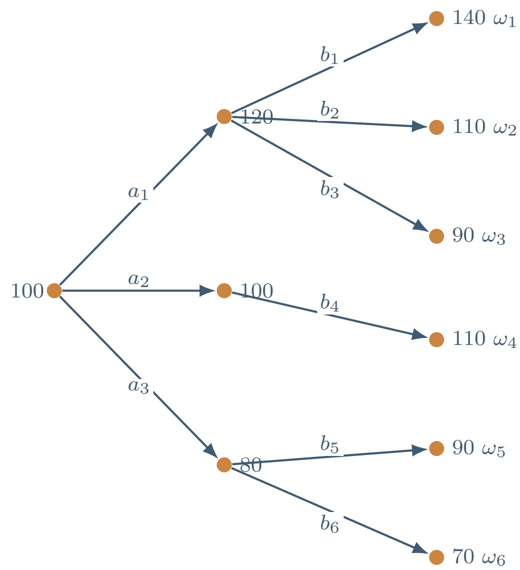

FIGURE 22

At time 1, all states with prefix a_1 must share the same value. In particular,

$$S_i(1,\omega_1) = S_i(1,\omega_2) = S_i(1,\omega_3) = 120 = S_i(1,a_1).$$

This identity is exactly the adaptedness condition: the value at time 1 depends only on the time-1 history $\omega|_1$, not on which state at maturity is realized at time 2.

The nonrecombining history tree in Figure 23 makes the structure of path prefixes explicit through three periods.

There are n assets in the model: one risk-free asset with continuously compounded risk-free rate r, and $n-1$ risky assets (again, we assume no dividends for simplicity). The risk-free asset price process is

$$S_1(t) = e^{rt}, \quad t \in I,$$

and for each risky asset $i = 2,\ldots,n$, the price at time t in state ω is denoted by $S_i(t,\omega)$ for $i = 2,\ldots,n$, which we again think of as a random variable on the state space Ω.

In a multi-period model, the first timing requirement is adaptedness of the price process.

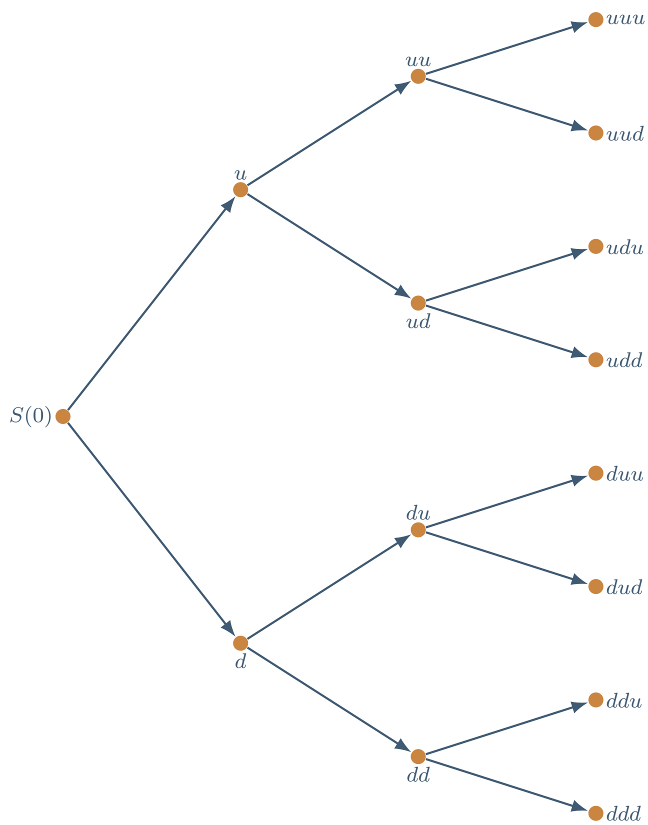

FIGURE 23

Definition 14.2 An asset price process $S(t, \omega)$ is called **adapted** if

$$S(t, \omega) = S(t, \tilde{\omega})$$

whenever

$$\omega|_t = \tilde{\omega}|_t \qquad \text{and} \qquad \omega|_t := \xi_1 \xi_2 \cdots \xi_t.$$

In words, the value at time t depends only on the information revealed up to time t.

If a process is adapted, then it is observable at time t, and we may write it as $S(t, \omega|_t) := S(t, \omega)$. Adapted processes are precisely the processes whose dynamics can be represented on a tree.

Example 14.3 The usual binomial stock price process is adapted. Indeed, if at time t the path prefix is $\omega|_t = u^k d^{t-k}$, then

$$S(t, \omega|_t) = S(0) u^k d^{t-k},$$

which depends only on the first t moves.

By contrast, the following process is not adapted:

$$X(0) = 1,$$

$$X(1, \omega) = \begin{cases} 2, & \omega = uu, \\ 1, & \omega = ud, \\ 0, & \omega = du, \\ 0, & \omega = dd, \end{cases} \qquad \text{and} \qquad X(2, \omega) = \begin{cases} 4, & \omega = uu, \\ 2, & \omega = ud, \\ 2, & \omega = du, \\ 0, & \omega = dd. \end{cases}$$

This process is not adapted because at time 1, the value distinguishes between uu and ud, even though both have the same time-1 history u.

For contrast, Figure 24 depicts an adapted process whose time-1 node values depend only on the observed first move.

Trading strategies have the opposite timing convention: positions must be chosen before the next price movement is observed.

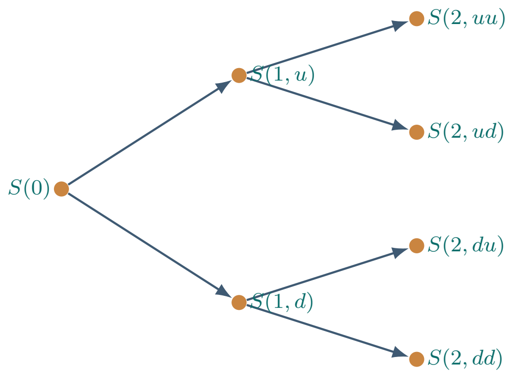

FIGURE 24

Definition 14.4 A **trading strategy** is a process

$$\boldsymbol{\pi}(t, \omega) : I \times \Omega \to \mathbb{R}^n.$$

Its ith component $\pi_i(t, \omega)$ denotes the number of shares of asset i held during period t. The strategy is **predictable** if

$$\pi_i(t, \omega) = \pi_i(t, \tilde{\omega})$$

whenever

$$\omega|_{t-1} = \tilde{\omega}|_{t-1}.$$

The reason is that the position for period t must be chosen at time $t - 1$, before the period-t uncertainty is resolved.

Trading time	Holdings before adjustment	Holdings after adjustment
time 0	not applicable	$\boldsymbol{\pi}(1, \omega)$
time 1	$\boldsymbol{\pi}(1, \omega)$	$\boldsymbol{\pi}(2, \omega)$
time 2	$\boldsymbol{\pi}(2, \omega)$	$\boldsymbol{\pi}(3, \omega)$
$\vdots$	$\vdots$	$\vdots$
time $T - 1$	$\boldsymbol{\pi}(T - 1, \omega)$	$\boldsymbol{\pi}(T, \omega)$
time T	$\boldsymbol{\pi}(T, \omega)$	not applicable

If $\boldsymbol{\pi}$ is a predictable trading strategy, then for $t = 1, \ldots, T$ the portfolio value before rebalancing at time t is

$$V^{\boldsymbol{\pi}}(t^-, \omega) = \sum_{i=1}^{n} \pi_i(t, \omega) S_i(t, \omega),$$

while for $t = 0, \dots, T-1$ the portfolio value after rebalancing at time t is

$$V^{\boldsymbol{\pi}}(t, \omega) = \sum_{i=1}^{n} \pi_i(t+1, \omega) S_i(t, \omega).$$

$V^{\boldsymbol{\pi}}(t^-, \omega)$ and $V^{\boldsymbol{\pi}}(t, \omega)$ are the **portfolio value processes**. They are both adapted, because they are observable at time t. At maturity no further rebalancing occurs, so

$$V^{\boldsymbol{\pi}}(T^-, \omega) = V^{\boldsymbol{\pi}}(T, \omega).$$

Time	Before adjustment	After adjustment
time 0	not applicable	$V^{\boldsymbol{\pi}}(0)$
time 1	$V^{\boldsymbol{\pi}}(1^-, \omega)$	$V^{\boldsymbol{\pi}}(1, \omega)$
time 2	$V^{\boldsymbol{\pi}}(2^-, \omega)$	$V^{\boldsymbol{\pi}}(2, \omega)$
$\vdots$	$\vdots$	$\vdots$
time $T-1$	$V^{\boldsymbol{\pi}}((T-1)^-, \omega)$	$V^{\boldsymbol{\pi}}(T-1, \omega)$
time T	$V^{\boldsymbol{\pi}}(T^-) = V^{\boldsymbol{\pi}}(T, \omega)$	not applicable

Definition 14.5 A predictable trading strategy $\boldsymbol{\pi}$ is called **self-financing** if for every $1 \le t \le T-1$ and every $\omega \in \Omega$,

$$\sum_{i=1}^{n} \pi_i(t, \omega) S_i(t, \omega) = \sum_{i=1}^{n} \pi_i(t+1, \omega) S_i(t, \omega).$$

Equivalently,

$$V^{\boldsymbol{\pi}}(t^-, \omega) = V^{\boldsymbol{\pi}}(t, \omega).$$

Thus, changes of portfolio value come only from changes in asset prices. No money is injected into or withdrawn from the portfolio at intermediate times.

Proposition 14.6 If $\boldsymbol{\pi}$ is self-financing, then for each $t = 1, \dots, T$,

$$V^{\boldsymbol{\pi}}(t, \omega) - V^{\boldsymbol{\pi}}(t-1, \omega) = \sum_{i=1}^{n} \pi_i(t, \omega) \big(S_i(t, \omega) - S_i(t-1, \omega) \big).$$

Proof. By self-financing,

$$V^{\boldsymbol{\pi}}(t,\omega) - V^{\boldsymbol{\pi}}(t-1,\omega) = V^{\boldsymbol{\pi}}(t^-,\omega) - V^{\boldsymbol{\pi}}(t-1,\omega).$$

Using the definition of $V^{\boldsymbol{\pi}}(t^-,\omega)$ gives

$$V^{\boldsymbol{\pi}}(t,\omega) - V^{\boldsymbol{\pi}}(t-1,\omega) = \sum_{i=1}^{n} \pi_i(t,\omega)S_i(t,\omega) - \sum_{i=1}^{n} \pi_i(t,\omega)S_i(t-1,\omega),$$

which simplifies to the stated formula. $\square$

With self-financing strategies in place, arbitrage and completeness are defined as follows.

Definition 14.7 An **arbitrage** is a self-financing trading strategy $\boldsymbol{\pi}$ such that the following hold:

1. $V^{\boldsymbol{\pi}}(0) = 0$.
2. $V^{\boldsymbol{\pi}}(T,\omega) \geq 0 \quad$ for all $\omega \in \Omega$.
3. $V^{\boldsymbol{\pi}}(T,\omega) > 0 \quad$ for at least one $\omega \in \Omega$.

Definition 14.8 A contingent claim with payoff at maturity $X(T,\omega)$ is called **attainable** if there exists a self-financing trading strategy $\boldsymbol{\pi}$ such that

$$V^{\boldsymbol{\pi}}(T,\omega) = X(T,\omega) \quad \text{for all } \omega \in \Omega.$$

If such a strategy exists, then by the law of one price the time-0 value of the claim is

$$X(0) = V^{\boldsymbol{\pi}}(0).$$

The multi-period market is called **complete** if every contingent claim is attainable.

Example 14.9 Consider the two-period binomial model with

$$r = 5\%, \qquad \delta = 0, \qquad T = 2, \qquad S(0) = 4, \qquad u = 2, \qquad \text{and} \qquad d = \tfrac{1}{2}.$$

Find a self-financing trading strategy $\boldsymbol{\pi}$ to replicate the European lookback option with payoff

$$X(2) = \max_{0 \le t \le 2} S(t) - S(2).$$

Solution. The stock prices at maturity are

$$S(2, uu) = 16, \qquad S(2, ud) = 4, \qquad S(2, du) = 4, \qquad \text{and} \qquad S(2, dd) = 1,$$

so the corresponding lookback payoffs are

$$X(2, uu) = 0, \qquad X(2, ud) = 4, \qquad X(2, du) = 0, \qquad \text{and} \qquad X(2, dd) = 3.$$

The one-step risk-neutral probabilities are

$$q_u = \frac{e^{0.05} - 1/2}{2 - 1/2} = 0.3675 \qquad \text{and} \qquad q_d = 0.6325.$$

Therefore

$$X(1, u) = e^{-0.05}(4q_d) = 2.4066 \qquad \text{and} \qquad X(1, d) = e^{-0.05}(3q_d) = 1.805,$$

and

$$X(0) = e^{-0.05}\big(q_u X(1, u) + q_d X(1, d)\big) = 1.9273.$$

The stock prices and corresponding lookback option values are displayed together in Figure 25.

