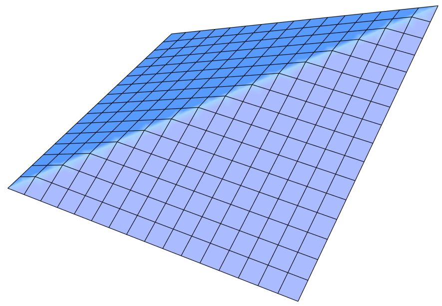

AMATH 350: Differential Equations for Business and Economics

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MWF 1:30–2:20PM in MC 4007

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Contents

Lecture 1 — Monday, January 05, 2015

1. First-Order Differential Equations

Basic Concepts and Terminology

A **differential equation** is an equation relating an unknown function to one or more of its derivatives. For example, $\frac{\mathrm{d}y}{\mathrm{d}x} = y$ has solutions $y = e^x$ and $y = 2e^x$. More generally, $y = Ce^x$, where $C \in \mathbb{R}$, is a family of solutions.

Differential equations typically have multiple solutions. The **general solution** is an expression which represents all, or nearly all, of the solutions. For instance, for $\frac{\mathrm{d}y}{\mathrm{d}x} = 2y + 1$, the general solution is $y = Ce^{2x} - 1/2$.

Occasionally, there will be **singular solutions**, which do not fit exactly into the general solution.

The **order** of a differential equation is the order of the highest derivative present. For example, $y^{(3)} + 3y' - 2y = 0$ is third order.

In multivariable calculus, we encounter **partial differential equations**, which involve partial derivatives. The **heat equation** is

$$u_t = u_{xx},$$

where $u = u(x, t)$. This is a second-order partial differential equation.

In applications, we usually look for one particular solution. To identify it, we need more information about the solution. Typically, for an ordinary differential equation, we are given values such as $y(x_0)$, $y'(x_0)$, $y''(x_0)$, and so on. An nth-order problem needs n initial conditions. Such a problem, together with its initial conditions, is an **initial value problem**. For example, consider $\frac{\mathrm{d}y}{\mathrm{d}x} = 2y$ with $y(0) = 5$. The general solution is $y = Ce^{2x}$, and substituting the initial condition gives $C = 5$. Thus the solution is $y = 5e^{2x}$.

An important distinction is that a **linear equation** is one which can be written in the form

$$f_n(x)y^{(n)} + \cdots + f_2(x)y'' + f_1(x)y' + f_0(x)y = g(x).$$

In particular, the unknown function and its derivatives occur only to the first power and are not multiplied together. For example, $y'' + x^2y' + xy = e^x$ is linear, whereas $y' + \sin(y) = 0$, $(y')^2 = y$, and $yy' = x$ are nonlinear.

We now turn to first-order equations. In general, these have the form $F(x, y, y') = 0$. We will consider only equations which can be solved for y'; that is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y).$$

Theorem 1.1 (Existence and uniqueness theorem) The initial value problem $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$, $y(x_0) = y_0$ has a unique solution, defined in some neighbourhood of x_0, if $f(x, y)$ and $f_y(x, y)$ are continuous within some neighbourhood of (x_0, y_0).

We skip the proof.

The Picard–Lindelöf theorem reaches the same conclusion when f is continuous in x and locally Lipschitz in y. This is weaker than requiring f_y to be continuous, but the derivative condition is often easier to check.

Lecture 2 — Wednesday, January 07, 2015

By the first-order existence and uniqueness theorem, the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$, $y(x_0) = y_0$ has a unique solution, defined on some interval containing x_0, if $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous in some neighbourhood of (x_0, y_0).

Example 2.1 $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2y + e^{xy}$ with $y(12) = 5$ has a unique solution because $f(x, y) = x^2y + e^{xy}$ and $f_y(x, y) = x^2 + xe^{xy}$ are continuous everywhere. Similarly, $y' = y/x$ has a unique solution for any initial condition with $x_0 \neq 0$ because $f(x, y) = y/x$ and $f_y(x, y) = 1/x$ are continuous when $x \neq 0$.

Example 2.2 For $y' = (xy)^{2/3}$, the function $f(x, y) = (xy)^{2/3}$ is continuous everywhere. Its partial derivative with respect to y fails to exist at $(x, 0)$ whenever $x \neq 0$; although $f_y(0, 0)$ itself exists, f_y is therefore not continuous on any neighbourhood of $(0, 0)$. Consequently, the problem has a unique solution for $y(0) = 1$, but Theorem 1.1 gives no conclusion about existence or uniqueness for the initial condition $y(0) = 0$.

Corollary 2.3 Different solution curves for a first-order ordinary differential equation can never intersect at points where the conditions of Theorem 1.1 are satisfied.

Separable Equations

Suppose $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$. If $f(x, y)$ can be factored so that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y)$, then the differential equation is called **separable**. We can solve it by writing

$$\frac{1}{h(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x) \implies \int \frac{1}{h(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int g(x)\,\mathrm{d}x.$$

This uses substitution: if $y = F(x)$, then $\mathrm{d}y = F'(x)\,\mathrm{d}x = \dfrac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x$. After integrating, we solve for y, if possible.

Example 2.4 Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^{x+y}$ with $y(0) = 0$.

Solution. Separate the variables:

$$\begin{aligned} \frac{\mathrm{d}y}{e^y} &= e^x\,\mathrm{d}x, \\ \int e^{-y}\,\mathrm{d}y &= \int e^x\,\mathrm{d}x, \\ -e^{-y} + C_1 &= e^x + C_2. \end{aligned}$$

Thus $e^{-y} = -e^x - C_3$, and hence

$$y = -\log(-e^x - C_3).$$

The condition $y(0) = 0$ gives $C_3 = -2$. Therefore the explicit solution is

$$y = -\log(2 - e^x) = \log\left(\frac{1}{2 - e^x}\right).$$

The existence and uniqueness theorem does not determine the domain of the solution; in this case the domain is $(-\infty, \log 2)$. $\square$

One danger of separation of variables is that certain **equilibrium solutions** may be lost when we divide by a factor involving y.

Example 2.5 Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -4xy^2$.

Solution. If $y \neq 0$, then separation gives $\mathrm{d}y/y^2 = -4x\,\mathrm{d}x$. Also, $y \equiv 0$ is a solution by inspection. Integrating the separated equation gives

$$\int y^{-2}\,\mathrm{d}y = -4\int x\,\mathrm{d}x \implies -\frac{1}{y} = -2x^2 + C \implies y = \frac{1}{2x^2 - C}.$$

The full solution set is $y \in \{1/(2x^2 - C), 0\}$. $\square$

Example 2.6 Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = (xy)^{2/3}$.

Solution. If $y \neq 0$, then

$$\int y^{-2/3}\,\mathrm{d}y = \int x^{2/3}\,\mathrm{d}x.$$

Also, $y = 0$ is a solution. For the separated equation,

$$3y^{1/3} = \frac{3}{5}x^{5/3} + C_1 \implies y = \left(\frac{1}{5}x^{5/3} + C\right)^3.$$

Thus the solutions obtained in this way are $y = \left(x^{5/3}/5 + C\right)^3$ and $y = 0$. Because uniqueness fails along $y = 0$, we can also obtain solutions by joining one of the nonzero branches to the zero solution where the branch reaches the axis. $\square$

First-Order Linear Ordinary Differential Equations

A first-order linear equation has the form

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = f(x).$$

Example 2.7 Solve $x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = e^x$.

Solution. The left-hand side is $\dfrac{\mathrm{d}}{\mathrm{d}x}(xy)$, so $\dfrac{\mathrm{d}}{\mathrm{d}x}(xy) = e^x$. Hence

$$xy = \int e^x \,\mathrm{d}x = e^x + C \implies y = \frac{e^x}{x} + \frac{C}{x}.$$

This is the general solution on any interval not containing 0. A solution that extends through $x = 0$ must have $C = -1$ on both sides; in that case $(e^x - 1)/x$ has the removable value $y(0) = 1$. In other words, this calculation is undoing the product rule. $\square$

Lecture 3 — Friday, January 09, 2015

Consider the equation

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = f(x).$$

If $a_0(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}a_1(x)$, then the left-hand side is an "exact differential", and we can write

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(a_1(x)y(x)\big) = f(x).$$

Even if $a_0 \neq \dfrac{\mathrm{d}a_1}{\mathrm{d}x}$, we can create this structure. First write the equation in standard form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x).$$

Then multiply through by an unknown function $\mu(x)$, called an **integrating factor**:

$$\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \mu(x)p(x)y = \mu(x)q(x).$$

We want the left-hand side to be the derivative of $\mu(x)y(x)$, so we require

$$\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \mu(x)p(x)y = \mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + y\frac{\mathrm{d}\mu}{\mathrm{d}x}.$$

This requires $\dfrac{\mathrm{d}\mu}{\mathrm{d}x} = \mu(x)p(x)$, which is separable:

$$\int \frac{\mathrm{d}\mu}{\mu} = \int p(x)\,\mathrm{d}x \implies \log|\mu| = \int p(x)\,\mathrm{d}x \implies \mu(x) = C\exp\left(\int p(x)\,\mathrm{d}x\right), \qquad C \neq 0.$$

Any nonzero choice of C works, so we usually take $C = 1$:

$$\mu(x) = \exp\left(\int p(x)\,\mathrm{d}x\right).$$

The differential equation is now

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(\mu(x)y(x)\big) = \mu(x)q(x),$$

so

$$\mu y = \int \mu q\,\mathrm{d}x.$$

Hence

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)q(x)\,\mathrm{d}x \qquad \text{and} \qquad \mu(x) = \exp\left(\int p(x)\,\mathrm{d}x\right).$$

Most simpletons remember only the formula for μ and the reason it is used.

The integrating-factor mechanism is summarized in Figure 1: multiplication by μ is chosen precisely so that the product rule collapses the left-hand side to one derivative.

Example 3.1 Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} + xy = x$.

Solution. This equation is linear and separable, and it is already in standard form. An integrating factor is

$$\mu(x) = \exp\left(\int x\,\mathrm{d}x\right) = \exp\left(\frac{1}{2}x^2\right).$$

$$\begin{tikzcd} y' + p(x)y = q(x) \arrow[r, "\times\mu"] & \mu y' + \mu p y = \mu q \arrow[d, "\mu' = \mu p"] \\ y = \dfrac{1}{\mu}\int \mu q \,\mathrm{d}x & \dfrac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu q \arrow[l, "\text{integrate}"] \end{tikzcd}$$

$\mu(x) = \exp\left(\int p(x)\,\mathrm{d}x\right)$ makes the product rule do the work

FIGURE 1

Multiplying by this integrating factor gives

$$\begin{aligned} e^{x^2/2}\frac{\mathrm{d}y}{\mathrm{d}x} + xe^{x^2/2}y &= xe^{x^2/2}, \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^2/2}y\right) &= xe^{x^2/2}, \\ e^{x^2/2}y &= \int xe^{x^2/2}\,\mathrm{d}x = e^{x^2/2} + C. \end{aligned}$$

Therefore

$$y(x) = 1 + Ce^{-x^2/2}.$$

□

Example 3.2 Solve $xy' - 2y = x^3 \cos x$.

Solution. On an interval where $x \neq 0$, convert to standard form:

$$y' - \frac{2}{x}y = x^2 \cos x.$$

An integrating factor is

$$\mu(x) = \exp\left(\int -2/x \,\mathrm{d}x\right) = e^{-2\log|x|} = x^{-2}.$$

Multiplying the standard-form equation by $1/x^2$ gives

$$\frac{1}{x^2}y' - \frac{2}{x^3}y = \cos x \implies \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) = \cos x.$$

Thus

$$\frac{y}{x^2} = \int \cos x \, \mathrm{d}x = \sin x + C,$$

and therefore

$$y(x) = x^2 \sin x + Cx^2 = x^2(\sin x + C).$$

This formula gives the solutions on intervals contained in $x > 0$ or $x < 0$. Solutions of the original equation can also pass through $x = 0$: they have $y(0) = 0$, and the constants on the two sides of 0 may be chosen independently. □

Example 3.3 Solve the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}x} + y = \sqrt{1 + \cos^2 x}$ subject to $y(1) = 4$.

Solution. An integrating factor is

$$\mu(x) = \exp\left(\int 1 \, \mathrm{d}x\right) = e^x.$$

Multiplying through by e^x gives

$$e^x \frac{\mathrm{d}y}{\mathrm{d}x} + e^x y = e^x \sqrt{1 + \cos^2 x} = \frac{\mathrm{d}}{\mathrm{d}x}(e^x y).$$

Therefore

$$e^x y = \int e^x \sqrt{1 + \cos^2 x} \, \mathrm{d}x.$$

The integral cannot be evaluated in elementary terms, but by the fundamental theorem of calculus we can write it as

$$\int_1^x e^t \sqrt{1 + \cos^2 t} \, \mathrm{d}t + C,$$

so

$$e^x y = \int_1^x e^t \sqrt{1 + \cos^2 t} \, \mathrm{d}t + C,$$

and therefore

$$y(x) = e^{-x} \int_1^x e^t \sqrt{1 + \cos^2 t} \, \mathrm{d}t + Ce^{-x}.$$

Now use the initial condition:

$$y(1) = 4 \implies 4 = Ce^{-1} \implies C = 4e.$$

Thus

$$y = e^{-x}\int_1^x e^t\sqrt{1+\cos^2 t}\,\mathrm{d}t + 4e^{1-x},$$

which is a totally legitimate function of x. $\square$

Lecture 4 — Monday, January 12, 2015

Direction Fields and Solution Families

It is possible to determine information about the graphs of solutions even without finding the solutions explicitly.

Example 4.1 Analyze and solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 4$.

Solution. We can see immediately that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ if and only if $y = \pm 2$. These equilibrium solutions are the only places where there are critical points. We can also see that $y' > 0$ when $|y| > 2$ and $y' < 0$ when $|y| < 2$. In general, we can determine the slope of the solution passing through any point and generate the **direction field** in Figure 2.

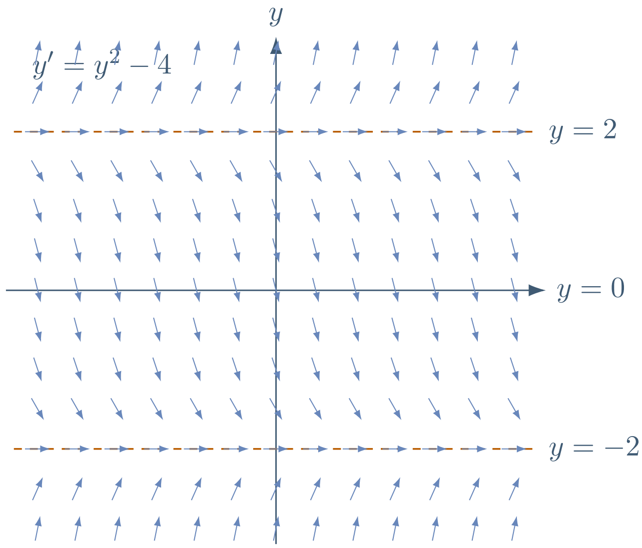

FIGURE 2

The non-equilibrium solution curves appear to have inflection points where $y = 0$. We can verify this:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 4 \implies \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2y(y^2 - 4) = 2y^3 - 8y.$$

The equilibrium solutions have zero curvature everywhere. Along any non-equilibrium solution, $y'' = 0$ when $y = 0$, and its sign changes there.

Of course, we can also solve this differential equation, since it is separable:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 4 \implies \int \frac{\mathrm{d}y}{y^2 - 4} = \int \mathrm{d}x \implies y = 2\left(\frac{C - e^{-4x}}{C + e^{-4x}}\right),$$

together with the equilibrium solutions $y = \pm 2$. This tells us something new: there are vertical asymptotes if $C < 0$, where $C + e^{-4x} = 0$. They occur at $x = -\log(-C)/4$. $\square$

If we have both the differential equation and its solution, then we can usually produce accurate sketches easily.

Example 4.2 Analyze $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y - x^2$, given that the solution is $y = Ce^x + x^2 + 2x + 2$.

Solution. From the solution, we can see an "exceptional" solution:

$$y = x^2 + 2x + 2,$$

which is the only parabolic solution. All other solutions move toward the parabola as $x \to -\infty$ and away from it as $x \to \infty$.

For critical points,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \implies y = x^2.$$

This curve is called the **horizontal isocline**. For inflection points,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y - x^2 \implies \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}y}{\mathrm{d}x} - 2x = y - x^2 - 2x.$$

Hence

$$y'' = 0 \implies y = x^2 + 2x = (x+1)^2 - 1.$$

The resulting solution family and its two isoclines are shown in Figure 3. □

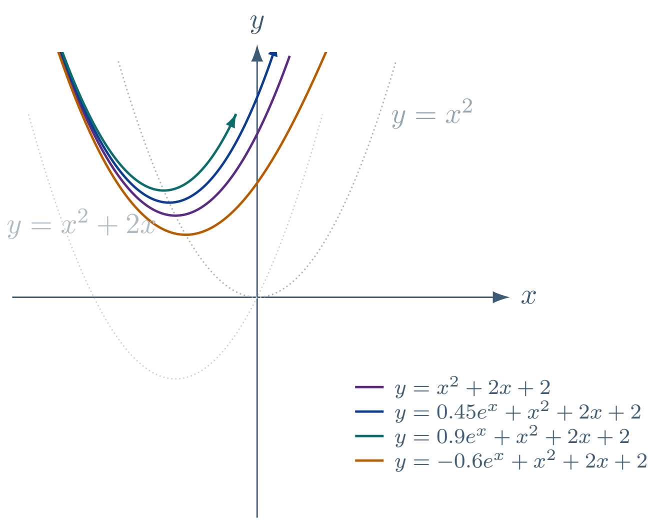

Figure 3

We use the following algorithm for sketching solutions:

Step 1. Solve the differential equation.

Step 2. Sketch the exceptional solutions.

Step 3. Consider limits of typical solutions.

Step 4. Find the horizontal isocline.

Step 5. Sketch the solutions. By Corollary 2.3, solution curves do not intersect where the hypotheses of Theorem 1.1 hold, so points on the isocline can be used as starting points.

Example 4.3 Solve and sketch $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2y^2$.

Solution. The constant function $y \equiv 0$ is a solution by inspection. If $y \neq 0$, then the equation is separable, and

$$\frac{\mathrm{d}y}{y^2} = 3x^2 \,\mathrm{d}x.$$

Integrating gives

$$\int y^{-2}\,\mathrm{d}y = \int 3x^2\,\mathrm{d}x \implies -\frac{1}{y} = x^3 + C_0.$$

Thus the nonzero solutions have the form

$$y = \frac{1}{C - x^3},$$

where $C = -C_0$, on any interval that does not contain the vertical asymptote $x = \sqrt[3]{C}$. The case $C = 0$ gives the curve $y = -1/x^3$. Therefore the full set of solutions consists of $y \equiv 0$ together with the nonzero branches $y = 1/(C - x^3)$.