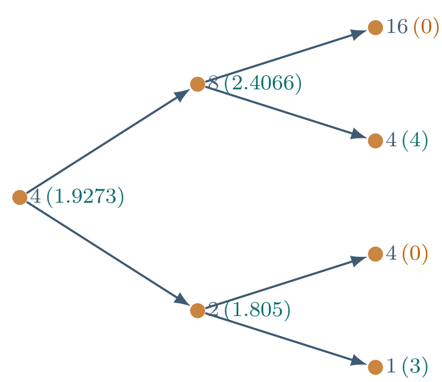

FIGURE 25

The replicating portfolios are found node by node. At the up node at time 1, we solve

$$\pi_1(2, u)e^{0.1} + \pi_2(2, u)16 = 0 \qquad \text{and} \qquad \pi_1(2, u)e^{0.1} + \pi_2(2, u)4 = 4,$$

which gives

$$\boldsymbol{\pi}(2,u) = \begin{pmatrix} 4.8258 \\ -0.3333 \end{pmatrix}.$$

At the down node at time 1, we solve

$$\pi_1(2,d)e^{0.1} + \pi_2(2,d)4 = 0 \qquad \text{and} \qquad \pi_1(2,d)e^{0.1} + \pi_2(2,d)1 = 3,$$

which gives

$$\boldsymbol{\pi}(2,d) = \begin{pmatrix} 3.6193 \\ -1 \end{pmatrix}.$$

At time 0, matching the two time-1 value of holdings gives

$$\pi_1(1)e^{0.05} + \pi_2(1)8 = 2.4066 \qquad \text{and} \qquad \pi_1(1)e^{0.05} + \pi_2(1)2 = 1.805,$$

so

$$\boldsymbol{\pi}(1) = \begin{pmatrix} 1.5262 \\ 0.1003 \end{pmatrix}.$$

The self-financing condition can then be checked directly at time 1 in both states. □

Lecture 15 — Wednesday, March 04, 2015

Martingales and Multi-Period Pricing

Definition 15.1 An integrable stochastic process $X : I \times \Omega \to \mathbb{R}$, adapted to a filtration $\{\mathcal{F}_t\}$, is a **martingale** if for every $t \geq 1$,

$$\mathbb{E}\left[X(t) \mid \mathcal{F}_{t-1}\right] = X(t-1).$$

This means that, given all information available through time $t-1$, the conditional expectation of the next value equals the current value. Standard examples are a constant process ($X(t) = c$ for every t) and a simple symmetric random walk.

Definition 15.2 A probability measure $\mathbb{Q}$ on Ω is called a **risk-neutral measure** if the following hold:

1. for every risky asset $i = 2, \ldots, n$, the discounted process S_i/S_1 is a martingale under $\mathbb{Q}$;
2. $\mathbb{Q}(\{\omega\}) > 0$ for every $\omega \in \Omega$.

Since $S_1(t) = e^{rt}$, where one unit of time is one compounding period, the martingale condition is equivalent to

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-rt}S_i(t)\,\middle|\,\mathcal{F}_{t-1}\right] = e^{-r(t-1)}S_i(t-1),$$

or, after multiplying by e^{rt},

$$\mathbb{E}^{\mathbb{Q}}\left[S_i(t)\,|\,\mathcal{F}_{t-1}\right] = e^{r}S_i(t-1).$$

In particular,

$$\mathbb{E}^{\mathbb{Q}}\left[S_i(t)\right] = e^{rt}S_i(0).$$

Thus under a risk-neutral measure, every asset has expected gross growth factor e^{rt} from time 0 to time t. Equivalently, its expected one-period simple return is $e^{r} - 1$, while r is the continuously compounded rate.

Proposition 15.3 Let $\mathbb{Q}$ be a risk-neutral measure and let $\boldsymbol{\pi}$ be a self-financing trading strategy. Then for every $t \geq 0$,

$$\mathbb{E}^{\mathbb{Q}}\left[V^{\boldsymbol{\pi}}(t)\right] = e^{rt}V^{\boldsymbol{\pi}}(0).$$

Proof. For $t \geq 1$,

$$\mathbb{E}^{\mathbb{Q}}\left[V^{\boldsymbol{\pi}}(t) \mid \mathcal{F}_{t-1}\right] = \mathbb{E}^{\mathbb{Q}}\left[V^{\boldsymbol{\pi}}(t^-) \mid \mathcal{F}_{t-1}\right],$$

because $\boldsymbol{\pi}$ is self-financing. By the definition of $V^{\boldsymbol{\pi}}(t^-)$,

$$V^{\boldsymbol{\pi}}(t^-) = \sum_{i=1}^{n} \pi_i(t)S_i(t).$$

Since $\boldsymbol{\pi}(t)$ is predictable, it can be taken outside the conditional expectation, so

$$\mathbb{E}^{\mathbb{Q}}\left[V^{\boldsymbol{\pi}}(t) \mid \mathcal{F}_{t-1}\right] = \sum_{i=1}^{n} \pi_i(t)\mathbb{E}^{\mathbb{Q}}\left[S_i(t) \mid \mathcal{F}_{t-1}\right].$$

Using the risk-neutral property gives

$$\mathbb{E}^{\mathbb{Q}}\left[V^{\boldsymbol{\pi}}(t) \mid \mathcal{F}_{t-1}\right] = e^{r} \sum_{i=1}^{n} \pi_i(t) S_i(t-1) = e^{r} V^{\boldsymbol{\pi}}(t-1).$$

Taking expectations and iterating proves the claim. □

Theorem 15.4 If a multi-period model admits a risk-neutral measure $\mathbb{Q}$, then any attainable contingent claim with payoff $X(T)$ satisfies

$$X(0) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[X(T)\right] = e^{-rT} \sum_{\omega \in \Omega} X(T,\omega) q(\omega).$$

Proof. If $X(T)$ is replicated by a self-financing strategy $\boldsymbol{\pi}$, then $X(T) = V^{\boldsymbol{\pi}}(T)$. By Proposition 15.3,

$$\mathbb{E}^{\mathbb{Q}}\left[X(T)\right] = \mathbb{E}^{\mathbb{Q}}\left[V^{\boldsymbol{\pi}}(T)\right] = e^{rT} V^{\boldsymbol{\pi}}(0) = e^{rT} X(0).$$

Rearranging gives the pricing formula. □

The multi-period versions of the first and second fundamental theorems of asset pricing have the same structure:

Theorem 15.5 The multi-period market is arbitrage-free if and only if there exists a risk-neutral measure.

Theorem 15.6 An arbitrage-free multi-period market is complete if and only if the risk-neutral measure is unique.

Lecture 16 — Monday, March 09, 2015

5. Continuous-Time Pricing and the Black–Scholes Model

Brownian Motion and Stochastic Calculus

Definition 16.1 A stochastic process $W = \{W(t) : t \geq 0\}$ is called a **Brownian motion** if it satisfies the following conditions:

1. $W(0) = 0$.
2. With probability one, the sample path $t \mapsto W(t, \omega)$ is continuous on $[0, \infty)$.
3. Whenever $t > s \geq 0$, the increment $W(t) - W(s)$ is normally distributed with mean 0 and variance $t - s$; that is,
$$W(t) - W(s) \sim \mathcal{N}(0, t - s).$$
4. Increments over disjoint time intervals are independent; that is, for every partition $0 = t_0 < t_1 < \cdots < t_m$, the random variables
$$W(t_1), \qquad W(t_2) - W(t_1), \qquad \ldots, \qquad \text{and} \qquad W(t_m) - W(t_{m-1})$$
are independent.

Brownian motion has continuous sample paths, but those paths are highly irregular. Over a short interval of length Δt, the increment has typical size on the order of $\sqrt{\Delta t}$, not Δt, which is why the usual rules of deterministic calculus do not apply. Two representative paths are shown in Figure 26.

Sample paths of Brownian motion

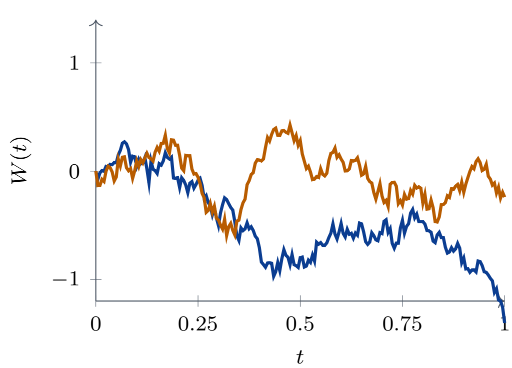

FIGURE 26

Example 16.2 Compute $\mathrm{Var}(W(1) + W(2) + W(3))$.

Solution. To start, it is convenient to rewrite the sum in terms of independent increments:

$$W(1) + W(2) + W(3) = \big(W(3) - W(2)\big) + 2\big(W(2) - W(1)\big) + 3W(1).$$

The three terms on the right-hand side are independent, and their variances are 1, 4, and 9, because

$$W(3) - W(2) \sim \mathcal{N}(0,1), \qquad W(2) - W(1) \sim \mathcal{N}(0,1), \qquad \text{and} \qquad W(1) \sim \mathcal{N}(0,1).$$

Therefore,

$$\mathrm{Var}(W(1) + W(2) + W(3)) = 1 + 4 + 9 = 14.$$

□

The martingale property is formulated using conditional expectation. If X is a stochastic process and $\mathcal{F}_s^X$ denotes the information (or more specifically, the **filtration**) generated by X up to time s, write

$$\mathbb{E}_s\left[X(t)\right] := \mathbb{E}\left[X(t) \mid \mathcal{F}_s^X\right].$$

The standard rules of conditional expectation remain in force. In particular, $\mathbb{E}_t[X(t)] = X(t)$; conditional expectation is linear; known quantities can be taken outside the operator; independent quantities reduce to ordinary expectations; and $\mathbb{E}[\mathbb{E}_s[X]] = \mathbb{E}[X]$. We also have $\mathbb{E}_0[X(t)] = \mathbb{E}[X(t)]$ whenever the initial sigma-field is trivial, as it is for the standard Brownian filtration after completion.

Proposition 16.3 If W is a Brownian motion, then for every $0 \le s \le t$,

$$\mathbb{E}_s\left[W(t)\right] = W(s).$$

In particular,

$$\mathbb{E}\left[W(t)\right] = W(0) = 0.$$

Proof. Write

$$W(t) = \big(W(t) - W(s)\big) + W(s).$$

The increment $W(t) - W(s)$ is independent of $\mathcal{F}_s^W$ and has mean 0, so

$$\mathbb{E}_s\left[W(t) - W(s)\right] = \mathbb{E}\left[W(t) - W(s)\right] = 0.$$

Since $W(s)$ is already known at time s,

$$\mathbb{E}_s[W(t)] = \mathbb{E}_s[W(t) - W(s)] + \mathbb{E}_s[W(s)] = 0 + W(s) = W(s).$$

$\square$

Lemma 16.4 If $\mathbb{E}[(X_n - X)^2] \longrightarrow 0$, then $X_n \xrightarrow{p} X$. In other words, convergence in mean square implies convergence in probability.

Proof. For any $\varepsilon > 0$, Markov's inequality gives

$$\mathbb{P}(|X_n - X| \geq \varepsilon) = \mathbb{P}\big((X_n - X)^2 \geq \varepsilon^2\big) \leq \frac{\mathbb{E}[(X_n - X)^2]}{\varepsilon^2} \longrightarrow 0.$$

$\square$

The distinctive calculus of Brownian motion comes from its quadratic variation.

Proposition 16.5 Let W be a Brownian motion. For any $T \geq 0$, the **quadratic variation** satisfies

$$[W,W](T) := \lim_{\Delta t \to 0} \sum_{j=1}^{n} \big(W(t_j) - W(t_{j-1})\big)^2 = T,$$

where the limit is in probability and

$$0 = t_0 < t_1 < \cdots < t_n = T$$

and

$$\Delta t = \max_{1 \leq j \leq n} (t_j - t_{j-1}).$$

Proof. Let

$$Q_\pi := \sum_{j=1}^{n} \big(W(t_j) - W(t_{j-1})\big)^2$$

for a partition π of $[0,T]$. Since each increment

$$\Delta W_j := W(t_j) - W(t_{j-1})$$

is normal with mean 0 and variance $\Delta t_j := t_j - t_{j-1}$, we have

$$\mathbb{E}\big[(\Delta W_j)^2\big] = \Delta t_j \qquad \text{and} \qquad \operatorname{Var}\big((\Delta W_j)^2\big) = 2(\Delta t_j)^2.$$

Therefore

$$\mathbb{E}[Q_\pi] = \sum_{j=1}^{n} \Delta t_j = T.$$

Because Brownian increments over disjoint intervals are independent, the random variables $(\Delta W_j)^2$ are independent, so

$$\operatorname{Var}(Q_\pi) = \sum_{j=1}^{n} \operatorname{Var}\big((\Delta W_j)^2\big) = 2\sum_{j=1}^{n}(\Delta t_j)^2 \leq 2\,\Delta t \sum_{j=1}^{n} \Delta t_j = 2T\,\Delta t \xrightarrow[\Delta t \to 0]{} 0.$$

Hence

$$\mathbb{E}\big[(Q_\pi - T)^2\big] = \operatorname{Var}(Q_\pi) \to 0,$$

so as $\Delta t \to 0$, $Q_\pi \to T$ in mean square, and therefore in probability by Lemma 16.4. This is exactly

$$[W,W](T) = T.$$

□

In differential notation, the quadratic variation identity is written as

$$\mathrm{d}W(t) \cdot \mathrm{d}W(t) = \mathrm{d}t.$$

Proposition 16.6 Let W be a Brownian motion and let $I(t) = t$ be the identity process. Then for every $T \geq 0$, the **quadratic covariation** satisfies

$$[W,I](T) := \lim_{\Delta t \to 0} \sum_{j=1}^{n} \big(W(t_j) - W(t_{j-1})\big)(t_j - t_{j-1}) = 0,$$

where

$$0 = t_0 < t_1 < \cdots < t_n = T$$

and

$$\Delta t = \max_{1 \leq j \leq n} (t_j - t_{j-1}).$$

Proof. For a partition π of $[0, T]$, define

$$S_\pi := \sum_{j=1}^{n} \big(W(t_j) - W(t_{j-1})\big)(t_j - t_{j-1}) = \sum_{j=1}^{n} \Delta W_j \, \Delta t_j.$$

Since each Brownian increment ΔW_j has mean 0, we get

$$\mathbb{E}[S_\pi] = \sum_{j=1}^{n} \Delta t_j \, \mathbb{E}[\Delta W_j] = 0.$$

Using independence of increments,

$$\operatorname{Var}(S_\pi) = \sum_{j=1}^{n} (\Delta t_j)^2 \operatorname{Var}(\Delta W_j) = \sum_{j=1}^{n} (\Delta t_j)^2 \, \Delta t_j = \sum_{j=1}^{n} (\Delta t_j)^3.$$

Now

$$\sum_{j=1}^{n} (\Delta t_j)^3 \le \Delta t \sum_{j=1}^{n} (\Delta t_j)^2 \le \Delta t \, \Delta t \sum_{j=1}^{n} \Delta t_j = T(\Delta t)^2 \xrightarrow[\Delta t \to 0]{} 0.$$

Therefore

$$\mathbb{E}[S_\pi^2] = \operatorname{Var}(S_\pi) \to 0,$$

so $S_\pi \to 0$ in mean square, and hence in probability by Lemma 16.4. This gives

$$[W, I](T) = 0.$$

$\square$

The corresponding differential rule is

$$\mathrm{d}W(t) \cdot \mathrm{d}t = 0.$$

The final rule,

$$\mathrm{d}t \cdot \mathrm{d}t = 0,$$

comes from the observation that the quadratic variation of the identity process satisfies

$$[I, I](T) = \lim_{\Delta t \to 0} \sum_{j=1}^{n} (t_j - t_{j-1})^2 \le \lim_{\Delta t \to 0} \Delta t \, T = 0.$$

The three identities $\mathrm{d}W(t) \cdot \mathrm{d}W(t) = \mathrm{d}t$, $\mathrm{d}W(t) \cdot \mathrm{d}t = 0$, and $\mathrm{d}t \cdot \mathrm{d}t = 0$ are the bookkeeping rules behind Ito's formula, which we will explore later.

Exponentiating Brownian motion produces the lognormal process used for stock prices.

Definition 16.7 A random variable X is **lognormally distributed** if $X = e^Y$ for some normal random variable $Y \sim \mathcal{N}(\mu, \sigma^2)$. We write

$$X \sim \text{Lognormal}(\mu, \sigma^2),$$

or equivalently

$$\log X \sim \mathcal{N}(\mu, \sigma^2).$$

If $X \sim \text{Lognormal}(\mu, \sigma^2)$, then

$$\mathbb{E}\left[X\right] = e^{\mu + \frac{1}{2}\sigma^2}.$$

Definition 16.8 The **lognormal stock-price model** is the process

$$S(t) = S(0)e^{\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)t+\sigma W(t)},$$

where α is the appreciation rate, δ is the continuous dividend yield, and σ is the volatility. This process is also called **geometric Brownian motion**.

For any fixed t, the exponent is normal, so

$$\log\left(\frac{S(t)}{S(0)}\right) \sim \mathcal{N}\left(\left(\alpha - \delta - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right).$$

Taking expectations gives

$$\mathbb{E}\left[S(t)\right] = S(0)e^{\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)t}\mathbb{E}\left[e^{\sigma W(t)}\right] = S(0)e^{(\alpha-\delta)t}.$$

Sample paths for two choices of drift and volatility are compared in Figure 27.

Example 16.9 Suppose $S(t)$ follows geometric Brownian motion with $\alpha = 3\%$, $\delta = 0$, and $\sigma = 20\%$. Find $\text{Var}\left(\log\sqrt{S(1)S(2)}\right)$.

Solution. We have

$$S(t) = S(0)e^{(0.03-\frac{1}{2}0.2^2)t+0.2W(t)} = S(0)e^{0.01t+0.2W(t)}.$$

Sample paths of geometric Brownian motion

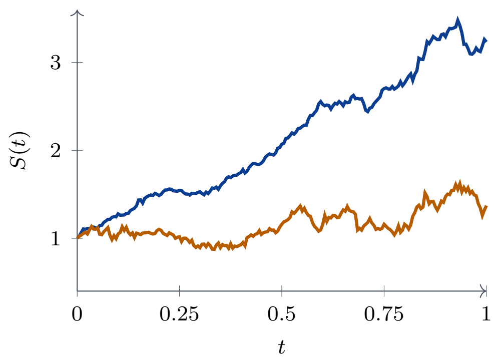

FIGURE 27

Hence

$$\sqrt{S(1)S(2)} = S(0)e^{0.015+0.1W(1)+0.1W(2)},$$

so

$$\log\sqrt{S(1)S(2)} = \log S(0) + 0.015 + 0.1W(1) + 0.1W(2).$$

The constant terms do not affect the variance, and

$$\operatorname{Var}(W(1)+W(2)) = \operatorname{Var}(2W(1) + (W(2) - W(1))) = 4 + 1 = 5.$$

Therefore,

$$\operatorname{Var}\Big(\log\sqrt{S(1)S(2)}\Big) = 0.1^2 \cdot 5 = 0.05.$$

$\square$

Example 16.10 Suppose $S(t)$ follows geometric Brownian motion with $S(0) = 100$, $\alpha = 10\%$, $\delta = 0$, and $\sigma = 30\%$. Find a central 95% prediction interval for $S(2)$.

Solution. We have

$$S(2) = 100e^{(0.10-\frac{1}{2}0.3^2)2+0.3W(2)} = 100e^{0.11+0.3W(2)}.$$

Since $W(2) \sim \mathcal{N}(0, 2)$,

$$\log\frac{S(2)}{100} \sim \mathcal{N}(0.11, 0.18).$$

For a central 95% interval, we use the quantile $z_{0.975} = 1.96$, so

$$\log \frac{U}{100} = 0.11 + 1.96\sqrt{0.18} \qquad \text{and} \qquad \log \frac{L}{100} = 0.11 - 1.96\sqrt{0.18}.$$

Exponentiating gives

$$U = 256.3972 \qquad \text{and} \qquad L = 48.5995.$$

Therefore a central 95% prediction interval for $S(2)$ is

$$[48.5995,\ 256.3972].$$

□

Lecture 17 — Wednesday, March 11, 2015

Ito's Formula

Stochastic integration is formalized through the Ito integral.

Definition 17.1 Let

$$0 = t_0 < t_1 < \cdots < t_n = T$$

and

$$\Delta t = \max_{1 \le j \le n} (t_j - t_{j-1}).$$

For a progressively measurable process X satisfying $\mathbb{E}[\int_0^T X(t)^2 \,\mathrm{d}t] < \infty$, the **Ito integral** is the mean-square limit of left-endpoint sums

$$\int_0^T X(t)\,\mathrm{d}W(t) := L^2\text{-}\lim_{\Delta t \to 0} \sum_{j=1}^{n} X(t_{j-1})\big(W(t_j) - W(t_{j-1})\big),$$

first for adapted step processes and then, by L^2 approximation, for general square-integrable integrands of this kind.

The use of $X(t_{j-1})$ instead of a midpoint or right endpoint reflects the economic requirement that the integrand be chosen using only information available before the increment $W(t_j) - W(t_{j-1})$ is observed.

Proposition 17.2 If X is progressively measurable and square-integrable, then

$$\mathbb{E}\left[\int_0^T X(t)\,\mathrm{d}W(t)\right] = 0$$

and

$$\mathbb{E}\left[\left(\int_0^T X(t)\,\mathrm{d}W(t)\right)^2\right] = \int_0^T \mathbb{E}\left[X(t)^2\right]\mathrm{d}t. \quad (\textbf{Ito's isometry})$$

Ito's isometry shows that stochastic integration preserves L^2 structure in a way that is closely analogous to orthogonality of independent increments.

Example 17.3 If $X(t) \equiv 1$, then

$$\int_0^T X(t)\,\mathrm{d}W(t) = \int_0^T 1\,\mathrm{d}W(t) = W(T) - W(0) = W(T).$$

From Proposition 16.3, we know that $\mathbb{E}[W(T)] = 0$. Moreover, $\mathbb{E}[W(T)^2] = \mathrm{Var}(W(T)) + \mathbb{E}[W(T)]^2 = T + 0 = T$. These agree with what we would obtain using Proposition 17.2, which gives

$$\mathbb{E}\left[\int_0^T 1\,\mathrm{d}W(t)\right] = 0$$

and

$$\mathbb{E}\left[\left(\int_0^T 1\,\mathrm{d}W(t)\right)^2\right] = \int_0^T \mathbb{E}\left[1^2\right]\mathrm{d}t = \int_0^T 1\,\mathrm{d}t = T.$$

Example 17.4 Taking $X(t) = W(t)$, Proposition 17.2 gives

$$\mathbb{E}\left[\int_0^T W(t)\,\mathrm{d}W(t)\right] = 0$$

and

$$\mathbb{E}\left[\left(\int_0^T W(t)\,\mathrm{d}W(t)\right)^2\right] = \int_0^T \mathbb{E}\left[W(t)^2\right]\mathrm{d}t = \int_0^T \mathrm{Var}(W(t))\,\mathrm{d}t = \int_0^T t\,\mathrm{d}t = \frac{1}{2}T^2.$$

The integrals above lead naturally to Ito processes.

Definition 17.5 A stochastic process X is called an **Ito process** if it can be written in the form

$$X(t) = X(0) + \int_0^t \mu(s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}W(s),$$

where μ and σ are progressively measurable processes satisfying the integrability conditions required for the two integrals. Its **stochastic differential equation** is written as

$$\mathrm{d}X(t) = \mu(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W(t).$$

Three standard examples are as follows. A **linear Brownian motion** satisfies

$$\mathrm{d}X(t) = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W(t).$$

A **geometric Brownian motion** satisfies

$$\mathrm{d}X(t) = \mu X(t)\,\mathrm{d}t + \sigma X(t)\,\mathrm{d}W(t).$$

More generally, a **diffusion process** can be written as

$$\mathrm{d}X(t) = \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W(t).$$

Lecture 18 — Monday, March 16, 2015

The central chain rule for Ito processes is **Ito's formula** (or **Ito's lemma**).

Theorem 18.1 (Ito's formula) Let X be an Ito process satisfying

$$\mathrm{d}X(t) = \mu(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W(t),$$

and let $f \in C^{1,2}$, so that f_t, f_x, and f_{xx} are continuous. Then

$$\mathrm{d}f(t, X(t)) = \frac{\partial f}{\partial t}(t, X(t))\,\mathrm{d}t + \frac{\partial f}{\partial x}(t, X(t))\,\mathrm{d}X(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X(t))[\mathrm{d}X(t)]^2.$$

Using the identities $[\mathrm{d}X(t)]^2 = \sigma(t)^2\,\mathrm{d}t$, $\mathrm{d}W(t)\,\mathrm{d}t = 0$, and $\mathrm{d}t\,\mathrm{d}t = 0$, we may also write

$$\begin{aligned}\mathrm{d}f(t, X(t)) = &\left(\frac{\partial f}{\partial t}(t, X(t)) + \frac{\partial f}{\partial x}(t, X(t))\mu(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X(t))\sigma^2(t)\right)\mathrm{d}t\\ &+ \frac{\partial f}{\partial x}(t, X(t))\sigma(t)\,\mathrm{d}W(t).\end{aligned}$$

Proof sketch. Fix a small increment Δt, and write

$$\Delta X := X(t+\Delta t) - X(t) = \mu(t)\Delta t + \sigma(t)\Delta W \qquad \text{and} \qquad \Delta W := W(t+\Delta t) - W(t).$$

Apply the second-order Taylor expansion of f at $(t, X(t))$:

$$\Delta f := f(t+\Delta t, X(t+\Delta t)) - f(t, X(t)) = \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x}\Delta X + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\Delta X)^2 + r(\Delta t, \Delta X),$$

where all derivatives are evaluated at $(t, X(t))$, and the remainder satisfies

$$\frac{r(\Delta t, \Delta X)}{|\Delta t| + |\Delta X|^2} \to 0, \quad \text{as } \Delta t \to 0.$$

Now compute

$$(\Delta X)^2 = \mu(t)^2(\Delta t)^2 + 2\mu(t)\sigma(t)\Delta t\,\Delta W + \sigma(t)^2(\Delta W)^2.$$

Using Brownian scaling, $\Delta W = O_p((\Delta t)^{1/2})$, so $(\Delta t)^2$ and $\Delta t\,\Delta W$ vanish after summation over a refining partition. The individual ratios $(\Delta W)^2/\Delta t$ do not converge to 1; the relevant fact is instead the quadratic-variation limit

$$\sum_j \sigma(t_{j-1})^2(\Delta W_j)^2 \longrightarrow \int_0^T \sigma(t)^2\,\mathrm{d}t.$$

Summing the Taylor expansions over the partition, using this limit, and then letting the mesh tend to zero gives the differential identity

$$\mathrm{d}f(t, X(t)) = \frac{\partial f}{\partial t}(t, X(t))\,\mathrm{d}t + \frac{\partial f}{\partial x}(t, X(t))\,\mathrm{d}X(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X(t))[\mathrm{d}X(t)]^2,$$

with

$$[\mathrm{d}X(t)]^2 = (\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W)^2 = \sigma(t)^2\,\mathrm{d}t,$$

because $\mathrm{d}W\,\mathrm{d}t = 0$ and $\mathrm{d}t\,\mathrm{d}t = 0$. Substituting $\mathrm{d}X(t) = \mu(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W(t)$ yields

$$\mathrm{d}f(t, X(t)) = \left(\frac{\partial f}{\partial t}(t, X(t)) + \frac{\partial f}{\partial x}(t, X(t))\mu + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X(t))\sigma^2\right)\mathrm{d}t + \frac{\partial f}{\partial x}(t, X(t))\sigma(t)\,\mathrm{d}W(t).$$

□

Theorem 18.2 (Ito's formula: integral form) If

$$X(T) = X(0) + \int_0^T \mu(t)\,\mathrm{d}t + \int_0^T \sigma(t)\,\mathrm{d}W(t),$$

then

$$\begin{aligned} f(T, X(T)) - f(0, X(0)) &= \int_0^T \frac{\partial f}{\partial t}(t, X(t))\,\mathrm{d}t + \int_0^T \frac{\partial f}{\partial x}(t, X(t))\,\mathrm{d}X(t) \\ &\quad + \frac{1}{2}\int_0^T \frac{\partial^2 f}{\partial x^2}(t, X(t))[\mathrm{d}X(t)]^2, \end{aligned}$$

and therefore

$$\begin{aligned} f(T, X(T)) - f(0, X(0)) &= \int_0^T \left(\frac{\partial f}{\partial t}(t, X(t)) + \frac{\partial f}{\partial x}(t, X(t))\mu(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X(t))\sigma^2(t)\right)\mathrm{d}t \\ &\quad + \int_0^T \frac{\partial f}{\partial x}(t, X(t))\sigma(t)\,\mathrm{d}W(t). \end{aligned}$$

The main uses of Ito's formula here are to derive stochastic differential equations from closed-form expressions and to solve stochastic differential equations by choosing a suitable transformation.