The horizontal isocline is determined by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \implies x = 0 \qquad \text{or} \qquad y = 0,$$

and $y' > 0$ everywhere else. Hence every nonconstant solution is increasing on its interval of definition, with a horizontal tangent at any point where it is defined on the line $x = 0$. Representative members of the solution family are plotted in Figure 4. □

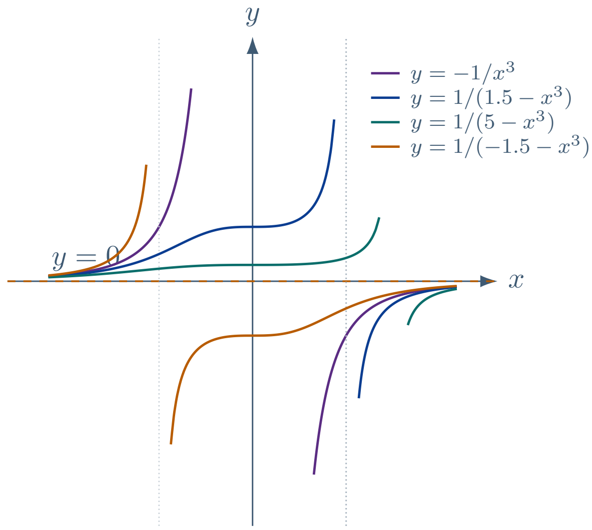

Figure 4

Lecture 5 — Wednesday, January 14, 2015

Mathematical Modelling With Differential Equations

Suppose we wish to predict future values of a population $P(t)$. To find $P(t)$, we make simple assumptions about its rate of change. In the **Malthusian model**, the rate of change of P is proportional to P itself:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP,$$

where r is the **growth rate**, which is taken to be constant. By inspection, $P(t) = Ke^{rt}$. If $P(0) = P_0$, then $K = P_0$, and therefore $P(t) = P_0e^{rt}$. To determine r, we would need to know the population at one positive time.

One way to improve this model is to suppose that resources are limited, so there is a maximum sustainable population $K > 0$, called the **carrying capacity**. For a positive intrinsic growth rate r, the resulting **logistic equation** is

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP\left(1 - \frac{P}{K}\right).$$

This gives $\dfrac{\mathrm{d}P}{\mathrm{d}t} \approx rP$ when $P \ll K$, but $\dfrac{\mathrm{d}P}{\mathrm{d}t} = 0$ when $P = K$. The equation is separable:

$$\begin{aligned}
\frac{\mathrm{d}P}{P(1-P/K)} &= r\,\mathrm{d}t \implies \int \frac{K\,\mathrm{d}P}{P(K-P)} = \int r\,\mathrm{d}t,\\
\frac{K}{P(K-P)} &= \frac{A}{P} + \frac{B}{K-P} \implies \int \left(\frac{1}{P} + \frac{1}{K-P}\right)\mathrm{d}P = rt + C,\\
\log(P) - \log|K-P| &= rt + C_1 \implies \log\left|\frac{P}{K-P}\right| = rt + C_1,\\
\frac{P}{K-P} &= C_2 e^{rt} \implies P(t) = \frac{C_2 K e^{rt}}{1 + C_2 e^{rt}} = \frac{K}{Ce^{-rt} + 1}.
\end{aligned}$$

For $P_0 \neq 0$, the initial condition $P(0) = P_0$ gives $P_0 = K/(C+1)$, so $C = (K - P_0)/P_0$. The division used above excludes $P = 0$ and $P = K$; the former must be added separately, while the latter is recovered by taking $C = 0$. Several representative logistic solution curves appear in Figure 5.

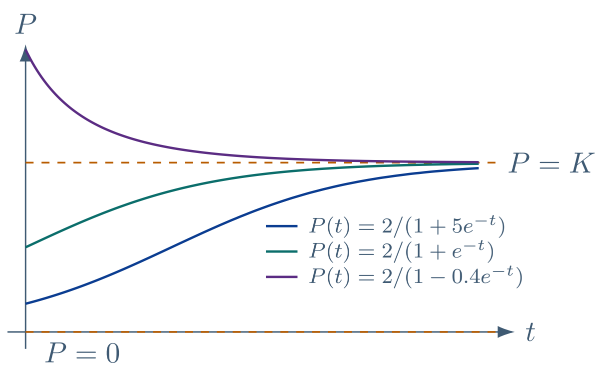

Figure 5

Example 5.1 Suppose a savings account earns interest at a continuous force of 5% per year. The account is opened with \$1000, and money is deposited continuously at a rate of \$365 per year. What is the account balance after 10 years, assuming the financial system (and indeed, civilization itself) is not first wiped out by a nuclear apocalypse?

Solution. Let $y(t)$ be the account balance after t years. The rate of change of the balance is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \text{interest rate} + \text{deposits} = 0.05y + 365.$$

This is a first-order linear differential equation. Rewriting it as $y' - y/20 = 365$, an integrating factor is

$$\mu(t) = \exp\left(-\int (1/20)\,\mathrm{d}t\right) = e^{-t/20}.$$

Multiplying through gives

$$\begin{aligned}
e^{-t/20}y' - \frac{1}{20}e^{-t/20}y &= 365e^{-t/20},\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t/20}y\right) &= 365e^{-t/20},\\
e^{-t/20}y &= -7300e^{-t/20} + C,\\
y(t) &= Ce^{t/20} - 7300.
\end{aligned}$$

The initial condition $y(0) = 1000$ gives $C = 8300$. Thus

$$y(t) = 8300e^{t/20} - 7300.$$

Finally,

$$y(10) = 8300\sqrt{e} - 7300 \approx \$6384.39.$$

For comparison, if the quoted 5% is a nominal annual rate compounded daily and \$1 is deposited at the end of each day, the exact discrete balance is approximately \$6383.92, less than \$0.50 below the continuous-model value. □

Lecture 6 — Friday, January 16, 2015

Substitutions

If we encounter first-order differential equations which are neither linear nor separable, we may be able to convert them into linear or separable equations by a change of variable. We will look at three such classes of equations.

1. For equations of the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} = F(ax + by)$, we try letting $u = ax + by$, thinking of u as a function of x.

Example 6.1 Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{x+y} - 1$ with $y(1) = 0$.

Solution. Let $u = x + y$, so $y = u - x$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}u}{\mathrm{d}x} - 1$. This gives

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x} - 1 = \sqrt{u} - 1 &\implies \frac{\mathrm{d}u}{\mathrm{d}x} = \sqrt{u}, \\
&\implies \int \frac{1}{\sqrt{u}}\,\mathrm{d}u = \int \mathrm{d}x, \\
&\implies 2\sqrt{u} = x + C_1, \\
&\implies 2\sqrt{x+y} = x + C_1, \\
&\implies x + y = \left(\frac{x + C_1}{2}\right)^2, \\
&\implies y = \frac{1}{4}(x + C)^2 - x.
\end{aligned}$$

Enforce the initial condition:

$$\begin{aligned}
y(1) = 0 &\implies 0 = \frac{1}{4}(1 + C)^2 - 1, \\
&\implies 0 = \frac{1}{4}C^2 + \frac{1}{2}C - \frac{3}{4}, \\
&\implies C = \frac{-2 \pm \sqrt{4 + 12}}{2}, \\
&\implies C = 1 \text{ or } C = -3.
\end{aligned}$$

By Theorem 1.1, at most one of these candidates can be the solution near $x = 1$. If $y = (x + C)^2/4 - x$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(x + C) - 1 \qquad \text{and} \qquad \sqrt{x+y} - 1 = \sqrt{\frac{1}{4}(x + C)^2} - 1.$$

These are equal when $x + C \geq 0$. At $x = 1$, we need $1 + C > 0$, so $C = 1$, not $C = -3$. Therefore

$$y = \frac{1}{4}(x + 1)^2 - x.$$

This formula satisfies the differential equation for $x \geq -1$. At $x = -1$ it joins smoothly to

$y = -x$, giving the global extension

$$y(x) = \begin{cases} -x, & x \leq -1, \\ (x+1)^2/4 - x, & x \geq -1. \end{cases}$$

$\square$

2. For equations of the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} = F(y/x)$, which are traditionally called homogeneous equations, we try $u = y/x$. Then $y = ux$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = u + x\dfrac{\mathrm{d}u}{\mathrm{d}x}$.

Example 6.2 Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = (x^2 - y^2)/(3yx)$.

Solution. This equation is neither linear nor separable, but we can write it as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}\left(\frac{x}{y} - \frac{y}{x}\right).$$

Let $u = y/x$. Then

$$\begin{aligned} u + x\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{3}\left(\frac{1}{u} - u\right) &\implies x\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{3u} - \frac{4}{3}u, \\ &\implies \int \frac{\mathrm{d}u}{\frac{1}{3u} - \frac{4}{3}u} = \int \frac{\mathrm{d}x}{x}, \\ &\implies 3\int \frac{u\,\mathrm{d}u}{1 - 4u^2} = \int \frac{\mathrm{d}x}{x}. \end{aligned}$$

With $t = 1 - 4u^2$, this becomes

$$\begin{aligned} -\frac{3}{8}\int \frac{\mathrm{d}t}{t} = \int \frac{\mathrm{d}x}{x} &\implies -\frac{3}{8}\log|t| = \log|x| + C_1, \\ &\implies -3\log\left|1 - 4u^2\right| = 8\log|x| + C_2, \\ &\implies (1 - 4u^2)^{-3} = C_3 x^8, \\ &\implies 1 - 4u^2 = C_4 x^{-8/3}, \\ &\implies u^2 = \frac{1}{4} + C_5 x^{-8/3}, \\ &\implies y^2 = \frac{1}{4}x^2 + Cx^{-2/3}, \\ &\implies y = \pm\sqrt{\frac{1}{4}x^2 + Cx^{-2/3}}. \end{aligned}$$

These real solutions are understood on intervals not containing 0 on which the radicand is positive and $y \neq 0$. The case $C = 0$ recovers the two constant-ratio solutions $y/x = \pm 1/2$ on each such interval. $\square$

3. A **Bernoulli equation** has the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)y^n$, where n is a constant and $n \neq 1$. These can be converted into linear equations by letting $v = y^{1-n}$.

Example 6.3 Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} - 5y = -(5/2)xy^3$.

Solution. Let $v = y^{-2}$. Then

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}x} = -2y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} &\implies \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}y^3\frac{\mathrm{d}v}{\mathrm{d}x}, \\
&\implies -\frac{1}{2}y^3\frac{\mathrm{d}v}{\mathrm{d}x} - 5y = -\frac{5}{2}xy^3, \\
&\implies -\frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}x} - 5v = -\frac{5}{2}x, \\
&\implies \frac{\mathrm{d}v}{\mathrm{d}x} + 10v = 5x.
\end{aligned}$$

An integrating factor is

$$\mu(x) = \exp\left(\int 10\,\mathrm{d}x\right) = e^{10x}.$$

Then

$$\begin{aligned}
e^{10x}v' + 10ve^{10x} &= 5xe^{10x}, \\
\frac{\mathrm{d}}{\mathrm{d}x}(ve^{10x}) &= 5xe^{10x}, \\
ve^{10x} &= \int 5xe^{10x}\,\mathrm{d}x \\
&= \frac{1}{2}xe^{10x} - \frac{1}{20}e^{10x} + C, \\
v &= \frac{1}{2}x - \frac{1}{20} + Ce^{-10x}, \\
y &= \pm\left(\frac{1}{2}x - \frac{1}{20} + Ce^{-10x}\right)^{-1/2} \qquad \text{or} \qquad y = 0.
\end{aligned}$$

The nonzero real solutions are defined on intervals where the expression in parentheses is strictly positive. $\square$

Lecture 7 — Monday, January 19, 2015

2. Higher-Order Linear Ordinary Differential Equations

We begin with the basic structure of higher-order linear differential equations.

An nth-order ordinary differential equation is **linear** if it has the form

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = F(x).$$

If $F(x) \equiv 0$, the equation is **homogeneous**. Otherwise, it is **inhomogeneous**. The function $F(x)$ is called the **inhomogeneous term** or the **forcing term**. If $a_n, \ldots, a_0$ are constants, then the equation is a **constant-coefficient linear ordinary differential equation**.

Theorem 7.1 (Existence and uniqueness theorem for IVPs for linear ODEs) Consider the initial value problem consisting of the equation

$$a_n(x)y^{(n)}(x) + \cdots + a_0(x)y(x) = F(x)$$

and prescribed values for $y(x_0)$, $y'(x_0)$, $\ldots$, $y^{(n-1)}(x_0)$. If there is an open interval I containing x_0 such that

1. the functions $a_0(x), \ldots, a_n(x)$ are continuous on I;
2. $a_n(x) \neq 0$ on I; and
3. $F(x)$ is continuous on I.

then the initial value problem has a unique solution on I.

We do not even contemplate the idea of considering the thought of proving this.

Example 7.2 The initial value problem

$$(x^2 - 1)y'' + xy' - y = \sin x, \qquad y(0) = 1, \qquad \text{and} \qquad y'(0) = 0$$

has a unique solution on $(-1, 1)$ by Theorem 7.1.

Notation 7.3 An **operator** is a transformation which maps functions to functions. The **differential operator** transforms f to f'. We may write this as

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] = f'(x),$$

or more concisely as $Df(x) = f'(x)$.

For example, $D(x^2 + 7) = 2x$. We can also define an **identity operator** by $If(x) = f(x)$. Operators can be combined: letting $D^2 := D \circ D$, we get

$$(D^2 + 7I)f(x) = f''(x) + 7f(x).$$

If we give the operator $D^2 + 7I$ its own name, say Φ, then we can write $y'' + 7y = \cos x$ as $\Phi[y] = \cos x$. Similarly, we could let

$$\Phi = a_n(x)\frac{\mathrm{d}^n}{\mathrm{d}x^n} + \cdots + a_0(x),$$

and then use $\Phi[y] = F(x)$ to represent a general nth-order linear ordinary differential equation.

Theorem 7.4 (Principle of superposition: first version) Let Φ be a linear differential operator. Let $y_1(x)$ be a solution to $\Phi[y] = F_1(x)$, and let $y_2(x)$ be a solution to $\Phi[y] = F_2(x)$. Then $y_1(x) + y_2(x)$ is a solution to $\Phi[y] = F_1(x) + F_2(x)$.

Proof. By linearity,

$$\Phi[y_1 + y_2] = \Phi[y_1] + \Phi[y_2] = F_1 + F_2.$$

□

Remark 7.5 For clarity, the first-order version of Theorem 7.4 says that if y_1 is a solution to $y' + p(x)y = F_1(x)$ and y_2 is a solution to $y' + p(x)y = F_2(x)$, then $y_1 + y_2$ is a solution to

$$y' + p(x)y = F_1(x) + F_2(x).$$

Example 7.6 Consider $y' + 2y = 6 + 3e^x$. For $y' + 2y = 6$, the function $y_1 = 3$ is a solution, and for $y' + 2y = 3e^x$, the function $y_2 = e^x$ is a solution. Hence, by Theorem 7.4, $y = 3 + e^x$ is a solution to the original equation.

Corollary 7.7 If y_h is a solution to $\Phi[y] = 0$ and y_p is a solution to $\Phi[y] = F(x)$, then $y_h + y_p$ is also a solution to $\Phi[y] = F(x)$. y_h is called a **homogeneous solution**, and y_p is called a **particular solution**.

Example 7.8 Consider again $y' + 2y = 6 + 3e^x$. The general solution to $y' + 2y = 0$ is $y_h = Ce^{-2x}$. Since $y_p = 3 + e^x$ is a particular solution, the general solution to $y' + 2y = 6 + 3e^x$ is

$$y = Ce^{-2x} + 3 + e^x.$$

Lecture 8 — Wednesday, January 21, 2015

Recall the principle of superposition and its consequence for adding homogeneous and particular solutions.

Theorem 8.1 (Principle of superposition: second version) If y_1 and y_2 are both solutions to the homogeneous equation $\Phi[y] = 0$, then $c_1y_1 + c_2y_2$ is also a solution for any $c_1, c_2 \in \mathbb{R}$.

Proof. For the first-order case, if $y = c_1y_1 + c_2y_2$, then

$$y'(x) + p(x)y(x) = (c_1y_1 + c_2y_2)' + p(x)(c_1y_1 + c_2y_2) = 0.$$

The same calculation applies to any linear differential operator. □

Linear Independence of Functions

Definition 8.2 The functions $f_1, f_2, \ldots, f_n$ are **linearly dependent** on an interval I if there exist constants $c_1, \ldots, c_n$, not all zero, such that

$$c_1 f_1(x) + \cdots + c_n f_n(x) = 0$$

for all $x \in I$. Otherwise, they are **linearly independent**.

Usually linear dependence will be obvious for small sets of functions.

Example 8.3

1. The functions $1, x, x^2, x^3, \ldots$ are independent: every linear relation among them involves only finitely many monomials, and a polynomial that vanishes on an interval has every coefficient equal to zero.
2. The functions e^x and e^{x+5} are linearly dependent.
3. The set $e^x, e^{-x}, \sinh x$ is linearly dependent, since $\sinh x = (e^x - e^{-x})/2$.
4. The functions $\sin x$ and $\cos(x - \pi/2)$ are linearly dependent, since they are the same function.
5. The functions $\cos^{-1} x$ and $\sin^{-1} x$ are linearly independent. However, the set $\{\cos^{-1} x, \sin^{-1} x, 1\}$ is linearly dependent, since

$$\cos^{-1} x + \sin^{-1} x = \pi/2$$

for all $x \in [-1, 1]$.

A test for independence does exist.

Definition 8.4 The **Wronskian** of the functions $f_1, f_2, \ldots, f_n$ is the determinant of the $n \times n$ matrix $\mathbf{W}(x)$ whose (i, j)-entry is $f_j^{(i-1)}(x)$:

$$\mathbf{W}(x) = \begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix}.$$

We write it as $W(x)$ or $W(f_1, \ldots, f_n)$.

Theorem 8.5 If $W(x_0) \neq 0$ for some $x_0 \in I$, then $f_1, \ldots, f_n$ are linearly independent on I.

Proof. Suppose

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all $x \in I$. Differentiating this identity k times, for $k = 0, 1, \ldots, n-1$, gives

$$c_1 f_1^{(k)}(x) + c_2 f_2^{(k)}(x) + \cdots + c_n f_n^{(k)}(x) = 0.$$

Evaluating these n equations at $x = x_0$ gives the homogeneous linear system

$$\begin{bmatrix} f_1(x_0) & f_2(x_0) & \cdots & f_n(x_0) \\ f_1'(x_0) & f_2'(x_0) & \cdots & f_n'(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x_0) & f_2^{(n-1)}(x_0) & \cdots & f_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The coefficient matrix has determinant $W(x_0)$, which is nonzero by assumption. Hence the matrix is invertible, so the only solution is $c_1 = c_2 = \cdots = c_n = 0$. Therefore $f_1, \ldots, f_n$ are linearly independent on I. □

Example 8.6

$$W(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2.$$

Although this vanishes at $x = 0$, it is nonzero at every other point. In particular, $W(1) \neq 0$, so x and x^2 are linearly independent on $\mathbb{R}$ by Theorem 8.5.

The converse of Theorem 8.5 is not generally true. That is, $W(f_1, f_2) = 0$ does not imply that f_1 and f_2 are dependent without further hypotheses.

Example 8.7 Let $f(x) = x^3$ and $g(x) = |x|x^2$. Then

$$W(f, g) = \begin{vmatrix} x^3 & |x|x^2 \\ 3x^2 & 3|x|x \end{vmatrix} = 3|x|x^4 - 3|x|x^4 = 0$$

for all $x \in \mathbb{R}$. However, f and g are independent on $\mathbb{R}$, since $cf(x) + dg(x) = 0$ for all x implies $c = d = 0$.