The following examples show how Ito's formula is used.

Example 18.3 Derive the stochastic differential equation for $W^3(t)$.

Solution. Take $f(x) = x^3$, so $\frac{\mathrm{d}f}{\mathrm{d}x}(x) = 3x^2$ and $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x) = 6x$. Applying Ito's formula to $f(W(t))$ gives

$$\begin{aligned} \mathrm{d}(W(t)^3) &= \mathrm{d}f(W(t)) \\ &= \frac{\mathrm{d}f}{\mathrm{d}x}(W(t))\,\mathrm{d}W(t) + \frac{1}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(W(t))[\mathrm{d}W(t)]^2 \\ &= 3W(t)^2\,\mathrm{d}W(t) + 3W(t)\,\mathrm{d}t. \end{aligned}$$

□

Example 18.4 Consider the lognormal stock price process

$$S(t) = S(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}.$$

Derive the stochastic differential equation for $S(t)$.

Solution. Set

$$X(t) = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t) \qquad \text{and} \qquad f(x) = e^x.$$

Then $\mathrm{d}X(t) = \left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\,\mathrm{d}W(t)$, $f'(x) = f''(x) = e^x$, and Ito's formula gives

$$\mathrm{d}S(t) = S(0)\ \mathrm{d}f(X(t)) = S(0)e^{X(t)}\left[\left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\,\mathrm{d}W(t) + \frac{1}{2}\sigma^2\,\mathrm{d}t\right].$$

Since $S(0)e^{X(t)} = S(t)$, this simplifies to

$$\mathrm{d}S(t) = \mu S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t).$$

□

Lecture 19 — Wednesday, March 18, 2015

Example 19.1 Solve the stochastic differential equation

$$\mathrm{d}S(t) = \mu S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t)$$

with initial condition $S(0)$.

Solution. To solve the geometric Brownian motion stochastic differential equation, apply Ito's formula to $f(x) = \log x$. Since $f'(x) = 1/x$ and $f''(x) = -1/x^2$,

$$\mathrm{d}\log S(t) = \frac{1}{S(t)}\,\mathrm{d}S(t) - \frac{1}{2S(t)^2}[\mathrm{d}S(t)]^2.$$

Substituting $\mathrm{d}S(t) = \mu S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t)$ and using $[\mathrm{d}S(t)]^2 = \sigma^2 S(t)^2\,\mathrm{d}t$ yields

$$\mathrm{d}\log S(t) = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W(t) - \frac{1}{2}\sigma^2\,\mathrm{d}t = \left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\,\mathrm{d}W(t).$$

Integrating from 0 to t gives

$$\log S(t) - \log S(0) = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t),$$

and hence

$$S(t) = S(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}.$$

$\square$

Example 19.2 Solve the stochastic differential equation

$$\mathrm{d}X(t) = \big(\alpha - \beta X(t)\big)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)$$

with initial condition $X(0)$, where α, β, and σ are positive constants.

Solution. The standard integrating factor is $f(t,x) = e^{\beta t}x$. Since

$$\frac{\partial f}{\partial t}(t,x) = \beta e^{\beta t}x, \qquad \frac{\partial f}{\partial x}(t,x) = e^{\beta t}, \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x^2}(t,x) = 0,$$

Ito's formula gives

$$\mathrm{d}\big(e^{\beta t}X(t)\big) = \beta e^{\beta t}X(t)\,\mathrm{d}t + e^{\beta t}\,\mathrm{d}X(t) = \alpha e^{\beta t}\,\mathrm{d}t + \sigma e^{\beta t}\,\mathrm{d}W(t).$$

Integrating from 0 to t yields

$$e^{\beta t}X(t) - X(0) = \alpha\int_0^t e^{\beta s}\,\mathrm{d}s + \sigma\int_0^t e^{\beta s}\,\mathrm{d}W(s) = \frac{\alpha}{\beta}\big(e^{\beta t} - 1\big) + \sigma\int_0^t e^{\beta s}\,\mathrm{d}W(s).$$

Therefore

$$X(t) = e^{-\beta t}X(0) + \frac{\alpha}{\beta}\big(1 - e^{-\beta t}\big) + \sigma\int_0^t e^{-\beta(t-s)}\,\mathrm{d}W(s).$$

$\square$

The Black–Scholes Model

We suppose the market contains a risk-free bond and a dividend-paying stock. The bond price process is

$$B(t) = e^{rt},$$

so its stochastic differential equation is

$$\mathrm{d}B(t) = rB(t)\,\mathrm{d}t.$$

The stock price is assumed to follow a geometric Brownian motion,

$$S(t) = S(0)e^{\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)t+\sigma W(t)},$$

which implies the stochastic differential equation

$$\mathrm{d}S(t) = (\alpha-\delta)S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t).$$

Here α is the **appreciation rate**, δ is the **continuous dividend yield**, and σ is the **volatility**.

The main claim of interest is a European call with strike K and maturity T, whose payoff is

$$C(S(T),T) = (S(T)-K)_+.$$

The goal is to determine the no-arbitrage price $C(S(0),0)$.

The Black–Scholes model is useful but highly idealized. It assumes no transaction costs, continuous trading, a known constant risk-free rate, a known constant volatility, and a known constant dividend yield. It also assumes that stock prices evolve continuously and that returns are lognormal, which excludes jumps and heavier-tailed behaviour often seen in real markets.

Pricing is obtained by forming a self-financing replicating portfolio.

Definition 19.3 Let (x,y) be a **trading strategy**, where $x(t)$ denotes the number of stock shares held at time t, and $y(t)$ denotes the number of risk-free bonds held at time t. The portfolio value under (x,y) at time t is

$$V(t) = x(t)S(t) + y(t)B(t).$$

The trading strategy (x, y) is called **self-financing** if for every $t \in (0, T)$,

$$\mathrm{d}V(t) = x(t)\big(\mathrm{d}S(t) + \delta S(t)\,\mathrm{d}t\big) + y(t)\ \mathrm{d}B(t).$$

In a self-financing portfolio, changes in the portfolio value are solely due to capital gains and dividend income from the assets held. There are no external cash flows into or out of the portfolio. The term $\delta S(t)\,\mathrm{d}t$ appears because a continuously dividend-paying stock contributes both capital gains and dividend income to the portfolio.

Definition 19.4 A self-financing portfolio $V(t)$ is called a **replicating portfolio** for the call price process $C(S(t), t)$ if

$$V(t) = C(S(t), t)$$

for all $t \in [0, T]$. In differential form, this means

$$\mathrm{d}V(t) = \mathrm{d}C(S(t), t).$$

Recall that

$$\mathrm{d}S(t) = (\alpha - \delta)S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t) \qquad \text{and} \qquad \mathrm{d}B(t) = rB(t)\,\mathrm{d}t.$$

The dynamics of a self-financing portfolio are

$$\begin{aligned}\mathrm{d}V(t) &= x(t)\big[\mathrm{d}S(t) + \delta S(t)\,\mathrm{d}t\big] + y(t)\ \mathrm{d}B(t)\\ &= \big[\alpha x(t)S(t) + ry(t)B(t)\big]\,\mathrm{d}t + \sigma S(t)x(t)\,\mathrm{d}W(t).\end{aligned}$$

On the other hand, Ito's formula gives

$$\begin{aligned}\mathrm{d}C(S(t), t) &= \frac{\partial C}{\partial t}(S(t), t)\,\mathrm{d}t + \frac{\partial C}{\partial S}(S(t), t)\ \mathrm{d}S(t) + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S(t), t)[\mathrm{d}S(t)]^2\\ &= \left(\frac{\partial C}{\partial t}(S(t), t) + (\alpha - \delta)S(t)\frac{\partial C}{\partial S}(S(t), t) + \frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 C}{\partial S^2}(S(t), t)\right)\mathrm{d}t\\ &\quad + \sigma S(t)\frac{\partial C}{\partial S}(S(t), t)\,\mathrm{d}W(t).\end{aligned}$$

Matching the $\mathrm{d}W(t)$ terms yields

$$x(t) = \frac{\partial C}{\partial S}(S(t), t).$$

Matching the dt terms then gives

$$y(t) = \frac{\dfrac{\partial C}{\partial t} - \delta S \dfrac{\partial C}{\partial S} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 C}{\partial S^2}}{rB(t)}.$$

The replication argument now yields the Black–Scholes partial differential equation and formula. Since a replicating portfolio must satisfy

$$C(S(t), t) = x(t)S(t) + y(t)B(t),$$

substituting the expressions for $x(t)$ and $y(t)$ yields the Black–Scholes partial differential equation.

Theorem 19.5 (Black–Scholes differential equation) The price $C(S, t)$ of a European call in the Black–Scholes model satisfies

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - \delta) S \frac{\partial C}{\partial S} - rC = 0,$$

for $0 \le t < T$ and $S > 0$, with condition at maturity

$$C(S, T) = (S - K)_+.$$

An important point is that the appreciation rate α disappears from the equation. The option price depends on the risk-free rate, not on the stock's actual expected return.

Theorem 19.6 (Black–Scholes formula) Under the Black–Scholes model, the European call price is

$$C(S(0), 0) = S(0)e^{-\delta T}\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

where

$$d_1 = \frac{\log(S(0)/K) + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}} \qquad \text{and} \qquad d_2 = d_1 - \sigma\sqrt{T},$$

and Φ is the cumulative distribution function of a standard normal random variable.

Proof. Under a risk-neutral measure $\mathbb{Q}$, the expected stock price must satisfy

$$\mathbb{E}^{\mathbb{Q}}[S(t)] = e^{rt}\mathrm{PV}_{0,t}(S(t)) = e^{(r-\delta)t}S(0),$$

because the prepaid forward value is $\mathrm{PV}_{0,t}(S(t)) = S(0)e^{-\delta t}$. Thus the expected price appreciation rate under $\mathbb{Q}$ is $r - \delta$; after continuously reinvested dividends are included, the expected total-return growth rate is r. From the mean of a lognormal random variable,

$$\mathbb{E}\left[S(0)e^{\left(r-\delta-\frac{1}{2}\sigma^2\right)t+\sigma W^{\mathbb{Q}}(t)}\right] = S(0)e^{(r-\delta)t},$$

the same price process has the representations

$$S(t) = S(0)e^{\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)t+\sigma W(t)} \quad \text{under } \mathbb{P}$$

and

$$S(t) = S(0)e^{\left(r-\delta-\frac{1}{2}\sigma^2\right)t+\sigma W^{\mathbb{Q}}(t)} \quad \text{under } \mathbb{Q}.$$

Equivalently, its dynamics are

$$\mathrm{d}S(t) = (\alpha-\delta)S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t) \quad \text{under } \mathbb{P}$$

and

$$\mathrm{d}S(t) = (r-\delta)S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W^{\mathbb{Q}}(t) \quad \text{under } \mathbb{Q}.$$

Here $W^{\mathbb{Q}}$ is a Brownian motion under the new measure. Comparing the two stochastic differentials gives

$$W^{\mathbb{Q}}(t) = \frac{\alpha-r}{\sigma}t + W(t). \tag{19.1}$$

This leads directly to the risk-neutral pricing representation

$$C(0) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S(T)-K)_+\right].$$

Using the fact that $W^{\mathbb{Q}}(T) \sim \mathcal{N}(0,T)$, we have

$$\begin{aligned}
C(0) &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S(T)-K)_+\right] \\
&= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\left(S(0)e^{\left(r-\delta-\frac{1}{2}\sigma^2\right)T+\sigma W^{\mathbb{Q}}(T)} - K\right)_+\right] \\
&= e^{-rT}\int_{-\infty}^{\infty}\left(S(0)e^{\left(r-\delta-\frac{1}{2}\sigma^2\right)T+\sigma x} - K\right)_+ \frac{1}{\sqrt{2\pi T}}e^{-\frac{x^2}{2T}}\,\mathrm{d}x \\
&= e^{-rT}\int_{-d_2\sqrt{T}}^{\infty}\left(S(0)e^{\left(r-\delta-\frac{1}{2}\sigma^2\right)T+\sigma x} - K\right) \frac{1}{\sqrt{2\pi T}}e^{-\frac{x^2}{2T}}\,\mathrm{d}x \\
&= S(0)e^{-rT}e^{\left(r-\delta-\frac{1}{2}\sigma^2\right)T}\int_{-d_2\sqrt{T}}^{\infty} e^{\sigma x}\frac{1}{\sqrt{2\pi T}}e^{-\frac{x^2}{2T}}\,\mathrm{d}x - Ke^{-rT}\int_{-d_2\sqrt{T}}^{\infty}\frac{1}{\sqrt{2\pi T}}e^{-\frac{x^2}{2T}}\,\mathrm{d}x \\
&= S(0)e^{-rT}e^{\left(r-\delta-\frac{1}{2}\sigma^2\right)T}e^{\frac{1}{2}\sigma^2 T}\int_{-d_2\sqrt{T}}^{\infty}\frac{1}{\sqrt{2\pi T}}e^{-\frac{(x-\sigma T)^2}{2T}}\,\mathrm{d}x - Ke^{-rT}\Phi(d_2) \\
&= S(0)e^{-\delta T}\int_{-(d_2+\sigma\sqrt{T})}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}\,\mathrm{d}u - Ke^{-rT}\Phi(d_2)
\end{aligned}$$

$$= S(0)e^{-\delta T}\Phi(d_1) - Ke^{-rT}\Phi(d_2).$$

□

(19.1) is a special case of Girsanov's theorem, which describes how to change the drift of a Brownian motion by changing the underlying probability measure:

Theorem 19.7 (Girsanov's theorem) Let W be a Brownian motion under the original probability measure $\mathbb{P}$. For any constant θ, define

$$W^{\mathbb{Q}}(t) = \theta t + W(t).$$

Then $W^{\mathbb{Q}}$ is a Brownian motion under the new probability measure $\mathbb{Q}$ defined by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(-\theta W(T) - \frac{1}{2}\theta^2 T\right).$$

Example 19.8 Let $S(0) = 20$, $\sigma = 24\%$, $r = 5\%$, and $\delta = 3\%$. Compute the 3-month 25-strike call and put prices under the Black–Scholes model.

Solution. We have

$$d_1 = \frac{\log(20/25) + (0.05 - 0.03 + \frac{1}{2}0.24^2)/4}{0.24\sqrt{1/4}} = -1.75786,$$

and

$$d_2 = d_1 - 0.24\sqrt{1/4} = -1.87786.$$

Using a standard normal table,

$$\Phi(d_1) \approx 0.0393854 \qquad \text{and} \qquad \Phi(d_2) \approx 0.0302000.$$

Therefore the call price is

$$C(S(0), 0) = 20e^{-0.03/4}\Phi(d_1) - 25e^{-0.05/4}\Phi(d_2) \approx 0.03620.$$

By put–call parity,

$$C(S(0), 0) - P(S(0), 0) = S(0)e^{-\delta T} - Ke^{-rT},$$

so
$$P(S(0), 0) = 0.03620 - 20e^{-0.03/4} + 25e^{-0.05/4} \approx 4.87509.$$
$\square$

Lecture 20 — Monday, March 23, 2015

Generalizations of Black–Scholes

For extensions to multiple assets, the Brownian drivers may be correlated.

Definition 20.1 The pair (W_1, W_2) consists of **correlated Brownian motions** with correlation coefficient $\rho \in [-1, 1]$ if it is a two-dimensional continuous Gaussian process starting at $(0, 0)$, with independent vector increments over disjoint intervals, and
$$\operatorname{Cov}(W_i(t) - W_i(s), W_j(t) - W_j(s)) = (t - s) \begin{cases} 1, & i = j, \\ \rho, & i \neq j, \end{cases} \qquad 0 \leq s < t.$$

Since $\operatorname{Var}(W_i(t)) = t$, we have
$$\mathbb{E}\left[W_1(t)W_2(t)\right] = \operatorname{Cov}(W_1(t), W_2(t)) = \rho t.$$

Theorem 20.2 If W_1 and W_2 are Brownian motions with constant correlation ρ, then their **quadratic covariation** is
$$[W_1, W_2](t) = \rho t,$$
so the differential form is
$$\mathrm{d}W_1(t)\,\mathrm{d}W_2(t) = \rho\,\mathrm{d}t.$$

Proof. Fix $t \geq 0$, and let
$$\pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$$
be a partition of $[0, t]$. Write
$$\Delta W_{1,j} = W_1(t_j) - W_1(t_{j-1}) \qquad \text{and} \qquad \Delta W_{2,j} = W_2(t_j) - W_2(t_{j-1}),$$

and define

$$S_\pi := \sum_{j=1}^{n} \Delta W_{1,j}\, \Delta W_{2,j}.$$

By definition, $[W_1, W_2](t)$ is the limit of S_π as the mesh of the partition tends to zero.

For each j, the pair $(\Delta W_{1,j}, \Delta W_{2,j})$ is jointly normal with means 0, variances Δt_j, and covariance $\rho\Delta t_j$, where $\Delta t_j = t_j - t_{j-1}$. Hence

$$\mathbb{E}\left[\Delta W_{1,j} \Delta W_{2,j}\right] = \rho \Delta t_j,$$

so

$$\mathbb{E}[S_\pi] = \sum_{j=1}^{n} \rho \Delta t_j = \rho t.$$

Now we compute the variance. Over disjoint intervals, Brownian increments are independent, so the random variables $\Delta W_{1,j}\Delta W_{2,j}$ are independent across j. Therefore

$$\mathrm{Var}(S_\pi) = \sum_{j=1}^{n} \mathrm{Var}(\Delta W_{1,j} \Delta W_{2,j}).$$

For centered jointly normal variables X, Y, $\mathbb{E}[X^2Y^2] = \mathbb{E}[X^2]\mathbb{E}[Y^2] + 2\mathbb{E}[XY]^2$. Applying this with $X = \Delta W_{1,j}$, $Y = \Delta W_{2,j}$, we get

$$\mathbb{E}\left[(\Delta W_{1,j} \Delta W_{2,j})^2\right] = (\Delta t_j)^2 + 2\rho^2 (\Delta t_j)^2,$$

and hence

$$\mathrm{Var}(\Delta W_{1,j} \Delta W_{2,j}) = \left(1 + \rho^2\right) (\Delta t_j)^2.$$

Thus

$$\mathrm{Var}(S_\pi) = (1 + \rho^2) \sum_{j=1}^{n} (\Delta t_j)^2 \le (1 + \rho^2)\, t\, \Delta t \xrightarrow[\Delta t \to 0]{} 0,$$

where $\Delta t = \max_j \Delta t_j$. So $S_\pi \to \rho t$ in mean square, and therefore in probability by Lemma 16.4. Consequently,

$$[W_1, W_2](t) = \rho t.$$

In differential notation, this is

$$\mathrm{d}W_1(t)\, \mathrm{d}W_2(t) = \rho\, \mathrm{d}t.$$

□

The cases $\rho = 0$, $\rho = 1$, and $\rho = -1$ correspond respectively to independent noise, perfectly positively aligned noise, and perfectly negatively aligned noise.

Correlated Brownian motions require the following product rule.

Theorem 20.3 (Ito product rule) If $X(t)$ and $Y(t)$ are Ito processes, then

$$\mathrm{d}(X(t)Y(t)) = X(t)\,\mathrm{d}Y(t) + Y(t)\,\mathrm{d}X(t) + \mathrm{d}X(t)\,\mathrm{d}Y(t).$$

This is the stochastic analogue of the ordinary product rule, except that it includes the extra quadratic covariation term $\mathrm{d}X(t)\,\mathrm{d}Y(t)$.

Example 20.4 Suppose

$$\mathrm{d}X(t) = (\alpha - \beta X(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W(t).$$

Derive the stochastic differential equation for $e^{\beta t}X(t)$.

Solution. Applying the Ito product rule to $e^{\beta t}X(t)$ gives

$$\begin{aligned}\mathrm{d}\big(e^{\beta t}X(t)\big) &= X(t)\ \mathrm{d}(e^{\beta t}) + e^{\beta t}\,\mathrm{d}X(t) + \mathrm{d}(e^{\beta t})\,\mathrm{d}X(t)\\ &= \beta e^{\beta t}X(t)\,\mathrm{d}t + e^{\beta t}\big[(\alpha - \beta X(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)\big]\\ &= \alpha e^{\beta t}\,\mathrm{d}t + \sigma e^{\beta t}\,\mathrm{d}W(t),\end{aligned}$$

because $\mathrm{d}(e^{\beta t})\,\mathrm{d}X(t) = 0$. □

Example 20.5 Suppose

$$\mathrm{d}S_1(t) = \alpha_1 S_1(t)\,\mathrm{d}t + \sigma_1 S_1(t)\,\mathrm{d}W_1(t) \qquad \text{and} \qquad \mathrm{d}S_2(t) = \alpha_2 S_2(t)\,\mathrm{d}t + \sigma_2 S_2(t)\,\mathrm{d}W_2(t),$$

Derive the stochastic differential equation for $S_1(t)/S_2(t)$.

Solution. To compute the dynamics of $S_1(t)/S_2(t)$, first apply Ito's formula to $1/S_2(t)$:

$$\mathrm{d}\left(\frac{1}{S_2(t)}\right) = -\frac{1}{S_2(t)^2}\,\mathrm{d}S_2(t) + \frac{1}{S_2(t)^3}(\mathrm{d}S_2(t))^2 = \frac{\sigma_2^2 - \alpha_2}{S_2(t)}\,\mathrm{d}t - \frac{\sigma_2}{S_2(t)}\,\mathrm{d}W_2(t).$$

Then Ito's product rule gives

$$\begin{aligned} \mathrm{d}\left(\frac{S_1(t)}{S_2(t)}\right) &= \frac{1}{S_2(t)}\,\mathrm{d}S_1(t) + S_1(t)\,\mathrm{d}\left(\frac{1}{S_2(t)}\right) + \mathrm{d}S_1(t)\,\mathrm{d}\left(\frac{1}{S_2(t)}\right) \\ &= \frac{S_1(t)}{S_2(t)}\Big[(\alpha_1 - \alpha_2 + \sigma_2^2 - \sigma_1\sigma_2\rho)\,\mathrm{d}t + \sigma_1\,\mathrm{d}W_1(t) - \sigma_2\,\mathrm{d}W_2(t)\Big]. \end{aligned}$$

□

These tools yield the generalized Black–Scholes formula. The extension to two assets prices the right to receive one unit of S_1 by giving up one unit of S_2 at time T.

Theorem 20.6 (Generalized Black–Scholes formula) Consider two stock prices $S_1(t)$ and $S_2(t)$ following Black–Scholes dynamics with volatilities σ_1 and σ_2, continuous dividend yields δ_1 and δ_2, correlation coefficient ρ, and risk-free rate r. Then the price of the European option to receive one unit of S_1 by giving up one unit of S_2 at time T with payoff

$$(S_1(T) - S_2(T))_+$$

is

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_1(T) - S_2(T))_+\right] = \mathrm{PV}(S_1)\Phi(d_1) - \mathrm{PV}(S_2)\Phi(d_2),$$

where

$$d_1 = \frac{\log(\mathrm{PV}(S_1)/\mathrm{PV}(S_2)) + \hat{\sigma}^2 T/2}{\hat{\sigma}\sqrt{T}} \qquad \text{and} \qquad d_2 = d_1 - \hat{\sigma}\sqrt{T},$$

and

$$\hat{\sigma} = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}.$$

Here

$$\mathrm{PV}(S_i) = S_i(0)e^{-\delta_i T}, \qquad i = 1, 2,$$

is the prepaid forward price of one unit of asset i delivered at maturity. This is also called **Margrabe's formula**.

The effective volatility $\hat{\sigma}$ is the volatility of the relative price S_1/S_2. Correlation reduces it when the two assets move together and increases it when they move in opposite directions.

Corollary 20.7 Under the same assumptions as Theorem 20.6,

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[S_1(T)\mathbb{1}_{\{S_1(T) > S_2(T)\}}\right] = \mathrm{PV}(S_1)\Phi(d_1)$$

and
$$e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[S_2(T)\mathbb{1}_{\{S_1(T)>S_2(T)\}}\right] = \mathrm{PV}(S_2)\Phi(d_2).$$

These identities allow a range of exotic payoffs to be decomposed into asset-or-nothing and cash-or-nothing pieces.

Example 20.8 Consider two non-dividend-paying stocks with

$$S_1(0) = 10, \qquad S_2(0) = 20, \qquad \sigma_1 = 18\%,$$
$$\sigma_2 = 25\%, \qquad \rho = -0.4, \qquad \text{and} \qquad r = 5\%.$$

Determine the price of an exchange option to receive two shares of S_1 by giving up one share of S_2 at time $T = 1$.

Solution. The payoff is
$$(2S_1(1) - S_2(1))_+.$$

Equivalently, apply the generalized formula with underlying values $2S_1$ and S_2. The effective volatility is
$$\hat{\sigma} = \sqrt{0.18^2 + 0.25^2 - 2(-0.4)(0.18)(0.25)} = 0.361801.$$

Since $2S_1(0) = 20 = S_2(0)$,

$$d_1 = \frac{\hat{\sigma}^2/2}{\hat{\sigma}} = 0.1809 \qquad \text{and} \qquad d_2 = d_1 - \hat{\sigma} = -0.1809.$$

Thus
$$\Phi(d_1) = 0.57178 \qquad \text{and} \qquad \Phi(d_2) = 0.42822,$$

and the exchange option price is

$$C(0) = 2S_1(0)\Phi(d_1) - S_2(0)\Phi(d_2) \approx 2.8711.$$

□

Example 20.9 Suppose $S(0) = 40$, $K = 40$, $\sigma = 30\%$, $r = 8\%$, $\delta = 0$, and $T = 0.25$. Calculate the premium $C_1(0)$ of a cash-or-nothing call paying 1 if $S(T) > K$ and the premium $C_2(0)$ of a cash-or-nothing put paying 1 if $S(T) < K$.

Solution. The cash-or-nothing call paying 1 if $S(T) > K$ has price

$$C_1(0) = e^{-rT}\Phi(d_2),$$

where
$$d_2 = \frac{\log\big(40/(40e^{-0.08 \cdot 0.25})\big) + 0.3^2 \cdot 0.25/2}{0.3\sqrt{0.25}} - 0.3\sqrt{0.25} = 0.0583.$$

Therefore,
$$C_1(0) = e^{-0.08 \cdot 0.25}\Phi(0.0583) = 0.5129.$$

The corresponding cash-or-nothing put paying 1 if $S(T) < K$ has price

$$C_2(0) = e^{-rT} - C_1(0) = 0.4673.$$

$\square$

Example 20.10 Suppose $S(0) = 40$, $K = 40$, $\sigma = 30\%$, $r = 8\%$, $\delta = 0$, and $T = 0.25$. Calculate the premium $C(0)$ of a gap call that pays $S(T) - 20$ if $S(T) > 40$ and 0 otherwise.