We need one more condition.

Theorem 8.8 Let $p(x)$ and $q(x)$ be continuous on an interval I, and let $y_1(x)$ and $y_2(x)$ be solutions on I to the homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0.$$

If $W(y_1, y_2) = 0$ for some $x_0 \in I$, then y_1 and y_2 are linearly dependent on I.

Lecture 9 — Friday, January 23, 2015

Proof. Since $W(x_0) = 0$, the linear system

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has a nonzero solution $(\bar{c}_1, \bar{c}_2)$. Thus

$$\bar{c}_1 y_1(x_0) + \bar{c}_2 y_2(x_0) = 0 \qquad \text{and} \qquad \bar{c}_1 y_1'(x_0) + \bar{c}_2 y_2'(x_0) = 0.$$

Let

$$u(x) = \bar{c}_1 y_1(x) + \bar{c}_2 y_2(x).$$

Then $u(x_0) = u'(x_0) = 0$. Since p and q are continuous, the hypotheses of Theorem 7.1 hold for the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \qquad y(x_0) = 0, \qquad \text{and} \qquad y'(x_0) = 0.$$

By Theorem 8.1, the function u is a solution, and the zero function is also a solution. Uniqueness gives $u \equiv 0$ on I. Since $(\bar{c}_1, \bar{c}_2)$ is nonzero, this proves that y_1 and y_2 are linearly dependent. □

Proposition 9.1 (Abel's formula) If p and q are continuous on an interval I and y_1 and y_2 are solutions there to $y'' + p(x)y' + q(x)y = 0$, then for $x, x_0 \in I$,

$$W(y_1, y_2) = W(x_0) \exp\left(-\int_{x_0}^{x} p(t)\,\mathrm{d}t\right).$$

Proof. If

$$y_1'' + p(x)y_1' + q(x)y_1 = 0$$

and

$$y_2'' + p(x)y_2' + q(x)y_2 = 0,$$

then multiplying the first equation by y_2 and the second by y_1, and then subtracting, gives

$$(y_1''y_2 - y_1y_2'') + p(x)(y_1'y_2 - y_1y_2') = 0.$$

Thus

$$W'(x) + p(x)W(x) = 0.$$

Solving this first-order differential equation gives

$$W(x) = C\exp\left(-\int p(x)\,\mathrm{d}x\right).$$

Applying the condition at x_0 gives the stated formula. □

Theorem 9.2 Let $p(x)$ and $q(x)$ be continuous on an interval I, and consider the differential equation

$$y'' + p(x)y' + q(x)y = 0.$$

If y_1 and y_2 are two linearly independent solutions on I, then the general solution is $y = c_1y_1 + c_2y_2$. There are no singular solutions.

Proof. Let $\Psi(x)$ be a solution on I. Consider the initial value problem consisting of the differential equation together with the initial conditions $y(x_0) = \Psi(x_0)$ and $y'(x_0) = \Psi'(x_0)$. By Theorem 7.1, Ψ is the unique solution to this initial value problem.

Choose c_1 and c_2 so that

$$c_1y_1(x_0) + c_2y_2(x_0) = \Psi(x_0) \qquad \text{and} \qquad c_1y_1'(x_0) + c_2y_2'(x_0) = \Psi'(x_0).$$

This can be done because $W(x_0) \neq 0$. By Theorem 8.1, $y = c_1y_1 + c_2y_2$ also satisfies the differential equation, and by construction it satisfies the same initial conditions as Ψ. Uniqueness gives $\Psi(x) = c_1y_1(x) + c_2y_2(x)$. □

It remains to find such functions y_1 and y_2. For now, assume that the differential equation has constant coefficients:

$$y'' + ay' + by = 0.$$

We solve this using characteristic equations.

Lecture 10 — Monday, January 26, 2015

Characteristic Equations

We continue by solving homogeneous second-order linear ordinary differential equations with constant coefficients. For example, the equation $y'' - y = 0$ has general solution $y = c_1e^x + c_2e^{-x}$.

Similarly, $y'' + y = 0$ has general solution $y = c_1 \cos x + c_2 \sin x$, which can also be expressed as $y = k_1e^{ix} + k_2e^{-ix}$. This uses Euler's formula:

$$e^{ix} = \cos x + i \sin x, \qquad \cos x = \frac{1}{2}(e^{ix} + e^{-ix}), \qquad \text{and} \qquad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

Perhaps other similar equations also have exponential solutions. Consider $y'' + ay' + by = 0$, and assume that $y = e^{mx}$ is a solution for some $m \in \mathbb{C}$. Then $y' = me^{mx}$ and $y'' = m^2e^{mx}$, so

$$y'' + ay' + by = 0 \implies m^2 + am + b = 0.$$

If m is a solution to the **characteristic equation** $m^2 + am + b = 0$, then e^{mx} is a solution to $y'' + ay' + by = 0$. There are three cases to consider.

Case 1. If the characteristic equation has distinct real roots m_1 and m_2, then there are two exponential solutions, and the general solution is

$$y = c_1e^{m_1x} + c_2e^{m_2x}.$$

Case 2. If the characteristic equation has complex conjugate roots $m = \alpha \pm i\beta$, then

$$y = c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x}.$$

This is the general complex-valued solution, with $c_1, c_2 \in \mathbb{C}$. For a real-valued solution the two coefficients must be complex conjugates. Equivalently, taking real and imaginary parts gives

$$\begin{aligned}
y &= e^{\alpha x}\left[c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right] \\
&= e^{\alpha x}\left[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\right] \\
&= e^{\alpha x}\left[(c_1+c_2)\cos\beta x + i(c_1-c_2)\sin\beta x\right] \\
&= e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right],
\end{aligned}$$

where $A, B \in \mathbb{R}$.

Case 3. If the characteristic equation has a repeated root, then the first guess gives only one solution, $y_1 = e^{mx}$. To find the other, d'Alembert argued that a repeated root should not differ much from two roots that are nearly equal. Suppose the roots are m and $m+\varepsilon$. The general solution is then

$$y = c_1 e^{mx} + c_2 e^{(m+\varepsilon)x}.$$

As $\varepsilon \to 0$, most choices of c_1 and c_2 make the two solutions merge. However, if $c_1 = -1/\varepsilon$ and $c_2 = 1/\varepsilon$, then

$$y = \frac{1}{\varepsilon}e^{(m+\varepsilon)x} - \frac{1}{\varepsilon}e^{mx}.$$

Taking the limit using L'Hôpital's rule gives

$$y = e^{mx}\left(\frac{e^{\varepsilon x}-1}{\varepsilon}\right) \xrightarrow{\varepsilon\to 0} xe^{mx}.$$

This limiting argument suggests the second solution. To verify it, note that a repeated root gives the characteristic polynomial $(r-m)^2$, so the equation is $y'' - 2my' + m^2 y = 0$; direct substitution shows that xe^{mx} satisfies it. Moreover,

$$W(e^{mx}, xe^{mx}) = e^{2mx} \neq 0.$$

Thus the general solution in the case of a repeated root is

$$y = c_1 e^{mx} + c_2 x e^{mx}.$$

The three possibilities for a quadratic characteristic polynomial are pictured in Figure 6. Their geometry in the complex plane determines whether the fundamental solutions are distinct exponentials, damped oscillations, or an exponential paired with xe^{mx}.

Example 10.1 Solve the following homogeneous equations:

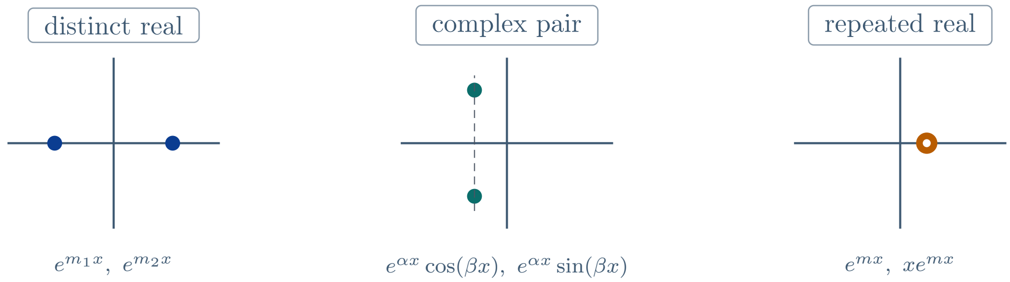

FIGURE 6

1. $y'' - y' - 2y = 0$.
2. $y'' + 2y' + 5y = 0$.
3. $y'' + 6y' + 9y = 0$.

Solution. For (1), the characteristic equation is

$$m^2 - m - 2 = 0 \implies (m-2)(m+1) = 0 \implies m = -1, 2.$$

Therefore the general solution is

$$y = c_1 e^{-x} + c_2 e^{2x}.$$

For (2), the characteristic equation is

$$m^2 + 2m + 5 = 0 \implies (m+1)^2 + 4 = 0 \implies m = -1 \pm 2i.$$

Therefore the general solution is

$$y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x).$$

For (3), the characteristic equation is

$$m^2 + 6m + 9 = (m+3)^2 = 0.$$

Thus $m = -3$ is a repeated root, so

$$y = c_1 e^{-3x} + c_2 x e^{-3x}.$$

□

Lecture 11 — Wednesday, January 28, 2015

Homogeneous Linear Equations

For a homogeneous nth-order linear ordinary differential equation with constant coefficients,

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

the characteristic equation is

$$m^n + a_{n-1}m^{n-1} + \cdots + a_1 m + a_0 = 0.$$

For every distinct real root m, the function e^{mx} is a solution. For every pair of complex conjugate roots $\alpha \pm i\beta$, the functions $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are solutions. For repeated roots, multiplying these functions by powers of x yields the additional solutions.

Example 11.1 Solve the following homogeneous equations:

1. $y^{(3)} - 4y'' + 7y' - 6y = 0.$
2. $y^{(3)} + 3y'' + 3y' + y = 0.$
3. $y^{(4)} + 2y'' + y = 0.$

Solution. For (1), the characteristic equation is

$$m^3 - 4m^2 + 7m - 6 = 0 \implies (m-2)(m^2 - 2m + 3) = 0.$$

Thus $m_1 = 2$ and $m_{2,3} = 1 \pm i\sqrt{2}$, so

$$y = c_1 e^{2x} + c_2 e^x \cos\left(\sqrt{2}x\right) + c_3 e^x \sin\left(\sqrt{2}x\right).$$

For (2), the characteristic equation is

$$m^3 + 3m^2 + 3m + 1 = 0 \implies (m+1)^3 = 0.$$

Thus $m = -1$ has multiplicity 3, so

$$y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}.$$

For (3), the characteristic equation is

$$m^4 + 2m^2 + 1 = 0 \implies (m^2+1)^2 = 0.$$

Thus $m = \pm i$ are repeated roots, so

$$y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x.$$

□

Inhomogeneous Equations and the Method of Undetermined Coefficients

By Corollary 7.7, if we can find one particular solution to $\Phi[y] = F(x)$ and add it to the general solution of $\Phi[y] = 0$, then we get solutions to $\Phi[y] = F(x)$. In fact, we get all of them: if y is any solution and y_p is one particular solution, then linearity gives $\Phi[y - y_p] = 0$. The remaining problem is how to find a particular solution. If $F(x)$ is of a certain type, we can make an educated guess.

Example 11.2 Solve $y'' + y' - 6y = e^{4x}$.

Solution. We make an educated guess that a particular solution has the form $y_p = Ae^{4x}$ for some $A \in \mathbb{R}$. Then $y_p' = 4Ae^{4x}$ and $y_p'' = 16Ae^{4x}$. Substituting gives

$$16Ae^{4x} + 4Ae^{4x} - 6Ae^{4x} = e^{4x},$$

which holds when $14A = 1$. Thus $A = 1/14$, and $y_p = e^{4x}/14$ is a particular solution.

For the homogeneous equation $y'' + y' - 6y = 0$, the characteristic equation is $m^2 + m - 6 = 0$, so $y_h = c_1 e^{-3x} + c_2 e^{2x}$. Therefore the original equation has general solution

$$y = c_1 e^{-3x} + c_2 e^{2x} + \frac{1}{14} e^{4x}.$$

□

Example 11.3 Find a particular solution to $y'' + y' - 4y = \cos 2x$.

Solution. We make the inspired guess that a particular solution has the form $y = A\cos 2x + B\sin 2x$. Then

$$y' = -2A\sin 2x + 2B\cos 2x \qquad \text{and} \qquad y'' = -4A\cos 2x - 4B\sin 2x.$$

Substituting gives

$$(-8A + 2B)\cos 2x + (-2A - 8B)\sin 2x = \cos 2x.$$

Comparing coefficients gives

$$-8A + 2B = 1 \qquad \text{and} \qquad -2A - 8B = 0.$$

Thus $A = -2/17$ and $B = 1/34$, so

$$y_p = \frac{1}{34}\sin 2x - \frac{2}{17}\cos 2x.$$

□

Example 11.4 Find a particular solution to $y'' + y' - 6y = 6x^2$.

Solution. Guess $y = Ax^2 + Bx + C$. Then $y' = 2Ax + B$ and $y'' = 2A$. Substituting gives

$$2A + 2Ax + B - 6Ax^2 - 6Bx - 6C = 6x^2.$$

Comparing coefficients gives

$$-6A = 6, \qquad 2A - 6B = 0, \qquad \text{and} \qquad 2A + B - 6C = 0.$$

Thus $A = -1$, $B = -1/3$, and $C = -7/18$, so

$$y_p = -x^2 - \frac{x}{3} - \frac{7}{18}.$$

□

The following table summarizes the method of undetermined coefficients for second-order equations. The trial function is the form of the particular solution we guess, and the inhomogeneous term is the right-hand side of the equation. For example, if $F(x) = e^{ax}$, then we try $y_p = Ae^{ax}$ for some $A \in \mathbb{R}$.

Inhomogeneous term	Trial function
e^{ax}	Ae^{ax}
$\sin kx + \cos kx$	$A \sin kx + B \cos kx$
x^n	$A_n x^n + \cdots + A_1 x + A_0$
$x^n e^{ax}$	$(A_n x^n + \cdots + A_0)e^{ax}$
$e^{ax} \sin kx$	$e^{ax}(A \cos kx + B \sin kx)$
$x^n \sin kx$	$(A_n x^n + \cdots + A_0) \cos kx + (B_n x^n + \cdots + B_0) \sin kx$

There is one problem with this method. For $y'' - y' - 2y = e^{2x}$, trying $y = Ae^{2x}$ leads to $0 = e^{2x}$, because e^{2x} already solves the homogeneous equation. That's not good! We will not be able to sleep until we find a workaround for this problem.

Lecture 12 — Friday, January 30, 2015

The problem in the last example was that our trial function was already a solution to the homogeneous equation. In that case, we multiply the usual guess by x. If there is still overlap, we keep multiplying by higher powers of x until the trial function is linearly independent from the homogeneous solution.

Example 12.1 Solve $y'' - y' - 2y = e^{2x}$. The usual guess $y_p = Ae^{2x}$ fails since e^{2x} is already part of the homogeneous solution. So instead we try

$$y_p = Axe^{2x}.$$

Then

$$y_p' = Ae^{2x} + 2Axe^{2x} \qquad \text{and} \qquad y_p'' = 4Ae^{2x} + 4Axe^{2x}.$$

Plugging in, we get

$$y_p'' - y_p' - 2y_p = \left(4Ae^{2x} + 4Axe^{2x}\right) - \left(Ae^{2x} + 2Axe^{2x}\right) - 2Axe^{2x} = 3Ae^{2x}.$$

Therefore $3A = 1$, so $A = 1/3$ and a particular solution is

$$y_p = \frac{1}{3}xe^{2x}.$$

Example 12.2 Solve $y' - y = e^x + x^2$. The homogeneous solution is $y_h = Ce^x$, so we cannot use Ae^x in our trial function. We would instead try

$$y_p = Axe^x + Bx^2 + Dx + E.$$

Substitution gives $A = 1$, $B = -1$, and $D = E = -2$. Therefore

$$y = Ce^x + xe^x - x^2 - 2x - 2.$$

Example 12.3 Solve $y'' + 2y' + y = e^{-x}$. Here the homogeneous solution is

$$y_h = c_1e^{-x} + c_2xe^{-x}.$$

Since both e^{-x} and xe^{-x} already appear, we must multiply by x^2, so we try

$$y_p = Ax^2e^{-x}.$$

Substitution gives $2Ae^{-x} = e^{-x}$, so $A = 1/2$ and

$$y = c_1e^{-x} + c_2xe^{-x} + \frac{1}{2}x^2e^{-x}.$$

Example 12.4 Choose a trial function for $y^{(4)} + 2y'' + y = x\cos x$. Previously, we found that the homogeneous solution is

$$y_h = c_1\cos x + c_2\sin x + c_3x\cos x + c_4x\sin x.$$

Since the forcing term already overlaps with this structure, we must guess

$$y_p = (Ax^3 + Bx^2)\cos x + (Cx^3 + Dx^2)\sin x.$$

Variation of Parameters

There is another method for finding particular solutions. It is called **variation of parameters**. For first-order equations, this is really just the integrating factor method in disguise, but it generalizes to solve equations of the form $\Phi[y] = F(x)$.

The basic idea is simple: if the homogeneous solution contains arbitrary constants, then for a particular solution we replace those constants with unknown functions and solve for them.

Example 12.5 Solve the differential equation

$$y' + xy = \frac{1}{x}.$$

Solution. Work on an interval which does not contain 0, and choose x_0 in that interval. By integrating factors, the homogeneous solution is

$$y_h = C_1 e^{-x^2/2}.$$

For a particular solution, we vary the parameter and try

$$y_p = u(x)e^{-x^2/2}.$$

Differentiating gives

$$y_p' = u'(x)e^{-x^2/2} - xu(x)e^{-x^2/2}.$$

Substituting into the differential equation gives

$$u'(x)e^{-x^2/2} - xu(x)e^{-x^2/2} + xu(x)e^{-x^2/2} = \frac{1}{x}.$$

Hence

$$u'(x) = \frac{1}{x}e^{x^2/2}.$$

Integrating, we obtain

$$u(x) = \int_{x_0}^{x} \frac{1}{t}e^{t^2/2}\,\mathrm{d}t.$$

Hence

$$y_p = e^{-x^2/2}\int_{x_0}^{x} \frac{1}{t}e^{t^2/2}\,\mathrm{d}t.$$

Therefore the general solution is

$$y = Ce^{-x^2/2} + e^{-x^2/2} \int_{x_0}^{x} \frac{1}{t} e^{t^2/2} \, \mathrm{d}t.$$

Alternatively, keeping a constant of integration inside $u(x)$ recovers the homogeneous part automatically. □

Lecture 13 — Monday, February 02, 2015

Now consider the general second-order equation

$$y'' + p(x)y' + q(x)y = F(x).$$

Suppose the homogeneous solution is

$$y_h = c_1 y_1(x) + c_2 y_2(x).$$

We vary both parameters and try

$$y = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Then

$$y' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'.$$

We now have two unknown functions, so we impose an extra condition:

$$u_1' y_1 + u_2' y_2 = 0.$$

This simplifies the derivative to

$$y' = u_1 y_1' + u_2 y_2'.$$

Differentiating again,

$$y'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''.$$

Substitute into the differential equation:

$$u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' + p(x)(u_1 y_1' + u_2 y_2') + q(x)(u_1 y_1 + u_2 y_2) = F(x).$$

Since y_1 and y_2 solve the homogeneous equation, all of the terms involving u_1 and u_2 vanish, leaving

$$u_1'y_1' + u_2'y_2' = F(x).$$

So we must solve the system

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0, \\ u_1'y_1' + u_2'y_2' &= F(x). \end{aligned}$$

Once we solve for u_1' and u_2', we integrate to obtain u_1 and u_2.

Example 13.1 Solve

$$y'' + 9y = 9\sec^2 3x.$$

Solution. Work on any interval that contains no zero of $\cos 3x$. The homogeneous solution is

$$y_h = c_1 \cos 3x + c_2 \sin 3x.$$

For a particular solution, try

$$y_p = u_1(x)\cos 3x + u_2(x)\sin 3x.$$

The two equations for u_1' and u_2' are

$$\begin{aligned} u_1' \cos 3x + u_2' \sin 3x &= 0, \\ -3u_1' \sin 3x + 3u_2' \cos 3x &= 9\sec^2 3x. \end{aligned}$$

Solving, we get

$$u_1' = -3\sec 3x \tan 3x \qquad \text{and} \qquad u_2' = 3\sec 3x.$$

Therefore,

$$u_1 = -\sec 3x + C_1 \qquad \text{and} \qquad u_2 = \log|\sec 3x + \tan 3x| + C_2.$$

Hence

$$\begin{aligned} y &= u_1y_1 + u_2y_2 \\ &= (-\sec 3x + C_1)\cos 3x + (\log|\sec 3x + \tan 3x| + C_2)\sin 3x \\ &= -1 + C_1\cos 3x + C_2 \sin 3x + \sin 3x \log|\sec 3x + \tan 3x|. \end{aligned}$$

□

Lecture 14 — Wednesday, February 04, 2015

Reduction of Order

The same idea helps in another situation. If we know a nonzero solution to a homogeneous linear equation, we may be able to find another by lowering the order of the problem. Locally, on an interval where that known solution does not vanish, an nth-order equation can be reduced to an $(n-1)$th-order equation. This is the method of **reduction of order**.

Example 14.1 Solve
$$x^2y'' + 2xy' - 2y = 0, \quad x > 0.$$

Solution. The function $y_1 = x$ is one solution. We guess a second solution of the form
$$y = u(x)x.$$
Then
$$y' = u'x + u \qquad \text{and} \qquad y'' = u''x + 2u'.$$
Substituting into the differential equation gives
$$\begin{aligned} x^2(u''x + 2u') + 2x(u'x + u) - 2ux &= 0 \\ x^3u'' + 4x^2u' &= 0. \end{aligned}$$
Dividing by x^2, we obtain
$$xu'' + 4u' = 0.$$
Let $v = u'$. Then
$$xv' + 4v = 0.$$
This is separable:
$$\int \frac{\mathrm{d}v}{v} = -4\int \frac{\mathrm{d}x}{x} \implies \log|v| = -4\log|x| + C.$$
So
$$v = C_0x^{-4} \qquad \text{and} \qquad u' = C_0x^{-4}.$$
Integrating once more,
$$u = C_1x^{-3} + C_2.$$

Therefore

$$y = ux = \frac{C_1}{x^2} + C_2 x.$$

□

The same substitution can also streamline higher-order equations.

Example 14.2 Solve

$$y^{(4)} + 4y^{(3)} + 6y'' + 4y' + y = 0.$$

Solution. The characteristic equation is $(m+1)^4 = 0$, so one solution is e^{-x}. We guess a second solution of the form

$$y = u(x)e^{-x}.$$

Differentiating repeatedly,

$$\begin{aligned} y' &= u'e^{-x} - ue^{-x}, \\ y'' &= u''e^{-x} - 2u'e^{-x} + ue^{-x}, \\ y^{(3)} &= u^{(3)}e^{-x} - 3u''e^{-x} + 3u'e^{-x} - ue^{-x}, \\ y^{(4)} &= u^{(4)}e^{-x} - 4u^{(3)}e^{-x} + 6u''e^{-x} - 4u'e^{-x} + ue^{-x}. \end{aligned}$$

Substituting into the differential equation, every term cancels except the one involving $u^{(4)}$, so

$$u^{(4)} = 0.$$

Thus

$$u^{(3)} = C_1, \qquad u'' = C_1 x + C_2, \qquad \text{and} \qquad u' = \frac{1}{2}C_1 x^2 + C_2 x + C_3,$$

and hence

$$u = \frac{1}{6}C_1 x^3 + \frac{1}{2}C_2 x^2 + C_3 x + C_4.$$

Absorbing the constants, we obtain

$$y = C_1 x^3 e^{-x} + C_2 x^2 e^{-x} + C_3 x e^{-x} + C_4 e^{-x}.$$

□

Lecture 15 — Friday, February 06, 2015

Example 15.1 Solve
$$x^2y'' + xy' - y = 72x^5.$$

Solution. Work on an interval contained in $x > 0$ or $x < 0$. Observe that $y_1 = x$ is a solution to the homogeneous equation, so let
$$y = u(x)x.$$

Then
$$y' = u'x + u \qquad \text{and} \qquad y'' = u''x + 2u'.$$

Substituting gives
$$\begin{aligned} x^2(u''x + 2u') + x(u'x + u) - ux &= 72x^5 \\ x^3u'' + 3x^2u' &= 72x^5. \end{aligned}$$

Dividing by x^2,
$$xu'' + 3u' = 72x^3.$$

Let $v = u'$. Then
$$xv' + 3v = 72x^3 \implies v' + \frac{3}{x}v = 72x^2.$$

The integrating factor is
$$\mu(x) = \exp\left(\int 3/x \,\mathrm{d}x\right) = x^3.$$

Thus
$$\frac{\mathrm{d}}{\mathrm{d}x}(x^3v) = 72x^5 \implies x^3v = 12x^6 + C.$$

So
$$v = u' = 12x^3 + Cx^{-3}.$$

Integrating,
$$u = 3x^4 + C_1x^{-2} + C_2.$$

Therefore
$$y = ux = 3x^5 + C_1x^{-1} + C_2x.$$

□

Boundary Value Problems

A **boundary value problem** is a differential equation together with side conditions given at different points. For example, we might have

$$y'' + y = 0, \qquad y(0) = 0, \qquad \text{and} \qquad y(\pi) = 1.$$

Unlike initial value problems, there is generally no existence and uniqueness theorem for boundary value problems.

Example 15.2 Analyze

$$y'' + \pi^2 y = 0, \qquad y(0) = 0, \qquad \text{and} \qquad y(1) = 1.$$

Lecture 16 — Monday, February 09, 2015

Solution. The general solution is

$$y = c_1 \cos \pi x + c_2 \sin \pi x.$$

The condition $y(0) = 0$ gives $c_1 = 0$, while $y(1) = 1$ gives

$$1 = c_2 \sin \pi = 0,$$

which is impossible. So this problem has no solution.

Now consider instead

$$y'' + \pi^2 y = 0, \qquad y(0) = 0, \qquad \text{and} \qquad y(1) = 0.$$

Again $c_1 = 0$, but now the second condition gives no information about c_2, so any multiple of $\sin \pi x$ is a solution. Thus this boundary value problem has infinitely many solutions. □

Since these problems often have either no solutions or infinitely many, we usually ask a different question.

Example 16.1 For which values of $k \in \mathbb{R}$ does the boundary value problem

$$y'' + ky = 0, \qquad y(0) = 0, \qquad \text{and} \qquad y(1) = 0$$

have nontrivial solutions?

Solution. *Case 1.* Suppose $k < 0$. Let $\lambda^2 = -k$. Then the characteristic equation becomes

$$m^2 - \lambda^2 = 0,$$

so

$$y = c_1 e^{\lambda x} + c_2 e^{-\lambda x}.$$

From $y(0) = 0$, we get $c_1 + c_2 = 0$. From $y(1) = 0$, we get

$$c_1 e^{\lambda} + c_2 e^{-\lambda} = 0 \implies c_1(e^{2\lambda} - 1) = 0.$$

Hence $c_1 = c_2 = 0$, so there are no nontrivial solutions.

Case 2. Suppose $k = 0$. Then

$$y = c_1 + c_2 x.$$

Applying the boundary conditions gives $c_1 = c_2 = 0$, so again only the trivial solution.

Case 3. Suppose $k > 0$. Let $k = \lambda^2$. Then the characteristic equation is

$$m^2 + \lambda^2 = 0,$$

so

$$y = c_1 \cos \lambda x + c_2 \sin \lambda x.$$

The condition $y(0) = 0$ gives $c_1 = 0$, and then $y(1) = 0$ gives

$$c_2 \sin \lambda = 0.$$

For a nontrivial solution, we need $\sin \lambda = 0$, so

$$\lambda = n\pi, \quad n \in \mathbb{N}.$$

Therefore

$$k = n^2 \pi^2,$$

and the corresponding solutions are

$$y = C \sin n\pi x.$$

□

The values $n^2\pi^2$ are the **eigenvalues** of the boundary value problem, and the functions $\sin n\pi x$ are the corresponding **eigenfunctions**.

Example 16.2 Find the eigenvalues and eigenfunctions of the boundary value problem

$$y'' + 2y' + ky = 0, \qquad y(0) = 0, \qquad \text{and} \qquad y(1) = 0.$$

Solution. The characteristic equation is

$$m^2 + 2m + k = 0 \implies (m+1)^2 + (k-1) = 0.$$

As before, we divide into cases.

Case 1. Suppose $k < 1$. Then the roots are real, and we find that the only solution satisfying the boundary conditions is the trivial solution.

Case 2. Suppose $k = 1$. Then

$$y = c_1 e^{-x} + c_2 x e^{-x}.$$

Applying $y(0) = 0$ and $y(1) = 0$ again forces $c_1 = c_2 = 0$.

Case 3. Suppose $k > 1$. Then

$$y = e^{-x}\left[c_1 \cos\left(\sqrt{k-1}\,x\right) + c_2 \sin\left(\sqrt{k-1}\,x\right)\right].$$

The condition $y(0) = 0$ gives $c_1 = 0$. Then $y(1) = 0$ gives

$$c_2 \sin\left(\sqrt{k-1}\right) = 0.$$

For a nontrivial solution, we need

$$\sqrt{k-1} = n\pi,$$

so

$$k = n^2\pi^2 + 1.$$

The corresponding eigenfunctions are

$$y = e^{-x}\sin(n\pi x).$$

$\square$

Lecture 17 — Wednesday, February 11, 2015

3. Systems of Ordinary Differential Equations

In some applications, we encounter systems of coupled equations, where each unknown function is related to the derivatives of the others.

Example 17.1 Let $x(t)$ and $y(t)$ denote the amounts of salt in two tanks, each of volume 100 L. Suppose tank 1 has 20 L/min of pure water flowing in, 10 L/min from tank 2 flowing in, and 30 L/min flowing into tank 2. Suppose tank 2 has 30 L/min from tank 1 flowing in and 20 L/min draining out. Then

$$\begin{aligned} x'(t) &= -30\left(\frac{x}{100}\right) + 10\left(\frac{y}{100}\right), \\ y'(t) &= 30\left(\frac{x}{100}\right) - 30\left(\frac{y}{100}\right). \end{aligned}$$

That is,

$$\begin{aligned} x'(t) &= -\frac{3}{10}x + \frac{1}{10}y, \\ y'(t) &= \frac{3}{10}x - \frac{3}{10}y. \end{aligned}$$

More generally, we might have

$$\begin{aligned} x' &= a_1x + b_1y + c_1z + f_1(t), \\ y' &= a_2x + b_2y + c_2z + f_2(t), \\ z' &= a_3x + b_3y + c_3z + f_3(t), \end{aligned}$$

which can be written in matrix form as

$$\boldsymbol{x}'(t) = \mathbf{A}\boldsymbol{x}(t) + \boldsymbol{f}(t),$$

where

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \qquad \text{and} \qquad \boldsymbol{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}.$$

Theorem 17.2 Any nth-order ordinary differential equation that can be solved for its highest derivative can be converted into a first-order vector differential equation. The same is true for higher-order systems whose equations can be solved for their highest derivatives.

Proof. Consider a scalar nth-order equation written in normal form as

$$y^{(n)} = F(t, y, y', \dots, y^{(n-1)}).$$

Define

$$x_1 = y, \qquad x_2 = y', \qquad \dots, \qquad \text{and} \qquad x_n = y^{(n-1)},$$

and let

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Then

$$\boldsymbol{x}' = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ F(t, x_1, x_2, \dots, x_n) \end{bmatrix}.$$

This is a first-order vector differential equation, and the construction is reversible because $x_1 = y$, $x_2 = y'$, $\dots$, $x_n = y^{(n-1)}$.

In particular, if

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \dots + a_0(t)y = g(t), \quad a_n(t) \neq 0,$$

then the last component is

$$x_n' = -\frac{a_0(t)}{a_n(t)}x_1 - \frac{a_1(t)}{a_n(t)}x_2 - \cdots - \frac{a_{n-1}(t)}{a_n(t)}x_n + \frac{g(t)}{a_n(t)},$$

so the resulting first-order vector equation is linear.

For a system, suppose that the unknown functions are $y_1, \dots, y_m$ and that the equation for y_j has order n_j. Introduce variables

$$x_{j,1} = y_j, \qquad x_{j,2} = y_j', \qquad \dots, \qquad \text{and} \qquad x_{j,n_j} = y_j^{(n_j-1)}$$

for each $j = 1, \dots, m$. The equations $x_{j,r}' = x_{j,r+1}$ for $r < n_j$, together with the original equations solved for the highest derivatives, give a first-order system in $n_1 + \cdots + n_m$ unknown functions. $\square$

Lecture 18 — Friday, February 13, 2015

Theorem 18.1 Conversely, every first-order constant-coefficient linear vector differential equation in n unknown functions, with sufficiently differentiable forcing term, can be reduced componentwise to nth-order linear ordinary differential equations.

Proof. Consider

$$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{f}(t) \qquad \text{and} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

where A is a constant $n \times n$ matrix. Let $D = \mathrm{d}/\,\mathrm{d}t$, and let

$$p(\lambda) = \det(\lambda \mathbf{I} - \mathbf{A}) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0$$

be the characteristic polynomial of $\mathbf{A}$. Set $\alpha_n = 1$. We first note that, for $k = 1, \dots, n$, repeated differentiation gives

$$D^k \boldsymbol{x} = \mathbf{A}^k \boldsymbol{x} + \sum_{r=0}^{k-1} \mathbf{A}^{k-1-r} \boldsymbol{f}^{(r)}(t).$$

Indeed, this follows by induction from $\boldsymbol{x}' = \mathbf{A}\boldsymbol{x} + \boldsymbol{f}(t)$ and the fact that $\mathbf{A}$ is constant. Hence

$$\begin{aligned} p(D)\boldsymbol{x} &= \left(D^n + \alpha_{n-1}D^{n-1} + \cdots + \alpha_0\right)\boldsymbol{x} \\ &= p(\mathbf{A})\boldsymbol{x} + \sum_{k=1}^{n} \alpha_k \sum_{r=0}^{k-1} \mathbf{A}^{k-1-r}\boldsymbol{f}^{(r)}(t). \end{aligned}$$

By the Cayley–Hamilton theorem, $p(\mathbf{A}) = 0$. Therefore

$$p(D)\boldsymbol{x} = \boldsymbol{h}(t),$$

where

$$\boldsymbol{h}(t) = \sum_{k=1}^{n} \alpha_k \sum_{r=0}^{k-1} \mathbf{A}^{k-1-r}\boldsymbol{f}^{(r)}(t).$$

Thus each component x_j satisfies

$$x_j^{(n)} + \alpha_{n-1}x_j^{(n-1)} + \cdots + \alpha_0 x_j = h_j(t),$$

which is an nth-order linear ordinary differential equation. In the homogeneous case, $\boldsymbol{h}(t) = \mathbf{0}$. $\square$

The theory of linear ordinary differential equations extends very naturally to vector differential equations. Existence and uniqueness still hold, the principle of superposition still holds, the general solution to a homogeneous problem contains linearly independent solutions, and the general solution to an inhomogeneous problem has the form $\boldsymbol{x} = \boldsymbol{x}_h + \boldsymbol{x}_p$. What about linear independence? We need a suitable definition first.

Definition 18.2 The **Wronskian** of vector functions $\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n : \mathbb{R} \to \mathbb{R}^n$ is the determinant of the matrix

$$\begin{bmatrix} \boldsymbol{f}_1 & \boldsymbol{f}_2 & \cdots & \boldsymbol{f}_n \end{bmatrix}.$$

Example 18.3 Are the vector functions

$$\begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}$$

linearly independent? Their Wronskian is

$$W(t) = \begin{vmatrix} \sin t & \sin 2t \\ \cos t & \cos 2t \end{vmatrix} = \sin t \cos 2t - \cos t \sin 2t.$$

This is nonzero on $(0, \pi/2)$, so the functions are linearly independent there.

Homogeneous Vector Differential Equations

Consider the homogeneous system

$$\boldsymbol{x}' = \mathbf{A}\boldsymbol{x}.$$

Since the scalar equation $y' = ay$ has solution $y = Ce^{at}$, we guess that vector systems should also have exponential solutions. Try

$$\boldsymbol{x} = \mathbf{c}e^{\lambda t}.$$

Then

$$\boldsymbol{x}' = \lambda\mathbf{c}e^{\lambda t},$$

so the differential equation becomes

$$\lambda\mathbf{c}e^{\lambda t} = \mathbf{A}\mathbf{c}e^{\lambda t}.$$

Thus we need

$$\mathbf{A}\mathbf{c} = \lambda\mathbf{c}.$$

So λ must be an eigenvalue of $\mathbf{A}$, and $\mathbf{c}$ must be a corresponding eigenvector.

We will need the following facts from linear algebra:

1. $\mathbf{A}\boldsymbol{v} = \lambda\boldsymbol{v}$ is equivalent to $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{v} = \mathbf{0}$. This has nonzero solutions precisely when $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ (that is, when $\mathbf{A} - \lambda\mathbf{I}$ is not invertible).
2. Once an eigenvalue λ is known, the corresponding eigenvectors are found from $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{v} = \mathbf{0}$.
3. If an eigenvalue λ has multiplicity m, then there are between 1 and m linearly independent eigenvectors corresponding to λ.
4. If $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of $\mathbf{A}$, then any corresponding eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are linearly independent.
5. If $\mathbf{A}$ is real and $\lambda = \alpha + i\beta$ is a complex eigenvalue, then its complex conjugate $\overline{\lambda} = \alpha - i\beta$ is also an eigenvalue, and if $\boldsymbol{v}$ is an eigenvector corresponding to λ, then $\overline{\boldsymbol{v}}$ is an eigenvector corresponding to $\overline{\lambda}$.

Lecture 19 — Monday, February 23, 2015

Example 19.1 Solve the system

$$x' = 2x + 3y,$$
$$y' = 2x + y.$$

In matrix form,

$$\boldsymbol{x}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \boldsymbol{x}.$$

The characteristic equation gives eigenvalues $\lambda = 4, -1$.