Solution. The gap call has payoff

$$S(T) - 20 \quad \text{if } S(T) > 40.$$

Its payoff can be decomposed as

$$S(T)\mathbb{1}_{\{S(T)>40\}} - 20\mathbb{1}_{\{S(T)>40\}}.$$

Therefore,

$$C(0) = \mathrm{PV}(S(T))\Phi(d_1) - \mathrm{PV}(20)\Phi(d_2) = 40\Phi(d_1) - 20e^{-0.08 \cdot 0.25}\Phi(d_2),$$

where

$$d_1 = \frac{\log\big(40/(40e^{-0.08 \cdot 0.25})\big) + 0.3^2 \cdot 0.25/2}{0.3\sqrt{0.25}} = 0.20833 \qquad \text{and} \qquad d_2 = d_1 - 0.3\sqrt{0.25} = 0.05833.$$

Hence
$$C(0) = 13.0427.$$

$\square$

Lecture 21 — Wednesday, March 25, 2015

Option Greeks

The **Greeks** measure how an option price changes when one of the underlying parameters changes. If X denotes the option price and the model inputs are $(S, \sigma, T, r, \delta)$, then the main Greeks are

$$\Delta = \frac{\partial X}{\partial S}, \qquad \Gamma = \frac{\partial^2 X}{\partial S^2}, \qquad \text{Vega} = \frac{\partial X}{\partial \sigma}, \qquad \text{and} \qquad \Theta = -\frac{\partial X}{\partial T}.$$

Thus **delta** measures first-order sensitivity to the stock price, **gamma** measures curvature with respect to the stock price, **vega** measures volatility sensitivity, and **theta** measures the rate of time decay in the convention used in finance. Annoyingly, "vega" is the name of a star, not a Greek letter, and there is no standard Greek letter for it. The convention of using a negative sign in the definition of theta is also somewhat arbitrary, but it is widely used because most options lose value as time passes.

For delta, starting from the Black–Scholes formulas for European calls and puts, we obtain the closed-form deltas

$$\Delta_{\text{call}} = \frac{\partial C}{\partial S} = e^{-\delta T}\Phi(d_1) > 0$$

and

$$\Delta_{\text{put}} = \frac{\partial P}{\partial S} = -e^{-\delta T}\Phi(-d_1) < 0.$$

The put formula also follows immediately from put–call parity:

$$P = C - Se^{-\delta T} + Ke^{-rT}.$$

Therefore the call price is increasing in S, while the put price is decreasing in S.

Delta is especially important for hedging. The Black–Scholes call price can be rewritten as

$$C = S\Delta_{\text{call}} - Ke^{-rT}\Phi(d_2),$$

which makes the stock holding inside the replicating strategy explicit.

For the representative parameter values $K = 40$, $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$, the delta curves in Figure 28 have the expected shape. Call delta rises from near 0 to near 1, put delta rises from near -1 to near 0, and longer maturities make both transitions less abrupt.

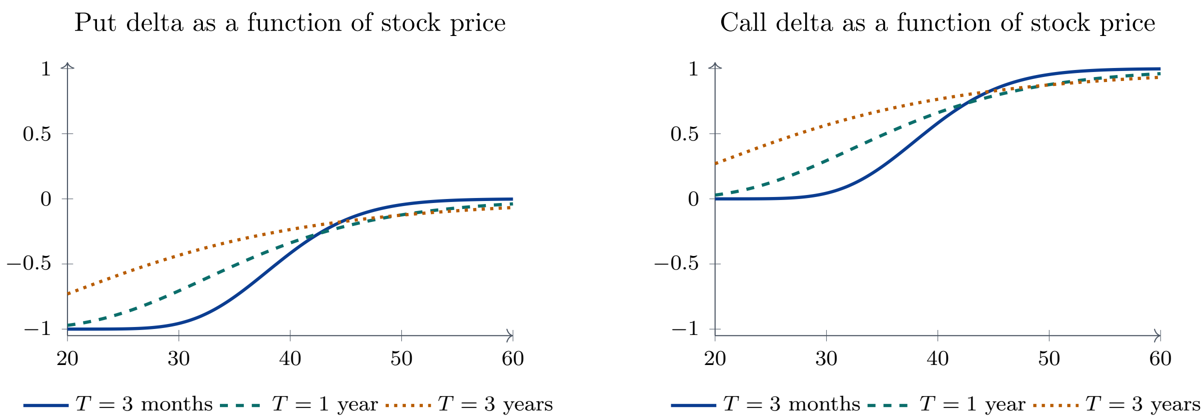

Figure 28

For gamma, the Black–Scholes model gives

$$\Gamma_{\text{call}} = \frac{\partial^2 C}{\partial S^2} = \frac{e^{-\delta T}\Phi'(d_1)}{S\sigma\sqrt{T}} > 0.$$

The put gamma is the same:

$$\Gamma_{\text{put}} = \frac{\partial^2 P}{\partial S^2} = \frac{\partial^2 C}{\partial S^2} > 0.$$

Hence both call and put prices are convex functions of the stock price. Gamma measures how quickly delta changes as S changes, so it captures the residual nonlinear risk left after first-order delta hedging.

For vega, the Black–Scholes formulas give

$$\text{Vega}_{\text{call}} = \frac{\partial C}{\partial \sigma} = Se^{-\delta T}\Phi'(d_1)\sqrt{T} > 0$$

and

$$\text{Vega}_{\text{put}} = \frac{\partial P}{\partial \sigma} = \frac{\partial C}{\partial \sigma} > 0.$$

Therefore both call and put prices increase with volatility. This matches the economic intuition that greater uncertainty increases the value of optionality.

The Black–Scholes call theta is

$$\Theta_{\text{call}} = \delta S e^{-\delta T}\Phi(d_1) - rKe^{-rT}\Phi(d_2) - Ke^{-rT}\Phi'(d_2)\frac{\sigma}{2\sqrt{T}},$$

and the put theta satisfies

$$\Theta_{\text{put}} = \Theta_{\text{call}} + rKe^{-rT} - \delta Se^{-\delta T}.$$

If the stock pays no dividend, then the call price increases with T, so $\partial C/\partial T > 0$ and therefore $\Theta_{\text{call}} < 0$ under this sign convention.

For the same representative parameter values $K = 40$, $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$, the corresponding gamma curves appear in Figure 29, and the vega curves appear in Figure 30.

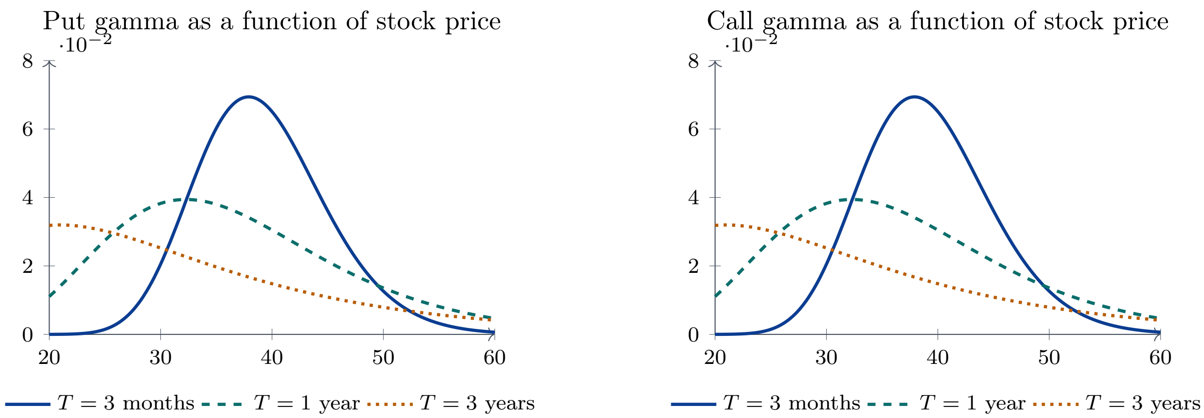

FIGURE 29

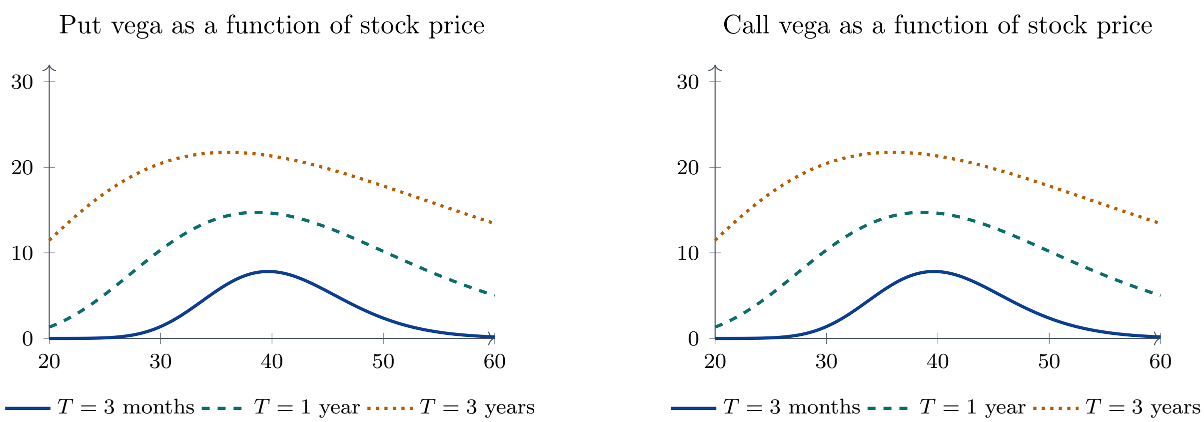

FIGURE 30

For theta, we now fix $S = 40$, $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$. Figure 31 plots Θ against time to expiry for three strikes $K \in \{30, 40, 50\}$.

For the representative parameters $S = 40$, $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$, call premiums increase with time to expiry for strikes 30, 40, and 50. Put premiums need not behave uniformly across strikes: the out-of-the-money and at-the-money puts increase with maturity in the plot, while the

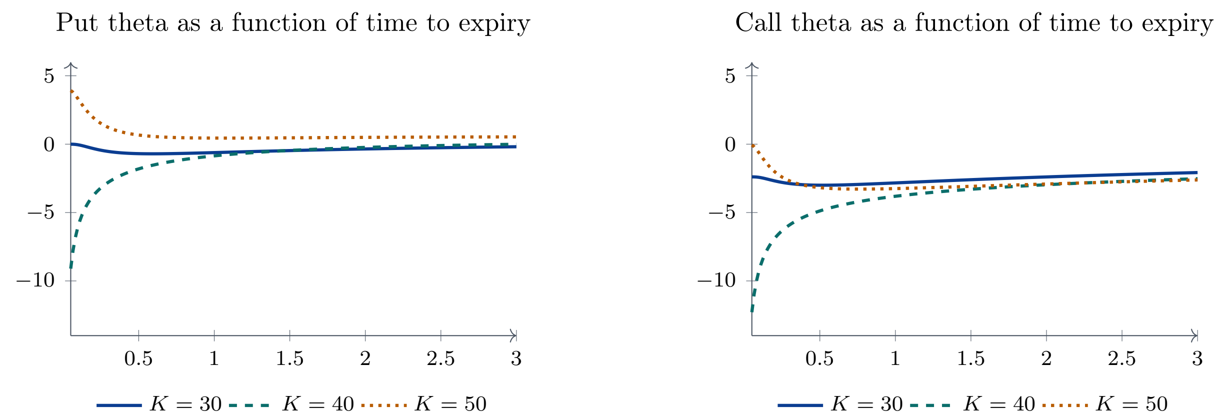

FIGURE 31

deep in-the-money put decreases.

Greeks also aggregate linearly across portfolios. If a portfolio contains N assets, if ω_i is the number of units of asset i, and if g_i denotes a given Greek for asset i, then the corresponding Greek of the portfolio is

$$g_{\text{portfolio}} = \sum_{i=1}^{N} \omega_i g_i.$$

Thus delta, gamma, vega, and theta all aggregate linearly across positions.

Hedging with Greeks

Delta-hedging is a strategy to reduce the risk associated with the change of the underlying price. The idea is to construct a portfolio that is delta-neutral, so that the first-order sensitivity to the stock price is eliminated.

Definition 21.1 A portfolio is **delta-neutral** if

$$\Delta_{\text{portfolio}} = 0.$$

Such a portfolio is relatively insensitive to small changes in the underlying stock price.

To delta-hedge a short position in one option, the market maker can buy Δ shares of stock. If the portfolio value is

$$V = -C + \Delta S,$$

then for a small change ε in the stock price,

$$\begin{aligned}
V(S+\varepsilon) - V(S) &= -\left(C(S+\varepsilon) - C(S)\right) + \Delta(S+\varepsilon-S) \\
&\approx -\left(C + \varepsilon\frac{\partial C}{\partial S}\right) + \Delta(S+\varepsilon) - (-C + \Delta S) \\
&= -\varepsilon\frac{\partial C}{\partial S} + \Delta\varepsilon \\
&= -\varepsilon\Delta + \Delta\varepsilon \\
&= 0.
\end{aligned}$$

This is only a first-order approximation, which is why gamma still matters.

How do we delta-hedge? If $\Delta = \partial V/\partial S$ is the delta of the original portfolio, then the portfolio $\tilde{V} = V - \Delta S$ is delta-neutral because

$$\tilde{\Delta} := \frac{\partial \tilde{V}}{\partial S} = \frac{\partial V}{\partial S} - \Delta = 0.$$

Thus, to delta-hedge, we subtract ΔS from the original portfolio V. For example, a short call has portfolio delta $-\Delta_{\text{call}}$, so its hedge buys Δ_{call} shares of the underlying stock.

Example 21.2 Consider a 91-day European call with

$$S_t = 40, \qquad K = 40, \qquad \sigma = 0.3, \qquad r = 0.08, \qquad \text{and} \qquad \delta = 0.$$

The Black–Scholes formula gives

$$C = 2.7804 \qquad \text{and} \qquad \Delta = 0.5824.$$

Suppose the stock price rises to 40.75. Then the option price increases to 3.2352, so the change in option value is

$$3.2352 - 2.7804 = 0.4548.$$

The percentage change in stock price is

$$\frac{0.75}{40} \times 100\% = 1.875\%,$$

while the percentage change in option price is

$$\frac{0.4548}{2.7804} \times 100\% = 16.36\%.$$

Thus the option position is much more sensitive than the stock position itself.