For $\lambda = 4$, we solve $(\mathbf{A} - 4\mathbf{I})\mathbf{v} = \mathbf{0}$ and obtain

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

So one solution is

$$\boldsymbol{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t}.$$

For $\lambda = -1$, we solve $(\mathbf{A} + \mathbf{I})\mathbf{v} = \mathbf{0}$ and may take

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Thus another solution is

$$\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}.$$

Hence the general solution is

$$\boldsymbol{x} = C_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}.$$

Suppose now that

$$\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = C_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Solving gives

$$C_1 = \frac{2}{5} \qquad \text{and} \qquad C_2 = -\frac{1}{5}.$$

Therefore

$$\boldsymbol{x}(t) = \frac{1}{5} \begin{bmatrix} 6e^{4t} - e^{-t} \\ 4e^{4t} + e^{-t} \end{bmatrix}.$$

The eigenvalues 4 and -1 make the origin a saddle. In Figure 7, trajectories move away from the origin along the eigenvector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and toward it along $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

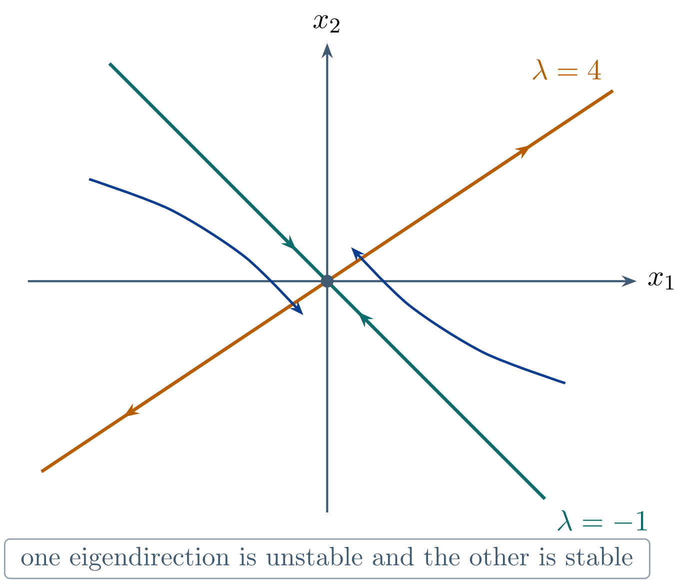

FIGURE 7

Example 19.2 Solve

$$\boldsymbol{x}' = \begin{bmatrix} -2 & 1 \\ -3 & -4 \end{bmatrix} \boldsymbol{x}.$$

The characteristic equation is

$$(\lambda+3)^2+2=0,$$

so the eigenvalues are

$$\lambda=-3\pm i\sqrt{2}.$$

For $\lambda=-3+i\sqrt{2}$, solving $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}$ gives an eigenvector

$$\mathbf{v}=\begin{bmatrix}1\\-1+i\sqrt{2}\end{bmatrix}.$$

Therefore one complex solution is

$$\boldsymbol{x}_1=\begin{bmatrix}1\\-1+i\sqrt{2}\end{bmatrix}e^{(-3+i\sqrt{2})t}.$$

Writing this as

$$\boldsymbol{x}_1=e^{-3t}\begin{bmatrix}1\\-1+i\sqrt{2}\end{bmatrix}\left(\cos\left(\sqrt{2}t\right)+i\sin\left(\sqrt{2}t\right)\right),$$

and expanding, we get

$$\boldsymbol{x}_1=e^{-3t}\left[\begin{bmatrix}\cos(\sqrt{2}t)\\-\cos(\sqrt{2}t)-\sqrt{2}\sin(\sqrt{2}t)\end{bmatrix}+i\begin{bmatrix}\sin(\sqrt{2}t)\\\sqrt{2}\cos(\sqrt{2}t)-\sin(\sqrt{2}t)\end{bmatrix}\right].$$

The real and imaginary parts are linearly independent real solutions. Therefore the general solution is

$$\boldsymbol{x}=e^{-3t}\left[C_1\begin{bmatrix}\cos(\sqrt{2}t)\\-\cos(\sqrt{2}t)-\sqrt{2}\sin(\sqrt{2}t)\end{bmatrix}+C_2\begin{bmatrix}\sin(\sqrt{2}t)\\\sqrt{2}\cos(\sqrt{2}t)-\sin(\sqrt{2}t)\end{bmatrix}\right].$$

Thus the case of complex eigenvalues works exactly as in scalar ordinary differential equations: complex exponentials combine into exponentially damped sines and cosines.

Example 19.3 Solve

$$\boldsymbol{x}'=\begin{bmatrix}-3&4\\-1&1\end{bmatrix}\boldsymbol{x}.$$

The characteristic equation is

$$(\lambda+1)^2=0,$$

so $\lambda = -1$ is a repeated eigenvalue.

Solving $(\mathbf{A} + \mathbf{I})\boldsymbol{v} = \mathbf{0}$, we obtain an eigenvector

$$\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Therefore one solution is

$$\boldsymbol{x}_1 = e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

We need a second solution. Can we just multiply by t? If we try

$$\boldsymbol{x} = \boldsymbol{v} t e^{\lambda t},$$

then

$$\boldsymbol{x}' = \boldsymbol{v} e^{\lambda t} + \lambda \boldsymbol{v} t e^{\lambda t},$$

while

$$\mathbf{A}\boldsymbol{x} = \mathbf{A}\boldsymbol{v} t e^{\lambda t} = \lambda \boldsymbol{v} t e^{\lambda t}.$$

These are not equal unless $\boldsymbol{v} = \mathbf{0}$, which is impossible. So this fails.

The issue is that this would force both components to remain multiples of one another. Instead, we try

$$\boldsymbol{x}_2 = \boldsymbol{v} t e^{\lambda t} + \boldsymbol{w} e^{\lambda t},$$

where $\boldsymbol{w}$ is to be determined. Substituting into the differential equation gives

$$(\mathbf{A} - \lambda \mathbf{I})\boldsymbol{w} = \boldsymbol{v}.$$

Such a vector $\boldsymbol{w}$ is called a **generalized eigenvector** corresponding to λ.

In our example, solving

$$(\mathbf{A} + \mathbf{I})\boldsymbol{w} = \boldsymbol{v}$$

gives

$$\boldsymbol{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Hence

$$\boldsymbol{x}_2 = te^{-t}\begin{bmatrix}2\\1\end{bmatrix} + e^{-t}\begin{bmatrix}-1\\0\end{bmatrix}.$$

Therefore the general solution is

$$\boldsymbol{x} = C_1e^{-t}\begin{bmatrix}2\\1\end{bmatrix} + C_2\left[te^{-t}\begin{bmatrix}2\\1\end{bmatrix} + e^{-t}\begin{bmatrix}-1\\0\end{bmatrix}\right].$$

Lecture 20 — Wednesday, February 25, 2015

Remark 20.1 What about the 3×3 case? The same method works, but we may have to find generalized eigenvectors of higher order.

1. If λ is a simple eigenvalue, then we get one solution from the corresponding eigenvector.
2. If λ has multiplicity 2 and two linearly independent eigenvectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, then the corresponding solutions are $\boldsymbol{u}_1e^{\lambda t}$ and $\boldsymbol{u}_2e^{\lambda t}$.
3. If λ has multiplicity 2 but only one independent eigenvector $\boldsymbol{u}$, then the solutions are $\boldsymbol{u}e^{\lambda t}$ and $(\boldsymbol{u}t + \boldsymbol{v})e^{\lambda t}$, where $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{v} = \boldsymbol{u}$.
4. If λ has multiplicity 3 and one eigenvector $\boldsymbol{u}$, then the solutions are $\boldsymbol{u}e^{\lambda t}$, $(\boldsymbol{u}t + \boldsymbol{v})e^{\lambda t}$, and $(\boldsymbol{u}t^2/2 + \boldsymbol{v}t + \boldsymbol{w})e^{\lambda t}$, where $\boldsymbol{v}$ and $\boldsymbol{w}$ are generalized eigenvectors satisfying $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{v} = \boldsymbol{u}$ and $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{w} = \boldsymbol{v}$.
5. If λ has multiplicity 3 and two linearly independent eigenvectors, one of them, say $\boldsymbol{v}$, belongs to the range of $\mathbf{A} - \lambda\mathbf{I}$. If $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{w} = \boldsymbol{v}$ and $\boldsymbol{u}$ is an independent eigenvector, then the solutions are $\boldsymbol{u}e^{\lambda t}$, $\boldsymbol{v}e^{\lambda t}$, and $(\boldsymbol{v}t + \boldsymbol{w})e^{\lambda t}$.

Example 20.2 Solve

$$\boldsymbol{x}' = \begin{bmatrix}1 & 2 & 3\\0 & 1 & 0\\2 & 1 & 2\end{bmatrix}\boldsymbol{x}.$$

Solution. The characteristic equation is

$$-(\lambda - 4)(\lambda - 1)(\lambda + 1) = 0$$

so the eigenvalues are $\lambda = 1, -1, 4$. The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 6 \\ -4 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Hence the general solution is

$$\boldsymbol{x} = C_1 e^{t} \begin{bmatrix} -1 \\ 6 \\ -4 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + C_3 e^{4t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$\square$

Example 20.3 Solve

$$\boldsymbol{x}' = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \boldsymbol{x}.$$

Solution. The characteristic equation is

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 1 \\ 0 & -1 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)\left((1-\lambda)^2 + 1\right) \\ &= -(\lambda - 1)(\lambda - (1+i))(\lambda - (1-i)). \end{aligned}$$

Therefore the eigenvalues are $\lambda = 1$, $\lambda = 1 + i$, and $\lambda = 1 - i$.

For $\lambda = 1$,

$$\mathbf{A} - \mathbf{I} = \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Thus $(\mathbf{A}-\mathbf{I})\boldsymbol{v}=\mathbf{0}$ implies $v_2=v_3=0$, so we may take

$$\boldsymbol{v}_1=\begin{bmatrix}1\\0\\0\end{bmatrix}.$$

This gives the real solution

$$\boldsymbol{x}_1(t)=e^t\begin{bmatrix}1\\0\\0\end{bmatrix}.$$

For $\lambda=1+i$,

$$\mathbf{A}-(1+i)\mathbf{I}=\begin{bmatrix}-i&2&-1\\0&-i&1\\0&-1&-i\end{bmatrix}.$$

Writing $\boldsymbol{v}=(v_1,v_2,v_3)^T$, the last two equations give $v_3=iv_2$. Choosing $v_2=1$, we get $v_3=i$, and the first equation gives

$$-iv_1+2-i=0,$$

so $v_1=-1-2i$. Hence an eigenvector for $\lambda=1+i$ is

$$\boldsymbol{v}_2=\begin{bmatrix}-1-2i\\1\\i\end{bmatrix}=\begin{bmatrix}-1\\1\\0\end{bmatrix}+i\begin{bmatrix}-2\\0\\1\end{bmatrix}.$$

The eigenvector for $\lambda=1-i$ is the complex conjugate of $\boldsymbol{v}_2$. Therefore two real solutions are obtained from the real and imaginary parts of $\boldsymbol{v}_2e^{(1+i)t}$:

$$\boldsymbol{x}_2(t)=e^t\begin{bmatrix}-\cos t+2\sin t\\\cos t\\-\sin t\end{bmatrix}\qquad\text{and}\qquad\boldsymbol{x}_3(t)=e^t\begin{bmatrix}-\sin t-2\cos t\\\sin t\\\cos t\end{bmatrix}.$$

Thus the general real solution is

$$\boldsymbol{x}(t)=C_1e^t\begin{bmatrix}1\\0\\0\end{bmatrix}+C_2e^t\begin{bmatrix}-\cos t+2\sin t\\\cos t\\-\sin t\end{bmatrix}+C_3e^t\begin{bmatrix}-\sin t-2\cos t\\\sin t\\\cos t\end{bmatrix}.$$

$\square$

Lecture 21 — Friday, February 27, 2015

Example 21.1 Solve

$$\boldsymbol{x}' = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix} \boldsymbol{x}.$$

Solution. The characteristic equation is

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ -2 & -2 & -1-\lambda \end{vmatrix} \\ &= -(\lambda - 1)^3. \end{aligned}$$

Thus $\lambda = 1$ is an eigenvalue of algebraic multiplicity 3. To find the eigenspace, compute

$$\mathbf{A} - \mathbf{I} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix}.$$

Therefore $(\mathbf{A} - \mathbf{I})\boldsymbol{u} = \mathbf{0}$ is equivalent to

$$u_1 + u_2 + u_3 = 0.$$

Thus u_2 and u_3 are free, and $u_1 = -u_2 - u_3$. Choosing $(u_2, u_3) = (0, 1)$ and $(u_2, u_3) = (1, 0)$ gives the independent eigenvectors

$$\boldsymbol{u}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

These give the two solutions

$$e^t \boldsymbol{u}_1 \qquad \text{and} \qquad e^t \boldsymbol{u}_2.$$

Since the algebraic multiplicity is 3, we need one generalized eigenvector. Choose the eigenvector

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},$$

and solve

$$(\mathbf{A} - \mathbf{I})\boldsymbol{w} = \boldsymbol{v}.$$

If $\boldsymbol{w} = (w_1, w_2, w_3)^T$, then

$$(\mathbf{A} - \mathbf{I})\boldsymbol{w} = (w_1 + w_2 + w_3) \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

Thus the equation becomes

$$(w_1 + w_2 + w_3) \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},$$

so it is enough to choose $w_1 + w_2 + w_3 = 1$. Taking $w_2 = w_3 = 0$, we may use

$$\boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then

$$\boldsymbol{x}_3(t) = e^t(t\boldsymbol{v} + \boldsymbol{w})$$

is a solution, because $\mathbf{A}\boldsymbol{v} = \boldsymbol{v}$ and $(\mathbf{A} - \mathbf{I})\boldsymbol{w} = \boldsymbol{v}$. Therefore the general solution is

$$\boldsymbol{x}(t) = C_1 e^t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + C_2 e^t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + C_3 \left(te^t \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right).$$

□

Lecture 22 — Monday, March 02, 2015

Inhomogeneous Vector Differential Equations

The method of undetermined coefficients works for simple vector systems as well.

Example 22.1 Solve
$$\boldsymbol{x}' = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Solution. First we solve the homogeneous problem. We get
$$\boldsymbol{x}_h = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For a particular solution, guess a constant vector
$$\boldsymbol{x}_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

Then $\boldsymbol{x}_p' = \mathbf{0}$, so
$$\mathbf{0} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

This gives
$$\begin{aligned} a_1 + a_2 &= -2, \\ 2a_2 &= -1. \end{aligned}$$

Hence
$$a_2 = -\frac{1}{2} \qquad \text{and} \qquad a_1 = -\frac{3}{2}.$$

Therefore
$$\boldsymbol{x}_p = \begin{bmatrix} -3/2 \\ -1/2 \end{bmatrix},$$

and the general solution is

$$\boldsymbol{x} = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -3/2 \\ -1/2 \end{bmatrix}.$$

$\square$

Variation of Parameters for Systems

If

$$\boldsymbol{x}_h = c_1 \boldsymbol{x}_1 + c_2 \boldsymbol{x}_2,$$

then we look for a solution of the form

$$\boldsymbol{x} = u_1(t)\boldsymbol{x}_1(t) + u_2(t)\boldsymbol{x}_2(t).$$

Differentiating,

$$\boldsymbol{x}' = u_1' \boldsymbol{x}_1 + u_1 \boldsymbol{x}_1' + u_2' \boldsymbol{x}_2 + u_2 \boldsymbol{x}_2'.$$

Substituting into

$$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{f}(t),$$

and using the fact that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ solve the homogeneous problem, we obtain

$$u_1' \boldsymbol{x}_1 + u_2' \boldsymbol{x}_2 = \boldsymbol{f}(t).$$

This is a linear system for u_1' and u_2'. If

$$\boldsymbol{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}, \qquad \boldsymbol{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix}, \qquad \text{and} \qquad \boldsymbol{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

then in component form this system is

$$\begin{aligned} u_1' x_{11} + u_2' x_{12} &= f_1, \\ u_1' x_{21} + u_2' x_{22} &= f_2. \end{aligned}$$

Since $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are linearly independent, the coefficient matrix is invertible, so we can solve for u_1' and u_2'. Thus we obtain

$$u_1' = G_1(t) \qquad \text{and} \qquad u_2' = G_2(t),$$

and therefore

$$\boldsymbol{x}(t) = \left(\int G_1(t)\,\mathrm{d}t\right)\boldsymbol{x}_1(t) + \left(\int G_2(t)\,\mathrm{d}t\right)\boldsymbol{x}_2(t).$$

If we keep the constants of integration, this gives the general solution.

Lecture 23 — Wednesday, March 04, 2015

Example 23.1 Solve

$$\boldsymbol{x}' = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}\boldsymbol{x} + \begin{bmatrix}te^{-2t}\\ 3e^{-2t}\end{bmatrix}.$$

Solution. From the calculation in Example 22.1, the homogeneous solution is

$$\boldsymbol{x}_h = c_1 e^t \begin{bmatrix}1\\0\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}1\\1\end{bmatrix}.$$

So let

$$\boldsymbol{x} = u_1 e^t \begin{bmatrix}1\\0\end{bmatrix} + u_2 e^{2t}\begin{bmatrix}1\\1\end{bmatrix}.$$

Then u_1' and u_2' must satisfy

$$u_1' e^t \begin{bmatrix}1\\0\end{bmatrix} + u_2' e^{2t}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}te^{-2t}\\ 3e^{-2t}\end{bmatrix},$$

which gives the system

$$\begin{aligned} u_1' e^t + u_2' e^{2t} &= te^{-2t},\\ u_2' e^{2t} &= 3e^{-2t}. \end{aligned}$$

From the second equation,

$$u_2' = 3e^{-4t} \implies u_2 = -\frac{3}{4}e^{-4t} + c_2.$$

Substituting into the first equation,

$$u_1' e^t + 3e^{-2t} = te^{-2t},$$

so

$$u_1' = (t-3)e^{-3t}.$$

Integrating by parts,

$$u_1 = -\frac{t}{3}e^{-3t} + \frac{8}{9}e^{-3t} + c_1.$$

Therefore

$$\boldsymbol{x}(t) = \left(-\frac{t}{3}e^{-3t} + \frac{8}{9}e^{-3t} + c_1\right)e^t \begin{bmatrix}1\\0\end{bmatrix} + \left(-\frac{3}{4}e^{-4t} + c_2\right)e^{2t}\begin{bmatrix}1\\1\end{bmatrix}.$$

In component form,

$$\begin{aligned} x(t) &= c_1e^t + c_2e^{2t} - \frac{t}{3}e^{-2t} + \frac{5}{36}e^{-2t}, \\ y(t) &= c_2e^{2t} - \frac{3}{4}e^{-2t}. \end{aligned}$$

□

4. Partial Differential Equations and Fourier Methods

Partial differential equations involve functions of more than one variable.

Example 23.2

$$u_{xx} = u_t + x\sin t \qquad \text{and} \qquad u = u(x,t).$$

Example 23.3 Three famous partial differential equations are the following:

1. The **heat equation** models a metal bar of length L with insulated sides. Let $u(x,t)$ be the temperature at position x and time t. Then

$$u_t = \gamma u_{xx}.$$

Roughly, the temperature increases where the temperature profile is concave up. To complete the problem, we need side conditions:

(a) An initial condition such as $u(x,0) = f(x)$.

(b) Boundary conditions at $x=0$ and $x=L$, for instance $u(0,t) = u(L,t) = 0$ if the ends are held in ice.

(c) If the ends are insulated instead, then we would use $u_x(0,t) = u_x(L,t) = 0$.

Other boundary conditions are possible, depending on the physical setup.

2. The **wave equation** describes a vibrating string of length L under tension. If $u(x,t)$ denotes vertical displacement, then

$$u_{tt} = \alpha^2 u_{xx}.$$

This model assumes that the displacement is small compared to the length of the string. Roughly, the concavity of the string determines the forces and hence the acceleration. Here we typically need two initial conditions,

$$u(x,0) = f(x) \qquad \text{and} \qquad u_t(x,0) = g(x),$$

together with boundary conditions at the endpoints.

3. **Laplace's equation** is

$$u_{xx} + u_{yy} = 0.$$

It is a kind of smoothness condition. One application comes from the two-dimensional heat equation

$$u_t = \gamma(u_{xx} + u_{yy}),$$

since in a steady state we expect $u_t \to 0$, which leaves Laplace's equation. Boundary conditions are usually specified on the boundary of the region, and bounded regions are often the easiest to treat.

Lecture 24 — Friday, March 06, 2015

First-Order Linear Partial Differential Equations

Some simple partial differential equations can be solved by **partial integration**, which is just antidifferentiation with respect to one variable while holding the others fixed.

Example 24.1 Solve

$$u_y = -e^{-y} \qquad \text{and} \qquad u(x,0) = \frac{1}{1+x^2}.$$

Solution. Antidifferentiate with respect to y:

$$u(x, y) = e^{-y} + g(x).$$

Apply the initial condition:

$$\frac{1}{1+x^2} = 1 + g(x),$$

so

$$g(x) = \frac{1}{1+x^2} - 1.$$

Hence

$$u(x, y) = e^{-y} + \frac{1}{1+x^2} - 1.$$

We essentially have an ordinary differential equation for each fixed value of x, so the lines $x = C$ are characteristic curves here. $\square$

If both u_x and u_y appear in a partial differential equation, direct partial integration will not work. If the equation is linear, however, we can often make a change of variables to eliminate one derivative.

Example 24.2 Solve the initial value problem

$$u_y = -2u_x \qquad \text{and} \qquad u(x, 0) = e^{-x^2}.$$

Solution. Let

$$\xi = x - 2y \qquad \text{and} \qquad \eta = y,$$

and write

$$u(x, y) = \hat{u}(\xi, \eta).$$

By the chain rule,

$$\begin{aligned} u_y &= \hat{u}_\xi \xi_y + \hat{u}_\eta \eta_y = -2\hat{u}_\xi + \hat{u}_\eta, \\ u_x &= \hat{u}_\xi \xi_x + \hat{u}_\eta \eta_x = \hat{u}_\xi. \end{aligned}$$

The partial differential equation becomes

$$-2\hat{u}_\xi + \hat{u}_\eta = -2\hat{u}_\xi,$$

so

$$\hat{u}_\eta = 0.$$

Antidifferentiate:

$$\hat{u} = g(\xi),$$

and hence

$$u(x, y) = g(x - 2y).$$

Apply the initial condition:

$$u(x, 0) = g(x) = e^{-x^2}.$$

Therefore

$$u(x, y) = e^{-(x-2y)^2}.$$

□

The lines

$$\xi = x - 2y = C$$

are called the **characteristics**. The solution behaves like an ordinary differential equation along those lines rather than along $x = C$.

How do we find the change of variables in general?

Lemma 24.3 Consider the ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y).$$

If its general solution can be written implicitly as $\phi(x, y) = K$ and $\phi_y \neq 0$, then

$$\frac{\phi_x}{\phi_y} = -f(x, y).$$

Proof. Differentiate $\phi(x, y) = K$ implicitly with respect to x:

$$\phi_x + \phi_y \frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Rearranging gives

$$\frac{\phi_x}{\phi_y} = -\frac{\mathrm{d}y}{\mathrm{d}x} = -f(x, y).$$

□

Example 24.4 For

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 y,$$

the general solution is

$$y = Ce^{x^3}.$$

Rewriting this as

$$C = ye^{-x^3},$$

we may take

$$\phi(x, y) = ye^{-x^3}.$$

Then

$$\phi_y = e^{-x^3} \qquad \text{and} \qquad \phi_x = -3x^2 ye^{-x^3},$$

and hence

$$\frac{\phi_x}{\phi_y} = -3x^2 y,$$

exactly as the lemma predicts.

Lecture 25 — Monday, March 09, 2015

Change of Basis

Consider a first-order linear partial differential equation

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y),$$

on a region where $a \neq 0$. We seek a change of variables

$$\xi = \xi(x, y) \qquad \text{and} \qquad \eta = \eta(x, y),$$

and write

$$u(x, y) = \hat{u}(\xi, \eta).$$

By the chain rule,

$$\begin{aligned} u_x &= \hat{u}_\xi \xi_x + \hat{u}_\eta \eta_x, \\ u_y &= \hat{u}_\xi \xi_y + \hat{u}_\eta \eta_y. \end{aligned}$$

Substituting into the partial differential equation gives

$$[a\xi_x + b\xi_y]\hat{u}_\xi + [a\eta_x + b\eta_y]\hat{u}_\eta + c\hat{u} = f.$$

We want one derivative term to vanish. Let us eliminate the $\hat{u}_\eta$ term by requiring

$$a\eta_x + b\eta_y = 0.$$

Assuming $\eta_y \neq 0$, this becomes

$$\frac{\eta_x}{\eta_y} = -\frac{b(x,y)}{a(x,y)}.$$

By Lemma 24.3, this means that $\eta(x,y) = K$ should be the general solution to the ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b(x,y)}{a(x,y)}.$$

This is called the **characteristic equation** of the partial differential equation.

What should ξ be? The only remaining requirement is that the transformation be invertible, so its Jacobian must be nonzero:

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x\eta_y - \xi_y\eta_x \neq 0.$$

The simplest choice is

$$\xi = x.$$

Example 25.1 Solve the initial value problem

$$xu_x + yu_y = 3u, \qquad u(x,1) = 1 - x^2, \qquad \text{and} \qquad x \geq 0,\ y \geq 1.$$

Solution. First work on the region $x > 0$. Start with the characteristic equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x}.$$

This is separable:

$$\int \frac{\mathrm{d}y}{y} = \int \frac{\mathrm{d}x}{x} \implies \log y - \log x = C.$$

So we take

$$\xi = x \qquad \text{and} \qquad \eta = \log x - \log y.$$

Then

$$u_x = \hat{u}_\xi + \frac{1}{x}\hat{u}_\eta,$$
$$u_y = -\frac{1}{y}\hat{u}_\eta.$$

The partial differential equation becomes

$$\xi \hat{u}_\xi = 3\hat{u}.$$

This is now an ordinary differential equation in ξ:

$$\xi \frac{\mathrm{d}\hat{u}}{\mathrm{d}\xi} = 3\hat{u} \implies \hat{u} = C_1 \xi^3.$$

Since the constant may depend on η, we write

$$\hat{u} = F(\eta)\xi^3.$$

Returning to the original variables,

$$u(x,y) = F(\log x - \log y)x^3 = H\left(\frac{x}{y}\right)x^3.$$

Now apply the boundary condition $u(x,1) = 1 - x^2$:

$$1 - x^2 = H(x)x^3.$$

So

$$H(x) = \frac{1-x^2}{x^3}.$$

Therefore

$$u(x,y) = x^3 \cdot \frac{1-(x/y)^2}{(x/y)^3} = y^3 - x^2 y.$$

The final expression also satisfies the equation and boundary data at $x = 0$, so it extends the solution to the stated domain. □

Note that we could also try to eliminate the $\hat{u}_\xi$ term instead. Depending on the partial differential equation, one choice may be easier than the other.

Lecture 26 — Wednesday, March 11, 2015

Second-Order Linear Partial Differential Equations

A second-order linear partial differential equation in two variables has the form

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y).$$

As with first-order equations, we try to simplify this by making a change of variables

$$\xi = \xi(x,y) \qquad \text{and} \qquad \eta = \eta(x,y),$$

and writing

$$u(x,y) = \hat{u}(\xi,\eta).$$

After a long chain rule calculation, the transformed partial differential equation has the form

$$A(\xi,\eta)\hat{u}_{\xi\xi} + B(\xi,\eta)\hat{u}_{\xi\eta} + C(\xi,\eta)\hat{u}_{\eta\eta} + D(\xi,\eta)\hat{u}_{\xi} + E(\xi,\eta)\hat{u}_{\eta} + F(\xi,\eta)\hat{u} = G(\xi,\eta),$$

where

$$\begin{aligned} A(\xi,\eta) &= a(\xi_x)^2 + b\xi_x\xi_y + c(\xi_y)^2, \\ B(\xi,\eta) &= 2a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + 2c\xi_y\eta_y, \\ C(\xi,\eta) &= a(\eta_x)^2 + b\eta_x\eta_y + c(\eta_y)^2. \end{aligned}$$

For example, the calculation begins with

$$u_x = \hat{u}_\xi\xi_x + \hat{u}_\eta\eta_x,$$

and differentiating once more gives

$$u_{xx} = \hat{u}_{\xi\xi}(\xi_x)^2 + 2\hat{u}_{\xi\eta}\xi_x\eta_x + \hat{u}_{\eta\eta}(\eta_x)^2 + \hat{u}_\xi\xi_{xx} + \hat{u}_\eta\eta_{xx}.$$

The analogous formulas for u_{xy} and u_{yy} produce the coefficients above.

The goal is to choose ξ and η so that one or more of these coefficients vanish. To eliminate the $\hat{u}_{\xi\xi}$ term, we set

$$A = a(\xi_x)^2 + b\xi_x\xi_y + c(\xi_y)^2 = 0.$$

Assuming $\xi_y \neq 0$, this becomes

$$a\left(\frac{\xi_x}{\xi_y}\right)^2 + b\left(\frac{\xi_x}{\xi_y}\right) + c = 0.$$

Therefore

$$\frac{\xi_x}{\xi_y} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This leads to the characteristic equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}.$$

There are three cases.

Case 1. The equation is **hyperbolic** if $b^2 - 4ac > 0$. Then there are two distinct characteristic families. We may choose $\xi = \phi_1$ and $\eta = \phi_2$, where $\phi_1 = K_1$ and $\phi_2 = K_2$ are the general solutions to the two characteristic equations. This eliminates both $\hat{u}_{\xi\xi}$ and $\hat{u}_{\eta\eta}$, leaving the canonical form

$$\hat{u}_{\xi\eta} + \Phi(\xi, \eta, \hat{u}, \hat{u}_\xi, \hat{u}_\eta) = 0.$$

Case 2. The equation is **parabolic** if $b^2 - 4ac = 0$. Then there is only one characteristic family. We can reduce the equation to the canonical form

$$\hat{u}_{\eta\eta} + \Phi(\xi, \eta, \hat{u}, \hat{u}_\xi, \hat{u}_\eta) = 0.$$

The mixed term also vanishes in this case. Indeed, if $\xi_x/\xi_y = -b/(2a)$, then

$$B = (2a\xi_x + b\xi_y)\eta_x + (b\xi_x + 2c\xi_y)\eta_y = 0,$$

since the first factor vanishes by the choice of ξ, and the second vanishes because $b^2 = 4ac$.

Case 3. The equation is **elliptic** if $b^2 - 4ac < 0$. Then we cannot eliminate both pure second derivative terms. However, it is possible to reduce the equation to

$$\hat{u}_{\xi\xi} + \hat{u}_{\eta\eta} + \Phi(\xi, \eta, \hat{u}, \hat{u}_\xi, \hat{u}_\eta) = 0.$$

The names mirror the classification of the quadratic form $aX^2 + bXY + cY^2$: the sign of the same discriminant, $b^2 - 4ac$, distinguishes hyperbolic, parabolic, and elliptic forms.

Example 26.1 The wave equation $u_{tt} = \alpha^2 u_{xx}$ is hyperbolic, the heat equation $u_t = \gamma u_{xx}$ is parabolic, and Laplace's equation $u_{xx} + u_{yy} = 0$ is elliptic.

Example 26.2 Solve

$$u_{xy} + u_y = xy, \qquad u(x,0) = 0, \qquad u(0,y) = \sin y, \qquad \text{and} \qquad x \geq 0,\ y \geq 0.$$

Solution. First we antidifferentiate with respect to y:

$$u_x + u = \frac{1}{2}xy^2 + f(x).$$

This is now a first-order linear ordinary differential equation in x. The integrating factor is e^x, so

$$\frac{\mathrm{d}}{\mathrm{d}x}(ue^x) = \frac{1}{2}xe^x y^2 + e^x f(x).$$

Integrating gives

$$ue^x = \frac{1}{2}y^2 \int xe^x \,\mathrm{d}x + \int e^x f(x)\,\mathrm{d}x + h(y),$$

and hence

$$u = \frac{1}{2}y^2(x-1) + g(x) + e^{-x}h(y),$$

where

$$g(x) = e^{-x}\int e^x f(x)\,\mathrm{d}x.$$

Now we apply the boundary conditions. From $u(x,0) = 0$, we get

$$0 = g(x) + e^{-x}h(0),$$

so

$$g(x) = -e^{-x}h(0).$$

From $u(0,y) = \sin y$, we get

$$\sin y = -\frac{1}{2}y^2 + g(0) + h(y).$$

Since $g(0) = -h(0)$, this determines

$$h(y) = \sin y + \frac{1}{2}y^2 - g(0) = \sin y + \frac{1}{2}y^2 + h(0).$$

Therefore

$$
\begin{aligned}
u(x,y) &= \frac{1}{2}y^2(x-1) - e^{-x}h(0) + e^{-x}\left[\sin y + \frac{1}{2}y^2 + h(0)\right] \\
&= \frac{1}{2}y^2(x-1) + e^{-x}\sin y + \frac{1}{2}y^2 e^{-x}.
\end{aligned}
$$

□

Lecture 27 — Friday, March 13, 2015

The Wave Equation

Now we consider the wave equation

$$u_{tt} = \alpha^2 u_{xx}, \qquad u(x,0) = \gamma(x), \qquad \text{and} \qquad u_t(x,0) = 0.$$

In standard form, this is

$$\alpha^2 u_{xx} - u_{tt} = 0.$$

To convert it to canonical form, we solve the characteristic equations. In the variables (x,t), the standard form has $a = \alpha^2$, $b = 0$, and $c = -1$, so

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\pm\sqrt{4\alpha^2}}{2\alpha^2} = \pm\frac{1}{\alpha}.$$

Equivalently, $\mathrm{d}x/\,\mathrm{d}t = \pm\alpha$, so

$$x = \pm\alpha t + C.$$

Thus we use $x \pm \alpha t$ as the new variables and let

$$\xi = x + \alpha t \qquad \text{and} \qquad \tau = x - \alpha t.$$

Let

$$\xi = x + \alpha t, \qquad \tau = x - \alpha t, \qquad \text{and} \qquad u(x,t) = \hat{u}(\xi,\tau).$$

Then

$$u_t = \hat{u}_\xi \xi_t + \hat{u}_\tau \tau_t$$

$$
\begin{aligned}
&= \alpha\hat{u}_\xi - \alpha\hat{u}_\tau, \\
u_x &= \hat{u}_\xi \xi_x + \hat{u}_\tau \tau_x \\
&= \hat{u}_\xi + \hat{u}_\tau.
\end{aligned}
$$

Differentiating again,

$$
\begin{aligned}
u_{tt} &= \frac{\partial}{\partial t}\left(\alpha\hat{u}_\xi - \alpha\hat{u}_\tau\right) \\
&= \alpha(\hat{u}_{\xi\xi}\xi_t + \hat{u}_{\xi\tau}\tau_t) - \alpha(\hat{u}_{\tau\xi}\xi_t + \hat{u}_{\tau\tau}\tau_t) \\
&= \alpha\left(\alpha\hat{u}_{\xi\xi} - \alpha\hat{u}_{\xi\tau}\right) - \alpha\left(\alpha\hat{u}_{\tau\xi} - \alpha\hat{u}_{\tau\tau}\right) \\
&= \alpha^2\hat{u}_{\xi\xi} - 2\alpha^2\hat{u}_{\xi\tau} + \alpha^2\hat{u}_{\tau\tau}, \\
u_{xx} &= \hat{u}_{\xi\xi} + 2\hat{u}_{\xi\tau} + \hat{u}_{\tau\tau}.
\end{aligned}
$$

Substituting into the wave equation gives

$$\alpha^2\hat{u}_{\xi\xi} - 2\alpha^2\hat{u}_{\xi\tau} + \alpha^2\hat{u}_{\tau\tau} = \alpha^2\hat{u}_{\xi\xi} + 2\alpha^2\hat{u}_{\xi\tau} + \alpha^2\hat{u}_{\tau\tau}.$$

Equivalently,

$$4\alpha^2\hat{u}_{\xi\tau} = 0,$$

so

$$\hat{u}_{\xi\tau} = 0.$$

Antidifferentiate:

$$\hat{u}_\xi = f(\xi) \implies \hat{u} = F(\xi) + G(\tau).$$

Therefore

$$u(x,t) = F(x+\alpha t) + G(x-\alpha t).$$

This is called **d'Alembert's solution**.

Now apply the initial conditions. From $u(x,0) = \gamma(x)$, we get

$$F(x) + G(x) = \gamma(x).$$

From $u_t(x,0) = 0$, we get

$$\alpha F'(x) - \alpha G'(x) = 0,$$

so

$$F'(x) = G'(x).$$

Differentiating the first relation yields

$$F'(x) + G'(x) = \gamma'(x).$$

Hence

$$F'(x) = G'(x) = \frac{1}{2}\gamma'(x).$$

Integrating,

$$F(x) = \frac{1}{2}\gamma(x) + c_1 \qquad \text{and} \qquad G(x) = \frac{1}{2}\gamma(x) + c_2,$$

with $c_1 + c_2 = 0$. Therefore

$$u(x,t) = \frac{1}{2}\left[\gamma(x+\alpha t) + \gamma(x-\alpha t)\right].$$

For zero initial velocity, d'Alembert's formula splits the initial profile into two half-amplitude copies travelling in opposite directions. This propagation mechanism is shown schematically in Figure 8.

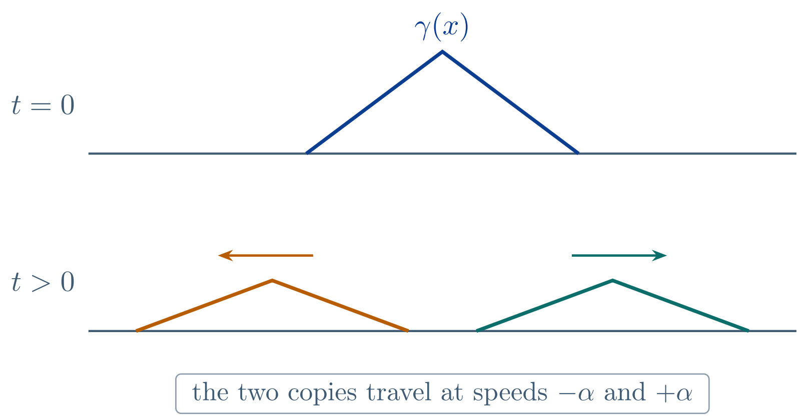

Figure 8

Lecture 28 — Monday, March 16, 2015

Example 28.1 Suppose

$$\gamma(x) = \begin{cases} 2 - x, & x \geq 0, \\ 2 + x, & x < 0. \end{cases}$$

Then at time $t = 2/\alpha$,

$$u(x,t) = \frac{1}{2}\left(\gamma(x+2) + \gamma(x-2)\right).$$

Thus two half-amplitude copies of the initial profile travel in opposite directions along the x-axis, each with speed α.