Example 21.3 Consider a market maker who writes one 91-day call on 100 shares with the same parameters as above. Since the option delta is 0.5824, the seller buys

$$100 \times 0.5824 = 58.24$$

shares of stock. The net investment at time 0 is

$$40 \times 58.24 - 2.7804 \times 100 = 2051.56.$$

Suppose on day 1 the stock price increases to 40.50. The updated option price is

$$C(40.50, 90/365) = 3.0621.$$

The gain on the stock position is

$$58.24 \times (40.50 - 40) = 29.12.$$

The gain on the 100 written calls is

$$100 \times (2.7804 - 3.0621) = -28.17.$$

The financing cost on the initial net investment is

$$-(e^{0.08/365} - 1) \times 2051.56 = -0.45.$$

Hence the overnight profit is

$$29.12 - 28.17 - 0.45 = 0.50.$$

At the new stock price, the updated delta is 0.6142, so the hedger must rebalance to

$$61.42$$

shares. The new net investment is

$$40.50 \times 61.42 - 3.0621 \times 100 = 2181.30.$$

Suppose on day 2 the stock price decreases to 39.25. The corresponding option price is

$$C(39.25, 89/365) = 2.3282.$$

The gain on the 61.42 shares is

$$61.42 \times (39.25 - 40.50) = -76.78.$$

The gain on the 100 written calls is

$$100 \times (3.0621 - 2.3282) = 73.39.$$

The financing cost is

$$-(e^{0.08/365} - 1) \times 2181.30 = -0.48.$$

Therefore the overnight profit is

$$-76.78 + 73.39 - 0.48 = -3.87.$$

The practical lesson is that, for a delta-hedged short option over a short rebalancing interval, realized stock movement that is small relative to the volatility priced into the option tends to help the seller, while unusually large movement tends to hurt. This is a local statement, not a guarantee of profit: financing, time decay, discrete rebalancing, model error, and changes in implied volatility also matter. Delta hedging reduces risk, but it does not eliminate it.

A second-order hedge is obtained by imposing gamma neutrality.

Definition 21.4 A portfolio is **gamma-neutral** if

$$\Gamma_{\text{portfolio}} = 0.$$

If a portfolio is both delta-neutral and gamma-neutral, then its value is much less sensitive to changes in the stock price than a portfolio that is only delta-neutral.

Gamma hedging cannot be done with stock alone, because the stock price is linear in itself and therefore has zero gamma:

$$\frac{\partial^2 S}{\partial S^2} = 0.$$

Consequently, gamma hedging requires additional options. Not every market maker has enough liquid option inventory to do this in practice.

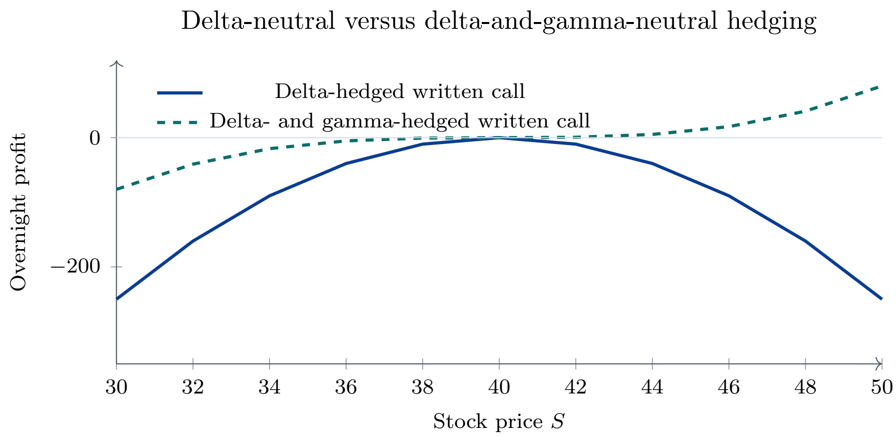

FIGURE 32

The geometry in Figure 32 illustrates the point. A delta-neutral portfolio still has noticeable curvature, so large stock moves can produce substantial losses. A portfolio that is both delta-neutral and gamma-neutral is much flatter in the neighbourhood of the current stock price.

Lecture 22 — Monday, March 30, 2015

6. Interest Rate Models

Continuous-Time Interest Rate Models

For a constant risk-free rate r, the risk-neutral pricing formula for a derivative with payoff $X(T)$ is, for $0 \leq t \leq T$,

$$X(t) = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}} \left[X(T) \right],$$

where $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ denotes conditional expectation under the risk-neutral measure $\mathbb{Q}$ given all information available up to time t. This formula assumes a constant risk-free rate r. However, in reality, interest rates fluctuate over time, and it is more realistic to model the short-term interest rate $r(t)$ as a stochastic process.

If the short-term interest rate $r(t)$ is stochastic, then the continuous-time risk-neutral pricing formula becomes

$$X(t) = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\,\mathrm{d}s\right) X(T)\right],$$

where $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ denotes conditional expectation under the risk-neutral measure $\mathbb{Q}$ given all information available up to time t.

The key application is the price of a zero-coupon bond with face value 1, which is

$$P(r,t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\,\mathrm{d}s\right)\right].$$

The central questions are how to identify the risk-neutral measure $\mathbb{Q}$ and how to derive the bond price $P(r,t,T)$.

Viewed from time 0, the future bond price $P(r(t),t,T)$ is random because the state $r(t)$ is not yet known. At time t, however, the price is $\mathcal{F}_t$-measurable: it is determined by the current state through a conditional expectation over possible future short-rate paths. At maturity, $P(r(T),T,T) = 1$.

The continuous-time bond pricing equation follows by applying Ito's formula to the bond price. Suppose the short rate follows the diffusion

$$\mathrm{d}r = a(r,t)\,\mathrm{d}t + \sigma(r,t)\,\mathrm{d}W(t),$$

and suppose the bond price process satisfies

$$\frac{\mathrm{d}P}{P} = \alpha(r,t,T)\,\mathrm{d}t - q(r,t,T)\,\mathrm{d}W(t).$$

Assume that the **Sharpe ratio**

$$\phi(r,t) := \frac{\alpha(r,t,T) - r}{q(r,t,T)}$$

is independent of maturity T; that is,

$$\frac{\alpha(r,t,T_1) - r}{q(r,t,T_1)} = \frac{\alpha(r,t,T_2) - r}{q(r,t,T_2)}$$

for any maturities T_1 and T_2 (without this assumption, arbitrage opportunities could exist).

Applying Ito's formula to $P(r,t,T)$ gives

$$\begin{aligned}\mathrm{d}P &= \frac{\partial P}{\partial t}\,\mathrm{d}t + \frac{\partial P}{\partial r}\,\mathrm{d}r + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}(\mathrm{d}r)^2 \\ &= \left(\frac{\partial P}{\partial t} + a(r,t)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma(r,t)^2\frac{\partial^2 P}{\partial r^2}\right)\mathrm{d}t + \sigma(r,t)\frac{\partial P}{\partial r}\,\mathrm{d}W(t).\end{aligned}$$

Comparing this with

$$\mathrm{d}P = \alpha(r,t,T)P\,\mathrm{d}t - q(r,t,T)P\,\mathrm{d}W(t)$$

shows that

$$\alpha(r,t,T) = \frac{1}{P}\left(\frac{\partial P}{\partial t} + a(r,t)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma(r,t)^2\frac{\partial^2 P}{\partial r^2}\right)$$

and

$$q(r,t,T) = -\frac{\sigma(r,t)}{P}\frac{\partial P}{\partial r}.$$

Substituting these expressions into the definition of $\phi(r,t)$ yields the **equilibrium equation for every zero-coupon bond**:

Theorem 22.1 For a short-rate model

$$\mathrm{d}r = a(r,t)\,\mathrm{d}t + \sigma(r,t)\,\mathrm{d}W(t),$$

the price $P(r,t,T)$ of a zero-coupon bond satisfies

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma(r,t)^2\frac{\partial^2 P}{\partial r^2} + \big(a(r,t) + \sigma(r,t)\phi(r,t)\big)\frac{\partial P}{\partial r} - rP = 0,$$

for $t < T$, with condition at maturity

$$P(r,T,T) = 1.$$

The corresponding risk-neutral measure is obtained by changing the Brownian drift. We seek a measure $\mathbb{Q}$ under which every bond has conditional drift equal to the short rate:

$$\mathbb{E}_t^{\mathbb{Q}}[\mathrm{d}P] = r(t)P\,\mathrm{d}t.$$

First choose a measure $\mathbb{Q}$ under which the process $W^{\mathbb{Q}}$, defined by

$$\mathrm{d}W^{\mathbb{Q}}(t) = -\phi(r,t)\,\mathrm{d}t + \mathrm{d}W(t).$$

is a Brownian motion. This is possible by Girsanov's theorem (Theorem 19.7), which allows us to change the drift of a Brownian motion by changing the probability measure, provided the usual integrability condition on ϕ holds (for example, Novikov's condition). Under the new measure $\mathbb{Q}$, the process $W^{\mathbb{Q}}$ is Brownian motion, and the bond dynamics become

$$\mathrm{d}P = \left(\frac{\partial P}{\partial t} + \big(a(r,t) + \sigma(r,t)\phi(r,t)\big)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma(r,t)^2\frac{\partial^2 P}{\partial r^2}\right)\mathrm{d}t + \sigma(r,t)\frac{\partial P}{\partial r}\,\mathrm{d}W^{\mathbb{Q}}(t).$$

Using the equilibrium equation for zero-coupon bonds (Theorem 22.1) simplifies the drift to rP, so

$$\mathrm{d}P = rP\,\mathrm{d}t + \sigma(r,t)\frac{\partial P}{\partial r}\,\mathrm{d}W^{\mathbb{Q}}(t).$$

Taking conditional expectations under $\mathbb{Q}$ gives

$$\mathbb{E}_t^{\mathbb{Q}}[\mathrm{d}P] = r(t)P\,\mathrm{d}t + \mathbb{E}_t^{\mathbb{Q}}\left[\sigma(r,t)\frac{\partial P}{\partial r}\,\mathrm{d}W^{\mathbb{Q}}(t)\right].$$

Since $\sigma(r,t)\cdot\partial P/\partial r$ is adapted and square-integrable and $W^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$, the stochastic-integral increment has conditional mean zero:

$$\mathbb{E}_t^{\mathbb{Q}}\left[\sigma(r,t)\frac{\partial P}{\partial r}\,\mathrm{d}W^{\mathbb{Q}}(t)\right] = 0.$$

Therefore

$$\mathbb{E}_t^{\mathbb{Q}}[\mathrm{d}P] = r(t)P\,\mathrm{d}t,$$

which confirms that $\mathbb{Q}$ is a risk-neutral measure for bond prices.

The Vasicek and Cox–Ingersoll–Ross Models

A first explicit short-rate specification is the Vasicek model.

Definition 22.2 The **Vasicek model** of the interest rate assumes

$$\mathrm{d}r(t) = a\big(b - r(t)\big)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t),$$

where a, b, and σ are positive constants.

The parameter a controls the speed of mean reversion, b is the long-run mean level, and σ is a constant volatility. In this model $r(t)$ has the mean-reverting property (which is good), but it can become negative (which is bad), as the sample paths in Figure 33 illustrate.

The **mean-reverting property** means that when the interest rate is above its long-run mean b, the drift term $a(b - r(t))$ is negative, which tends to pull the rate back down. Conversely, when

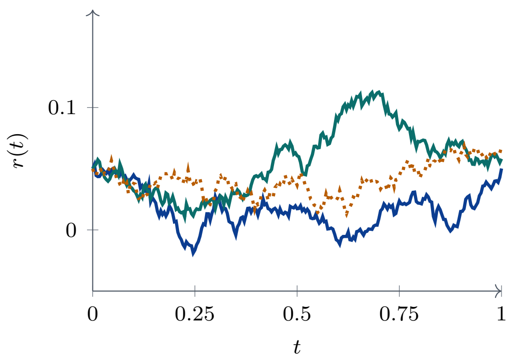

FIGURE 33

the interest rate is below b, the drift term is positive, which tends to push the rate back up. This feature helps to capture the empirical behavior of interest rates, which tend to fluctuate around a long-term average level.

If the Sharpe ratio ϕ is taken to be constant, then the bond price satisfies

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2} + \big(a(b-r) + \sigma\phi\big)\frac{\partial P}{\partial r} - rP = 0$$

subject to the boundary condition $P(r, T, T) = 1$.

Theorem 22.3 In the Vasicek model,

$$P(r, t, T) = A(t, T)e^{-B(t,T)r(t)},$$

where

$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a},$$

$$A(t, T) = \exp\left(\bar{r}\big(B(t, T) + t - T\big) - \frac{\sigma^2}{4a}B(t, T)^2\right),$$

and

$$\bar{r} = b + \frac{\sigma\phi}{a} - \frac{\sigma^2}{2a^2}.$$

Proof. By Theorem 22.1, the Vasicek bond price must satisfy

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2} + \big(a(b-r) + \sigma\phi\big)\frac{\partial P}{\partial r} - rP = 0$$

with condition at maturity $P(r,T,T) = 1$. Using the ansatz $P(r,t,T) = A(t,T)e^{-B(t,T)r}$, we have

$$\frac{\partial P}{\partial r} = -BP, \qquad \frac{\partial^2 P}{\partial r^2} = B^2 P, \qquad \text{and} \qquad \frac{\partial P}{\partial t} = \left(\frac{1}{A}\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\right)P.$$

Substituting into the PDE and dividing by P gives

$$\left(-\frac{\partial B}{\partial t} + aB - 1\right)r + \left(\frac{1}{A}\frac{\partial A}{\partial t} + \frac{1}{2}\sigma^2 B^2 - (ab + \sigma\phi)B\right) = 0.$$

Hence the coefficients of r and the constant term must vanish:

$$\frac{\partial B}{\partial t} = aB - 1 \qquad \text{and} \qquad \frac{1}{A}\frac{\partial A}{\partial t} = (ab + \sigma\phi)B - \frac{1}{2}\sigma^2 B^2,$$

with conditions at maturity $B(T,T) = 0$ and $A(T,T) = 1$. Solving the linear ODE for B yields

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a}.$$

Now define

$$\bar{r} = b + \frac{\sigma\phi}{a} - \frac{\sigma^2}{2a^2} \qquad \text{and} \qquad A(t,T) = \exp\left(\bar{r}\big(B(t,T) + t - T\big) - \frac{\sigma^2}{4a}B(t,T)^2\right).$$

Since $B(T,T) = 0$, we have $A(T,T) = 1$. Also, from $\frac{\partial B}{\partial t} = aB - 1$,

$$\begin{aligned}
\frac{1}{A}\frac{\partial A}{\partial t} &= \bar{r}\left(\frac{\partial B}{\partial t} + 1\right) - \frac{\sigma^2}{2a}B\frac{\partial B}{\partial t} \\
&= a\bar{r}B - \frac{\sigma^2}{2a}B(aB - 1) \\
&= \left(a\bar{r} + \frac{\sigma^2}{2a}\right)B - \frac{1}{2}\sigma^2 B^2 \\
&= (ab + \sigma\phi)B - \frac{1}{2}\sigma^2 B^2,
\end{aligned}$$

because $a\bar{r} + \sigma^2/2a = ab + \sigma\phi$. Thus A satisfies the required ODE. Therefore,

$$P(r,t,T) = A(t,T)e^{-B(t,T)r}$$

solves the equilibrium equation with the condition at maturity. Evaluating at the short rate $r = r(t)$ gives

$$P(r, t, T) = A(t, T)e^{-B(t,T)r(t)},$$

which is the Vasicek bond pricing formula. $\square$

The **yield to maturity** of a bond maturing at time T is

$$y(r, t, T) := -\frac{\log P(r, t, T)}{T - t}.$$

For an infinitely long bond, the Vasicek model gives

$$\lim_{T\to\infty} y(r, t, T) = -\lim_{T\to\infty} \frac{\log P(r, t, T)}{T - t} = \bar{r}.$$

A second short-rate specification is the Cox–Ingersoll–Ross model.