Separation of Variables

Here is the most commonly used approach: assume that the solution can be written in the form

$$u(x,t) = F(x)G(t).$$

Example 28.2 Solve

$$u_t = \gamma u_{xx}, \quad \gamma > 0,$$

with

$$u(x,0) = 20\sin\left(\frac{3\pi x}{L}\right), \qquad u(0,t) = 0, \qquad \text{and} \qquad u(L,t) = 0.$$

Assume

$$u(x,t) = F(x)G(t).$$

Solution. Then

$$u_t = F(x)G'(t) \qquad \text{and} \qquad u_{xx} = F''(x)G(t).$$

Since $u_t = \gamma u_{xx}$,

$$F(x)G'(t) = \gamma F''(x)G(t).$$

Rearranging,

$$\frac{G'(t)}{\gamma G(t)} = \frac{F''(x)}{F(x)}.$$

Since the left side depends only on t and the right side only on x, both must equal a constant c_0. Hence

$$\begin{cases} G'(t) = c_0\gamma G(t), \\ F''(x) = c_0 F(x). \end{cases}$$

The boundary conditions give

$$\begin{aligned} F(0)G(t) = 0 &\implies F(0) = 0, \\ F(L)G(t) = 0 &\implies F(L) = 0. \end{aligned}$$

So F satisfies the boundary value problem

$$F'' - c_0 F = 0, \qquad F(0) = 0, \qquad \text{and} \qquad F(L) = 0.$$

As before, there are no nontrivial solutions if $c_0 \geq 0$. Indeed, if $c_0 = 0$, then $F(x) = Ax + B$, and the boundary conditions force $A = B = 0$. If $c_0 = \lambda^2 > 0$, then $F(x) = Ae^{\lambda x} + Be^{-\lambda x}$, and the boundary conditions again force the trivial solution. Hence we assume $c_0 < 0$ and let

$$c_0 = -\lambda^2.$$

Then

$$F'' + \lambda^2 F = 0,$$

whose general solution is

$$F(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x).$$

Since $F(0) = 0$, we get $c_1 = 0$. Since $F(L) = 0$,

$$c_2 \sin(\lambda L) = 0.$$

For a nontrivial solution we need

$$\lambda L = n\pi \implies \lambda = \frac{n\pi}{L}, \quad n \in \mathbb{N}.$$

Thus

$$F_n(x) = c_n \sin\left(\frac{n\pi x}{L}\right).$$

Returning to G,

$$G'(t) = -\lambda^2 \gamma G(t) = -\frac{n^2\pi^2}{L^2}\gamma G(t),$$

so

$$G_n(t) = A_n e^{-\frac{n^2\pi^2}{L^2}\gamma t}.$$

Therefore

$$u_n(x,t) = B_n e^{-\frac{n^2\pi^2}{L^2}\gamma t} \sin\left(\frac{n\pi x}{L}\right).$$

By the principle of superposition, we get

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\frac{n^2\pi^2}{L^2}\gamma t} \sin\left(\frac{n\pi x}{L}\right).$$

Now apply the initial condition
$$u(x,0) = 20\sin\left(\frac{3\pi x}{L}\right).$$
This forces $B_3 = 20$ and all other $B_n = 0$. So
$$u(x,t) = 20e^{-\frac{9\pi^2}{L^2}\gamma t}\sin\left(\frac{3\pi x}{L}\right).$$

□

Lecture 29 — Wednesday, March 18, 2015

The Fourier Transform

This is a technique designed for problems with infinite spatial domains, for example the heat equation on an infinitely long bar:
$$u_t = \gamma u_{xx}, \qquad u(x,0) = f(x), \qquad \text{and} \qquad x \in \mathbb{R}.$$

Definition 29.1 The **Fourier transform** of a function $f : \mathbb{R} \to \mathbb{R}$ is
$$\mathcal{F}(f(x)) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,\mathrm{d}x,$$
provided the integral converges.

With this convention, the corresponding inverse transform is
$$\mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}\,\mathrm{d}\omega.$$

For example, if both f and $\hat{f}$ are absolutely integrable, Fourier inversion recovers $f(x)$ at every point where f is continuous. Whenever we use an inverse transform below, we either assume hypotheses that justify this formula or interpret the calculation formally and verify the resulting solution directly.

This transforms a real-valued function of x into a complex-valued function of ω. We will also use Euler's formula:

$$e^{x+iy} = e^x(\cos y + i \sin y).$$

Example 29.2 Find $\mathcal{F}(f(x))$ if

$$f(x) = \begin{cases} 0, & x < 0, \\ e^{-ax}, & x \geq 0, \end{cases}$$

where $a > 0$.

Solution. We have

$$\begin{aligned} \hat{f}(\omega) &= \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, \mathrm{d}x \\ &= \int_0^{\infty} e^{-(a+i\omega)x} \, \mathrm{d}x \\ &= \lim_{t \to \infty} \int_0^t e^{-(a+i\omega)x} \, \mathrm{d}x \\ &= \lim_{t \to \infty} \left[\frac{e^{-(a+i\omega)x}}{-(a+i\omega)} \right]_0^t \\ &= \lim_{t \to \infty} \left(\frac{-e^{-(a+i\omega)t}}{a+i\omega} + \frac{1}{a+i\omega} \right). \end{aligned}$$

Since

$$\left| \frac{e^{-(a+i\omega)t}}{a+i\omega} \right| = \frac{e^{-at}}{|a+i\omega|} \to 0 \quad \text{as } t \to \infty,$$

it follows that

$$\hat{f}(\omega) = \frac{1}{a+i\omega}.$$

□

Remark 29.3 The Fourier transform is one example of an **integral transform**:

$$\hat{f}(\omega) = \int_{\alpha}^{\beta} f(x) K(\omega, x) \, \mathrm{d}x,$$

where K is called the **kernel** of the transform. The Fourier transform has kernel $e^{-i\omega x}$, while the Laplace transform has kernel e^{-sx}.

Example 29.4 If $f(x) = 1$, then

$$\mathcal{F}(1) = \int_{-\infty}^{\infty} e^{-i\omega x}\,\mathrm{d}x,$$

which does not converge.

Theorem 29.5 Let $f : \mathbb{R} \to \mathbb{R}$. If

1. f is piecewise continuous on $\mathbb{R}$, and
2. f is absolutely integrable on $\mathbb{R}$,

then $\mathcal{F}(f(x))$ exists.

Proof. If f is absolutely integrable, then

$$\int_{-\infty}^{\infty} |f(x)|\,\mathrm{d}x < \infty.$$

Since $|e^{-i\omega x}| = 1$, we have

$$\left|\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,\mathrm{d}x\right| \leq \int_{-\infty}^{\infty} |f(x)|\,\mathrm{d}x.$$

Hence the Fourier transform converges. □

The functions x, $\sin x$, and e^x are not absolutely integrable on $\mathbb{R}$, so this theorem does not give them ordinary Fourier transforms. Some non-integrable functions do have Fourier transforms in a generalized sense, but that requires distribution theory.

Theorem 29.6 If $\hat{f}$ and $\hat{g}$ exist, then for $\alpha, \beta \in \mathbb{R}$,

$$\mathcal{F}(\alpha f + \beta g) = \alpha\hat{f} + \beta\hat{g}.$$

Proof. By linearity of the integral,

$$\begin{aligned}\mathcal{F}(\alpha f+\beta g) &= \int_{-\infty}^{\infty}(\alpha f(x)+\beta g(x))e^{-i\omega x}\,\mathrm{d}x \\ &= \alpha\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,\mathrm{d}x+\beta\int_{-\infty}^{\infty} g(x)e^{-i\omega x}\,\mathrm{d}x \\ &= \alpha\hat{f}(\omega)+\beta\hat{g}(\omega).\end{aligned}$$

□

Lecture 30 — Friday, March 20, 2015

Theorem 30.1 Let f be continuously differentiable on $\mathbb{R}$, and suppose that both f and f' are absolutely integrable. Then

$$\mathcal{F}(f'(x)) = i\omega\hat{f}(\omega).$$

Proof. Integration by parts gives

$$\begin{aligned}\mathcal{F}(f'(x)) &= \int_{-\infty}^{\infty} f'(x)e^{-i\omega x}\,\mathrm{d}x \\ &= \left[f(x)e^{-i\omega x}\right]_{-\infty}^{\infty}+i\omega\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,\mathrm{d}x.\end{aligned}$$

Under the stated hypotheses, $f(x)\to 0$ as $x\to\pm\infty$, so the boundary term vanishes and

$$\mathcal{F}(f'(x)) = i\omega\hat{f}(\omega).$$

□

Corollary 30.2 If f is n times continuously differentiable and $f, f', \ldots, f^{(n)}$ are absolutely integrable, then

$$\mathcal{F}(f^{(n)}(x)) = (i\omega)^n\hat{f}(\omega).$$

Conversely, if f is continuous and absolutely integrable and $F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$ is also absolutely integrable, then

$$\hat{F}(\omega) = \frac{1}{i\omega}\hat{f}(\omega), \qquad \omega \neq 0.$$

In principle, these identities let us convert ordinary differential equations into algebraic equations. For an arbitrary ordinary differential equation, however, its solutions need not be absolutely integrable, so an ordinary Fourier transform may be unavailable.

Example 30.3 Solve

$$y'' + ay' + by = f(x).$$

Solution. Applying the Fourier transform gives

$$(i\omega)^2\hat{y} + a(i\omega)\hat{y} + b\hat{y} = \hat{f}(\omega),$$

so formally

$$\hat{y}(\omega) = \frac{\hat{f}(\omega)}{-\omega^2 + ia\omega + b}.$$

Hence

$$y(x) = \mathcal{F}^{-1}\left\{\frac{\hat{f}(\omega)}{-\omega^2 + ia\omega + b}\right\}.$$

The difficulty is that $\hat{y}$ will not usually exist. However, the same procedure can work well for partial differential equations. □

Theorem 30.4 Let f be continuous and absolutely integrable on $\mathbb{R}$ with Fourier transform $\hat{f}(\omega)$. Then

$$\mathcal{F}(f(x-a)) = e^{-i\omega a}\hat{f}(\omega).$$

Proof. Let $t = x - a$. Then

$$\begin{aligned}
\mathcal{F}(f(x-a)) &= \int_{-\infty}^{\infty} f(x-a)e^{-i\omega x}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} f(t)e^{-i\omega(t+a)}\,\mathrm{d}t \\
&= e^{-i\omega a}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t
\end{aligned}$$

$$= e^{-i\omega a}\hat{f}(\omega).$$

□

Corollary 30.5

$$\mathcal{F}^{-1}\{e^{-i\omega a}\hat{f}(\omega)\} = f(x-a).$$

Proof. This follows from Theorem 30.4 and the fact that $\mathcal{F}^{-1}\{\hat{f}(\omega)\} = f(x)$. □

Example 30.6 Solve the unidirectional wave equation

$$u_t + \alpha u_x = 0.$$

Solution. Define

$$\hat{u}(\omega,t) = \mathcal{F}(u(x,t)) = \int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\,\mathrm{d}x.$$

Then

$$\mathcal{F}(u_t) = \hat{u}_t \qquad \text{and} \qquad \mathcal{F}(\alpha u_x) = \alpha(i\omega)\hat{u}.$$

So the partial differential equation becomes

$$\hat{u}_t + i\alpha\omega\hat{u} = 0.$$

For fixed ω, this is an ordinary differential equation in t, so

$$\hat{u}(\omega,t) = \hat{G}(\omega)e^{-i\alpha\omega t}.$$

Therefore

$$u(x,t) = \mathcal{F}^{-1}\{\hat{G}(\omega)e^{-i\alpha\omega t}\} = G(x-\alpha t).$$

This agrees with the solution found by characteristics. In that sense, the answer remains valid even in cases where $\hat{G}$ is only formal and does not exist as an ordinary Fourier transform. □

Definition 30.7 The **convolution** of two functions f and g is

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-\tau)g(\tau)\,\mathrm{d}\tau.$$

When f and g are absolutely integrable, the convolution exists almost everywhere. For **causal functions**, which vanish for $x < 0$, the integral reduces to a finite interval:

$$(f * g)(x) = \int_0^x f(x - \tau) g(\tau) \, \mathrm{d}\tau, \qquad x \geq 0,$$

provided the integrand is locally integrable.

Lecture 31 — Monday, March 23, 2015

Example 31.1 Let

$$f(x) = \begin{cases} 0, & x < 0, \\ x, & x \geq 0, \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 0, & x < 0, \\ e^{-x}, & x \geq 0. \end{cases}$$

Then

$$\begin{aligned} (f * g)(x) &= \int_0^x (x - \tau) e^{-\tau} \, \mathrm{d}\tau \\ &= \left[-(x - \tau) e^{-\tau} \right]_0^x - \int_0^x e^{-\tau} \, \mathrm{d}\tau \\ &= x + \left[e^{-\tau} \right]_0^x \\ &= x + e^{-x} - 1. \end{aligned}$$

Like multiplication, convolution is commutative and distributes over addition:

$$f * g = g * f \qquad \text{and} \qquad (\alpha f + \beta g) * h = \alpha (f * h) + \beta (g * h).$$

Theorem 31.2 (Convolution theorem) If f and g are absolutely integrable, then

$$\mathcal{F}(f * g) = \hat{f}(\omega) \hat{g}(\omega).$$

At points where Fourier inversion applies, equivalently,

$$\mathcal{F}^{-1}\{\hat{f}(\omega) \hat{g}(\omega)\} = f * g.$$

Proof. By definition and the substitution $\xi = x - \tau$,

$$
\begin{aligned}
\mathcal{F}(f * g) &= \int_{-\infty}^{\infty} (f * g)(x) e^{-i\omega x}\, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-\tau) g(\tau) e^{-i\omega x}\, \mathrm{d}\tau\, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi) g(\tau) e^{-i\omega(\xi+\tau)}\, \mathrm{d}\xi\, \mathrm{d}\tau \\
&= \left(\int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi}\, \mathrm{d}\xi \right) \left(\int_{-\infty}^{\infty} g(\tau) e^{-i\omega\tau}\, \mathrm{d}\tau \right) \\
&= \hat{f}(\omega)\hat{g}(\omega).
\end{aligned}
$$

□

Example 31.3 Solve

$$u_t = \gamma u_{xx}, \qquad u(x,0) = f(x), \qquad \text{and} \qquad u(x,t) \to 0 \text{ as } x \to \pm\infty.$$

Solution. Assume f and the resulting solution are sufficiently regular and integrable to justify the transforms and differentiations below. Decay to zero at spatial infinity alone would not be enough to guarantee existence of a Fourier transform. Let

$$\hat{u}(\omega, t) = \mathcal{F}(u(x,t)).$$

Then

$$u_t \mapsto \hat{u}_t \qquad \text{and} \qquad u_{xx} \mapsto (i\omega)^2 \hat{u} = -\omega^2 \hat{u},$$

so the transformed initial value problem is

$$\hat{u}_t = -\gamma\omega^2 \hat{u} \qquad \text{and} \qquad \hat{u}(\omega, 0) = \hat{f}(\omega).$$

Solving the ordinary differential equation in t gives

$$\hat{u}(\omega, t) = H(\omega) e^{-\gamma\omega^2 t}.$$

The initial condition gives $H(\omega) = \hat{f}(\omega)$, so

$$\hat{u}(\omega, t) = \hat{f}(\omega) e^{-\gamma\omega^2 t}.$$

Hence
$$u(x,t) = \mathcal{F}^{-1}\{\hat{f}(\omega)e^{-\gamma\omega^2 t}\} = f * \mathcal{F}^{-1}\{e^{-\gamma\omega^2 t}\}.$$

Let
$$g(x) = \mathcal{F}^{-1}\{e^{-\gamma\omega^2 t}\}.$$

Using the inverse transform convention compatible with our definition and the standard Fourier transform of a Gaussian,
$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\gamma t\omega^2 + i\omega x}\,\mathrm{d}\omega = \frac{1}{2\sqrt{\pi\gamma t}}e^{-x^2/(4\gamma t)}.$$

Therefore
$$u(x,t) = (f*g)(x) = \frac{1}{2\sqrt{\pi\gamma t}}\int_{-\infty}^{\infty} f(x-\tau)e^{-\tau^2/(4\gamma t)}\,\mathrm{d}\tau.$$

Equivalently, let $\tau = -2y\sqrt{\gamma t}$, so $\mathrm{d}\tau = -2\sqrt{\gamma t}\,\mathrm{d}y$. As τ goes from $-\infty$ to ∞, y goes from ∞ to $-\infty$, and hence
$$\begin{aligned}
u(x,t) &= \frac{1}{2\sqrt{\pi\gamma t}}\int_{\infty}^{-\infty} f(x+2y\sqrt{\gamma t})e^{-y^2}(-2\sqrt{\gamma t})\,\mathrm{d}y \\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+2y\sqrt{\gamma t})e^{-y^2}\,\mathrm{d}y.
\end{aligned}$$

□

Lecture 32 — Wednesday, March 25, 2015

Example 32.1 Suppose $f(x) = \mathbb{1}_{[-1,1]}(x)$. Then
$$f(x+2y\sqrt{\gamma t}) = \begin{cases} 1, & y \in \left[-\frac{1+x}{2\sqrt{\gamma t}}, \frac{1-x}{2\sqrt{\gamma t}}\right], \\ 0, & \text{otherwise.}\end{cases}$$

So
$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-(1+x)/(2\sqrt{\gamma t})}^{(1-x)/(2\sqrt{\gamma t})} e^{-y^2}\,\mathrm{d}y$$

$$= \frac{1}{2}\left[\operatorname{erf}\left(\frac{1-x}{2\sqrt{\gamma t}}\right) - \operatorname{erf}\left(\frac{-1-x}{2\sqrt{\gamma t}}\right)\right],$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{d}t.$$

5. Black–Scholes and Derivative Pricing

The Black–Scholes Equation

In this section, we focus on one particular stochastic differential equation: the Black–Scholes equation.

A **European call option** is the right to purchase an underlying asset for an agreed-upon price K at an agreed-upon time T. We call K the **strike price** and T the **expiry time**. The value of the option depends on the stock price S and on time t, so we write the option value as

$$F(S, t).$$

More naturally, it depends on the time remaining, $T - t$.

We make the following simplifying assumptions:

Assumption 1. Trading is continuous.

Assumption 2. Assets are infinitesimally divisible.

Assumption 3. There are no transaction costs.

Assumption 4. Short selling is permitted.

Assumption 5. Assets pay no dividends.