Definition 22.4 The **Cox–Ingersoll–Ross** (**CIR**) model assumes

$$\mathrm{d}r(t) = a\big(b - r(t)\big)\,\mathrm{d}t + \sigma\sqrt{r(t)}\,\mathrm{d}W(t),$$

where a, b, and σ are positive constants.

Like the Vasicek model, the Cox–Ingersoll–Ross model is mean-reverting, but its volatility $\sigma\sqrt{r(t)}$ is state dependent. Starting from a nonnegative rate, the CIR process remains nonnegative. The stronger Feller condition $2ab \geq \sigma^2$ makes zero inaccessible when the initial rate is strictly positive. Representative paths are shown in Figure 34.

If we choose the Sharpe ratio

$$\phi(r, t) = \frac{\bar{\phi}\sqrt{r}}{\sigma}$$

for a constant $\bar{\phi}$, then the bond price solves the equilibrium equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 r \frac{\partial^2 P}{\partial r^2} + \big(a(b - r) + r\bar{\phi}\big)\frac{\partial P}{\partial r} - rP = 0$$

subject to the condition at maturity $P(r, T, T) = 1$. For the standard mean-reverting risk-neutral specification, we also assume $a - \bar{\phi} > 0$.

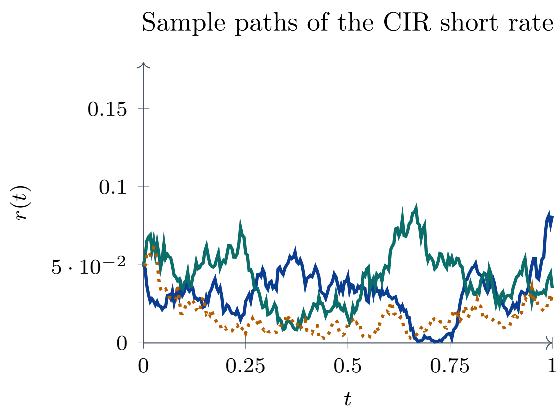

FIGURE 34

Theorem 22.5 In the Cox–Ingersoll–Ross model,

$$P(r,t,T) = A(t,T)e^{-B(t,T)r(t)},$$

where

$$\gamma = \sqrt{(a-\bar{\phi})^2 + 2\sigma^2},$$

$$A(t,T) = \left(\frac{2\gamma e^{(a-\bar{\phi}+\gamma)(T-t)/2}}{(a-\bar{\phi}+\gamma)(e^{\gamma(T-t)}-1)+2\gamma} \right)^{2ab/\sigma^2},$$

and

$$B(t,T) = \frac{2(e^{\gamma(T-t)}-1)}{(a-\bar{\phi}+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}.$$

The proof is similar to the Vasicek case but somewhat more involved, so we omit it.

The yield to maturity of an infinitely-lived bond under the Cox–Ingersoll–Ross model is

$$\lim_{T\to\infty} y(r,t,T) = -\lim_{T\to\infty} \frac{\log P(r,t,T)}{T-t} = \frac{2ab}{a-\bar{\phi}+\gamma}.$$

Lecture 23 — Wednesday, April 01, 2015

Discrete-Time Interest Rate Models

We now return to a finite tree model under a risk-neutral measure.

Example 23.1 Consider the three-period continuously compounded short-rate tree in Figure 35. Each period has length $h = 1$, and the risk-neutral up and down probabilities are both 0.5.

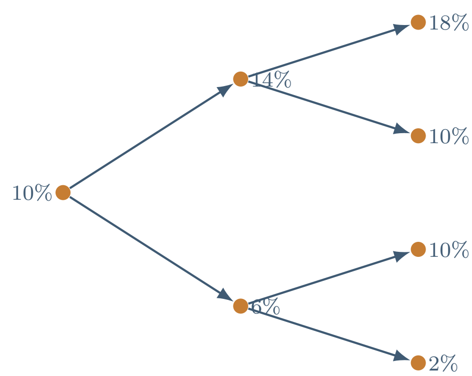

Figure 35

1. Calculate the zero-coupon bond prices $P(0, 1)$, $P(0, 2)$, and $P(0, 3)$.
2. Calculate the price of a two-year put on a one-year zero-coupon bond with strike 0.88.

Solution. For (1), from Figure 35, the one-year zero-coupon bond price is

$$P(0, 1) = e^{-0.1} = 0.9048.$$

The two-year price is the risk-neutral average of the two possible discount factors:

$$P(0, 2) = 0.5e^{-0.1}e^{-0.14} + 0.5e^{-0.1}e^{-0.06} = 0.8194.$$

Similarly,

$$P(0,3) = 0.25e^{-0.1}e^{-0.14}e^{-0.18} + 0.25e^{-0.1}e^{-0.14}e^{-0.10} + 0.25e^{-0.1}e^{-0.06}e^{-0.10} + 0.25e^{-0.1}e^{-0.06}e^{-0.02} = 0.7438.$$

Hence the yields to maturity are

$$\mathrm{YTM}_{\text{1-year}} = -\log P(0,1) = 0.1,$$

$$\mathrm{YTM}_{\text{2-year}} = -\frac{1}{2}\log P(0,2) = 0.0996,$$

and

$$\mathrm{YTM}_{\text{3-year}} = -\frac{1}{3}\log P(0,3) = 0.0987.$$

For (2), the payoff at time 2 is

$$(0.88 - P(2,3))_+.$$

At time 2, the possible one-year bond prices are

$$e^{-0.18}, \qquad e^{-0.10}, \qquad \text{and} \qquad e^{-0.02}.$$

Only the uu state gives a positive payoff, namely

$$X(2,uu) = 0.88 - e^{-0.18} = 0.0447.$$

Since the risk-neutral probability of reaching uu is 0.25, the time-0 price is

$$X(0) = e^{-0.14-0.10}\cdot 0.25\cdot 0.0447 = 0.0088.$$

□

The main discrete-time short-rate model considered here is the Black–Derman–Toy model.

Definition 23.2 In the **Black–Derman–Toy** (**BDT**) model, each period has two parameters (R_i, σ_i), and the effective interest rates evolve on a recombining tree according to

$$r_0, \qquad r_u = R_1e^{2\sigma_1}, \qquad \text{and} \qquad r_d = R_1,$$

$$r_{uu} = R_2e^{4\sigma_2}, \qquad r_{ud} = R_2e^{2\sigma_2}, \qquad \text{and} \qquad r_{dd} = R_2,$$

$$r_{uuu} = R_3e^{6\sigma_3}, \qquad r_{uud} = R_3e^{4\sigma_3}, \qquad r_{udd} = R_3e^{2\sigma_3}, \qquad \text{and} \qquad r_{ddd} = R_3,$$

and so on, as summarized in Figure 36:

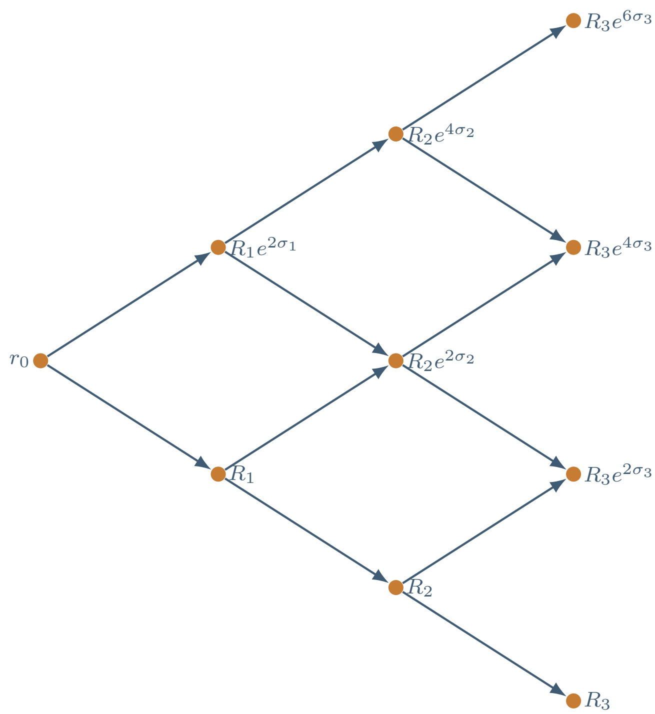

Figure 36

Each node rate is an **effective interest rate**. For example,

$$P(0,1) = \frac{1}{1+r_0} \qquad \text{and} \qquad P(r_u,1,2) = \frac{1}{1+R_1e^{2\sigma_1}}.$$

The yield to maturity is defined in terms of effective rates by

$$y(r,t,T) = P(r,t,T)^{-1/(T-t)} - 1.$$

The risk-neutral probability of going up or down for each period is 0.5. The Black–Derman–Toy parameters (R_i, σ_i) are chosen to fit both observed bond prices and the observed volatilities of yields. In particular, if σ denotes the one-year-ahead yield volatility in this binomial convention, then calibration uses

$$e^{2\sigma} = \frac{y(r_u,1,T)}{y(r_d,1,T)} = \frac{P(r_u,1,T)^{-1/(T-1)} - 1}{P(r_d,1,T)^{-1/(T-1)} - 1}.$$

Example 23.3 Consider an interest rate cap on notional 100 with strike rate 12%. Calculate the time-0 value of a 3-year cap under the Black–Derman–Toy model calibrated to the zero-coupon bond prices

$$r_0 = 10\%,$$

$$r_1^u = 13.22\% \qquad \text{and} \qquad r_1^d = 10.82\%,$$

$$r_2^{uu} = 20.17\%, \qquad r_2^{ud} = 13.66\%, \qquad \text{and} \qquad r_2^{dd} = 9.25\%.$$

Solution. The cap payoff at the end of year i is

$$100(r(i) - 0.12)_+, \quad i = 0, 1, 2.$$

The present value of the year-0 payment is

$$\mathrm{PV}_{\text{year } 0} = \frac{100(0.10 - 0.12)_+}{1.1} = 0.$$

For year 1, only the up state contributes, so

$$\mathrm{PV}_{\text{year } 1} = \frac{0.5 \cdot 100(0.1322 - 0.12)}{1.1322 \cdot 1.1} \approx 0.4898.$$

For year 2,

$$\begin{aligned}\mathrm{PV}_{\text{year } 2} = &\frac{0.25 \cdot 100(0.2017 - 0.12)}{1.2017 \cdot 1.1322 \cdot 1.1} + \frac{0.25 \cdot 100(0.1366 - 0.12)}{1.1366 \cdot 1.1322 \cdot 1.1} \\ &+ \frac{0.25 \cdot 100(0.1366 - 0.12)}{1.1366 \cdot 1.1082 \cdot 1.1} \approx 1.9574.\end{aligned}$$

Hence the time-0 cap value is

$$0 + 0.4898 + 1.9574 \approx 2.4472.$$

□

Example 23.4 Suppose the market zero-coupon bond data are

$$P(0,1) = 0.9091 \qquad \text{and} \qquad P(0,2) = 0.8116,$$

and the volatility of the one-year-ahead one-period yield is 10%. Construct a 2-period Black–Derman–Toy tree consistent with these data.

Solution. Since

$$0.9091 = \frac{1}{1+r_0},$$

we obtain

$$r_0 = 0.1.$$

Next set

$$r_u = R_1 e^{2\sigma_1} \qquad \text{and} \qquad r_d = R_1.$$

The two-year bond price condition gives

$$0.8116 = 0.5 \frac{1}{1.1} \frac{1}{1+R_1 e^{2\sigma_1}} + 0.5 \frac{1}{1.1} \frac{1}{1+R_1}.$$

The condition on bond price volatility gives

$$e^{2\cdot 0.1} = \frac{y(r_u,1,2)}{y(r_d,1,2)} = \frac{\left(\frac{1}{1+R_1 e^{2\sigma_1}}\right)^{-1} - 1}{\left(\frac{1}{1+R_1}\right)^{-1} - 1} = e^{2\sigma_1}.$$

Therefore

$$\sigma_1 = 0.1 \qquad \text{and} \qquad R_1 = 0.1082.$$

So the calibrated two-period tree in Figure 37 is

$$r_u = 13.22\%, \qquad r_0 = 10\%, \qquad \text{and} \qquad r_d = 10.82\%.$$

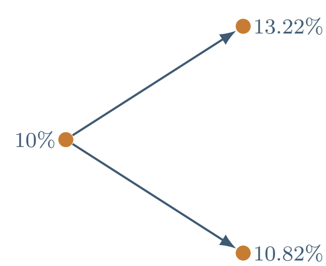

FIGURE 37

□

Appendix: Lecture and Tutorial Index