The stock price depends on time, so write $S = S(t)$. We assume the change in S has both a deterministic and a random component, and that the deterministic part is Malthusian. Thus if

$$\Delta S = S(t + \Delta t) - S(t),$$

then

$$\Delta S = f_{\text{det}}(S, t, \Delta t) + f_{\text{rand}}(S, t, \Delta t).$$

The Malthusian assumption gives

$$f_{\text{det}} = \mu S \Delta t.$$

For the random contribution we assume

$$f_{\text{rand}} = \sigma S \Delta W(t),$$

where $\Delta W(t) = W(t + \Delta t) - W(t)$. A **Wiener process** has continuous paths and independent increments, with

$$\Delta W(t) \sim \mathcal{N}(0, \Delta t).$$

Hence

$$\Delta S \approx S\big(\mu \Delta t + \sigma \Delta W(t)\big).$$

Letting $\Delta t \to 0$, it is customary to write this as the stochastic differential equation

$$\mathrm{d}S = \mu S \,\mathrm{d}t + \sigma S \,\mathrm{d}W(t).$$

We call μ the **drift** (the stock's expected instantaneous rate of return) and σ the **volatility**. This is our sixth assumption:

Assumption 6. The stock price S follows the stochastic differential equation

$$\mathrm{d}S = \mu S \,\mathrm{d}t + \sigma S \,\mathrm{d}W(t).$$

Here $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants.

One useful fact is that $[\Delta W(t)]^2$ has mean Δt and variance $2(\Delta t)^2$.

Lecture 33 — Friday, March 27, 2015

To incorporate our assumptions about $S(t)$ into $F(S, t)$, recall the second-order part of the Taylor expansion of a function of two variables:

$$f(x, y) = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot (x - x_0, y - y_0)$$

$$+\frac{1}{2}(x-x_0,y-y_0)\nabla^2 f(x_0,y_0)(x-x_0,y-y_0)^T + \text{higher-order terms.}$$

In increment form, this says

$$\Delta f = \nabla f \cdot \Delta x + \frac{1}{2}\Delta x^T \nabla^2 f\, \Delta x + \text{higher-order terms.}$$

Apply this to $F(S,t)$, using

$$\Delta S = \mu S \Delta t + \sigma S \Delta W(t).$$

Then

$$\begin{aligned}
\Delta F &= F_S[\mu S\Delta t + \sigma S \Delta W(t)] + F_t \Delta t \\
&\quad + \frac{1}{2}F_{SS}[\mu S\Delta t + \sigma S\Delta W(t)]^2 + F_{St}[\mu S\Delta t + \sigma S \Delta W(t)]\Delta t + \frac{1}{2}F_{tt}(\Delta t)^2 + \cdots \\
&= F_S \mu S \Delta t + F_S \sigma S \Delta W(t) + F_t \Delta t + \frac{1}{2}F_{SS}\mu^2 S^2 (\Delta t)^2 \\
&\quad + F_{SS}\mu\sigma S^2 \Delta t\, \Delta W(t) + \frac{1}{2}F_{SS}\sigma^2 S^2 [\Delta W(t)]^2 \\
&\quad + F_{St}\mu S(\Delta t)^2 + F_{St}\sigma S \Delta W(t)\Delta t + \frac{1}{2}F_{tt}(\Delta t)^2 + \cdots .
\end{aligned}$$

The rigorous bookkeeping is supplied by Itô calculus. A Brownian increment is of stochastic order $\sqrt{\Delta t}$, so terms of orders $(\Delta t)^2$ and $\Delta t\, \Delta W(t)$ disappear in the limit. Brownian quadratic variation satisfies

$$\sum_j [W(t_{j+1}) - W(t_j)]^2 \longrightarrow t$$

as the mesh of a partition of $[0,t]$ tends to zero; this is encoded by the Itô rule $(\mathrm{d}W)^2 = \mathrm{d}t$. It does *not* mean that $[\Delta W(t)]^2/\Delta t$ converges to 1 for each individual increment. With this interpretation, the surviving terms give

$$\Delta F = F_S \mu S \Delta t + F_S \sigma S \Delta W(t) + F_t \Delta t + \frac{1}{2}F_{SS}\sigma^2 S^2 \Delta t.$$

In stochastic differential form,

$$\mathrm{d}F = \sigma S F_S \ \mathrm{d}W(t) + \left(F_t + \mu S F_S + \frac{1}{2}\sigma^2 S^2 F_{SS}\right)\mathrm{d}t.$$

To eliminate the stochastic term, we make two more assumptions:

Assumption 7. Money can be borrowed or lent at the same constant continuously compounded interest rate r.

Assumption 8. Arbitrage is immediately eliminated, so no arbitrage is possible.

The last assumption rules out a trading strategy with zero initial cost, no possibility of loss, and a positive probability of profit.

Derivation of the Black–Scholes Partial Differential Equation

Consider a self-financing portfolio containing Δ shares of stock and an amount B in the risk-free account. Its value and change are

$$V = \Delta S + B \qquad \text{and} \qquad \mathrm{d}V = \Delta\, \mathrm{d}S + rB\,\mathrm{d}t.$$

Although Δ and B may change with time, self-financing means that purchases and sales of stock are paid for from within the portfolio, which is why no separate $S\,\mathrm{d}\Delta$ or $\mathrm{d}B$ term appears in the gain equation.

To replicate the option, require $V = F$ and choose

$$\Delta = F_S.$$

The stochastic term in $\mathrm{d}V$ is then $\sigma S F_S\,\mathrm{d}W$, exactly the stochastic term in $\mathrm{d}F$. Since $V = F$, the risk-free holding must be

$$B = F - \Delta S = F - SF_S.$$

Equating the drift terms in $\mathrm{d}V$ and $\mathrm{d}F$ gives

$$\mu S F_S + r(F - SF_S) = F_t + \mu S F_S + \frac{1}{2}\sigma^2 S^2 F_{SS}.$$

After cancelling the stock-drift term,

$$r(F - SF_S) = F_t + \frac{1}{2}\sigma^2 S^2 F_{SS},$$

or equivalently

$$\frac{1}{2}\sigma^2 S^2 F_{SS} + rSF_S + F_t - rF = 0.$$

This is the Black–Scholes equation. Under the stated market assumptions it applies to any sufficiently smooth derivative written on the stock; the terminal and boundary conditions specify that the derivative here is a European call.

Boundary and Terminal Conditions

What is the domain of $F(S,t)$? We must have

$$S \geq 0 \qquad \text{and} \qquad 0 \leq t \leq T.$$

At time $t = T$: If $S(T) > K$, the owner exercises the option and receives the payoff $S - K$.

If $S(T) < K$, the owner lets the option expire, so it is worthless. Therefore

$$F(S,T) = \max(S-K, 0) = (S-K)^+.$$

Next, if $S = 0$, then from

$$\mathrm{d}S = \mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}W(t),$$

the price remains 0, so

$$F(0,t) = 0.$$

On the other hand, a call with a very large stock price behaves like the stock minus the present value of the strike:

$$F(S,t) - S \longrightarrow -Ke^{-r(T-t)} \qquad \text{as } S \to \infty.$$

Thus the problem is

$$\begin{aligned} F_t &= -\frac{1}{2}\sigma^2 S^2 F_{SS} - rSF_S + rF, \\ F(S,T) &= (S-K)^+, \\ F(0,t) &= 0, \\ F(S,t) - S &\to -Ke^{-r(T-t)} \quad \text{as } S \to \infty. \end{aligned}$$

The partial differential equation is therefore posed on a half-strip, with a terminal payoff at $t = T$ and spatial boundary behaviour at $S = 0$ and $S \to \infty$. Figure 9 collects these conditions in one picture.

There are several ways to simplify this problem.

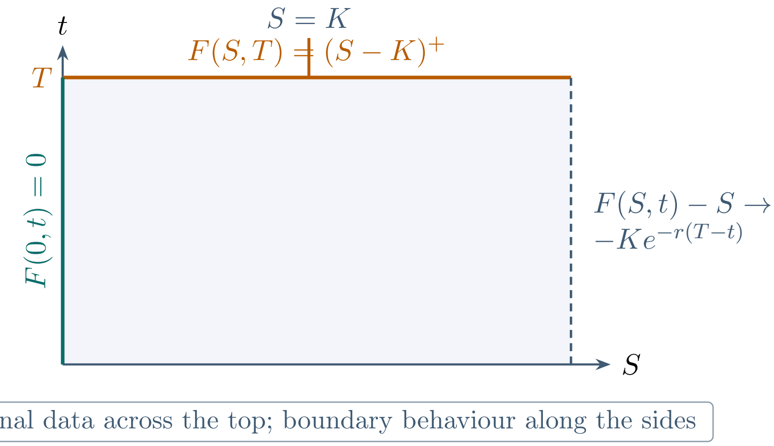

FIGURE 9

Lecture 34 — Monday, March 30, 2015

Reduction to a Constant-Coefficient Partial Differential Equation

We now transform the Black–Scholes partial differential equation into the heat equation. There are several useful simplifications behind the substitution.

- Normalizing time by a factor of $\sigma^2/2$ normalizes the coefficient of S^2F_{SS}. For example, if $\tau = \frac{1}{2}\sigma^2 t$, then
$$F_t = \frac{\partial F}{\partial t} = \frac{\partial F}{\partial \tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{1}{2}\sigma^2 F_\tau,$$
so the equation has the form $F_\tau = -S^2F_{SS} + \cdots$.
- Reversing time turns the terminal condition at $t = T$ into an initial condition. Combining the first two ideas by taking $\tau = \frac{1}{2}\sigma^2(T-t)$ changes the sign and gives $F_\tau = S^2F_{SS} + \cdots$.
- Dividing both F and S by K removes the strike price from the terminal condition, since
$$\frac{F(S,T)}{K} = \left(\frac{S}{K} - 1\right)^+.$$
- The substitution $S/K = e^x$ is analogous to the standard ordinary differential equation substitution which turns Euler-type equations into constant-coefficient equations.

We combine these ideas and set

$$x = \log\left(\frac{S}{K}\right), \qquad \tau = \frac{\sigma^2}{2}(T-t), \qquad \text{and} \qquad F(S,t) = Kv(x,\tau).$$

Equivalently,

$$S = Ke^x, \qquad t = T - \frac{2\tau}{\sigma^2}, \qquad \text{and} \qquad F = Kv.$$

Compute the derivatives:

$$\begin{aligned}
F_t &= Kv_\tau \frac{\mathrm{d}\tau}{\mathrm{d}t} = -\frac{\sigma^2 K}{2} v_\tau, \\
F_S &= Kv_x \frac{\mathrm{d}x}{\mathrm{d}S} = \frac{K}{S} v_x, \\
F_{SS} &= \frac{\partial}{\partial S}\left(\frac{K}{S} v_x\right) = -\frac{K}{S^2} v_x + \frac{K}{S}\frac{\partial v_x}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}S} = \frac{K}{S^2}(v_{xx} - v_x).
\end{aligned}$$

Substituting into

$$\frac{1}{2}\sigma^2 S^2 F_{SS} + rSF_S + F_t - rF = 0$$

gives

$$\frac{\sigma^2 K}{2}(v_{xx} - v_x) + rKv_x - \frac{\sigma^2 K}{2} v_\tau - rKv = 0.$$

After dividing by $K\sigma^2/2$, we obtain

$$v_{xx} + \left(\frac{2r}{\sigma^2} - 1\right) v_x - v_\tau - \frac{2r}{\sigma^2} v = 0.$$

Let

$$\delta = \frac{2r}{\sigma^2}.$$

Then

$$v_\tau = v_{xx} + (\delta - 1)v_x - \delta v.$$

The conditions transform as follows:

$$\begin{aligned}
F(S,T) = (S-K)^+ &\implies v(x,0) = (e^x - 1)^+, \\
F(0,t) = 0 &\implies \lim_{x\to-\infty} v(x,\tau) = 0, \\
F(S,t) - S \to -Ke^{-r(T-t)} \text{ as } S \to \infty &\implies v(x,\tau) - e^x \to -e^{-\delta\tau} \quad \text{as } x \to \infty.
\end{aligned}$$

Now let

$$v(x,\tau) = e^{ax-b\tau}u(x,\tau),$$

where

$$a = \frac{1-\delta}{2} \qquad \text{and} \qquad b = \frac{(1+\delta)^2}{4}.$$

Then

$$\begin{aligned} v_\tau &= e^{ax-b\tau}(u_\tau - bu), \\ v_x &= e^{ax-b\tau}(u_x + au), \\ v_{xx} &= e^{ax-b\tau}(u_{xx} + 2au_x + a^2u). \end{aligned}$$

Substituting into

$$v_\tau = v_{xx} + (\delta-1)v_x - \delta v$$

and cancelling the common factor $e^{ax-b\tau}$ gives

$$u_\tau - bu = u_{xx} + 2au_x + a^2u + (\delta-1)(u_x + au) - \delta u.$$

Thus

$$u_\tau = u_{xx} + \big(2a + \delta - 1\big)u_x + \big(a^2 + (\delta-1)a - \delta + b\big)u.$$

With

$$a = \frac{1-\delta}{2} \qquad \text{and} \qquad b = \frac{(1+\delta)^2}{4},$$

the coefficients of u_x and u vanish. Thus

$$u_\tau = u_{xx}.$$

So we have reduced the Black–Scholes partial differential equation to the heat equation.

The initial condition becomes

$$e^{ax}u(x,0) = (e^x - 1)^+,$$

so

$$u(x,0) = \left(e^{(1-a)x} - e^{-ax}\right)^+.$$

We will now solve this heat equation.

Lecture 35 — Wednesday, April 01, 2015

Heat Equation Solution

Recall the heat equation solution on the whole line:

$$u(x,\tau) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2y\sqrt{\tau}) e^{-y^2}\, \mathrm{d}y,$$

where

$$f(x) = \left(e^{(1-a)x} - e^{-ax}\right)^{+}.$$

This transformed payoff is not absolutely integrable, so the earlier Fourier-transform derivation does not apply to it verbatim. Nevertheless, the Gaussian integral converges because f grows at most exponentially, and the verification at the end of the section justifies the heat-kernel solution directly (or, equivalently, one may first truncate the payoff and pass to a limit).

The Gaussian kernel spreads as τ grows, averaging the initial condition over an increasingly wide neighbourhood. Figure 10 shows the accompanying decrease in peak height; the area under each schematic kernel is kept on the same visual scale.

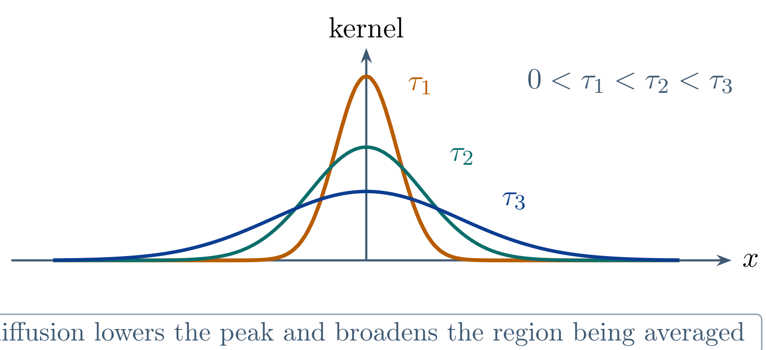

FIGURE 10

Hence

$$u(x,\tau) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(e^{(1-a)(x+2y\sqrt{\tau})} - e^{-a(x+2y\sqrt{\tau})}\right)^{+} e^{-y^2}\, \mathrm{d}y.$$

The integrand is positive exactly when

$$\begin{aligned} e^{(1-a)(x+2y\sqrt{\tau})} - e^{-a(x+2y\sqrt{\tau})} > 0 \\ &\iff (1-a)(x+2y\sqrt{\tau}) > -a(x+2y\sqrt{\tau}) \\ &\iff x + 2y\sqrt{\tau} > 0, \end{aligned}$$

so the lower limit becomes $-x/(2\sqrt{\tau})$. Therefore

$$u(x,\tau) = \frac{e^{(1-a)x}}{\sqrt{\pi}} \int_{-x/(2\sqrt{\tau})}^{\infty} e^{2(1-a)\sqrt{\tau}\,y - y^2}\,\mathrm{d}y \qquad - \frac{e^{-ax}}{\sqrt{\pi}} \int_{-x/(2\sqrt{\tau})}^{\infty} e^{-2a\sqrt{\tau}\,y-y^2}\,\mathrm{d}y.$$

For the first integral, completing the square gives

$$2(1-a)\sqrt{\tau}\,y - y^2 = -(y-(1-a)\sqrt{\tau})^2 + (1-a)^2\tau.$$

With

$$w = -\sqrt{2}\big(y - (1-a)\sqrt{\tau}\big),$$

the lower limit $y = -x/(2\sqrt{\tau})$ becomes

$$w = \frac{x + 2(1-a)\tau}{\sqrt{2\tau}},$$

and $y \to \infty$ corresponds to $w \to -\infty$. Therefore

$$\frac{e^{(1-a)x}}{\sqrt{\pi}} \int_{-x/(2\sqrt{\tau})}^{\infty} e^{2(1-a)\sqrt{\tau}\,y-y^2}\,\mathrm{d}y = e^{(1-a)x+(1-a)^2\tau}\,\Phi\left(\frac{x+2(1-a)\tau}{\sqrt{2\tau}}\right).$$

For the second integral,

$$-2a\sqrt{\tau}\,y - y^2 = -(y + a\sqrt{\tau})^2 + a^2\tau.$$

With

$$w = -\sqrt{2}\big(y + a\sqrt{\tau}\big),$$

the lower limit $y = -x/(2\sqrt{\tau})$ becomes

$$w = \frac{x - 2a\tau}{\sqrt{2\tau}},$$

so the second term becomes

$$\frac{e^{-ax}}{\sqrt{\pi}} \int_{-x/(2\sqrt{\tau})}^{\infty} e^{-2a\sqrt{\tau}\,y-y^2}\,\mathrm{d}y = e^{-ax+a^2\tau}\,\Phi\left(\frac{x-2a\tau}{\sqrt{2\tau}}\right).$$

Converting to the standard normal cumulative distribution function

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-w^2/2}\,\mathrm{d}w,$$

we get

$$u(x,\tau) = e^{(1-a)x+(1-a)^2\tau}\Phi\left(\frac{x+2(1-a)\tau}{\sqrt{2\tau}}\right) - e^{-ax+a^2\tau}\Phi\left(\frac{x-2a\tau}{\sqrt{2\tau}}\right).$$

Lecture 36 — Monday, April 06, 2015

Now return to the original variables. Since

$$F(S,t) = Ke^{ax-b\tau}u(x,\tau), \qquad x = \log\left(\frac{S}{K}\right), \qquad \text{and} \qquad \tau = \frac{\sigma^2}{2}(T-t),$$

the first prefactor simplifies as

$$Ke^{ax-b\tau}e^{(1-a)x+(1-a)^2\tau} = Ke^{x}e^{((1-a)^2-b)\tau} = S,$$

because $b = (1-a)^2$. The second prefactor is

$$Ke^{ax-b\tau}e^{-ax+a^2\tau} = Ke^{(a^2-b)\tau} = Ke^{-\delta\tau} = Ke^{-r(T-t)}.$$

Also,

$$\frac{x+2(1-a)\tau}{\sqrt{2\tau}} = \frac{\log(S/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

and

$$\frac{x-2a\tau}{\sqrt{2\tau}} = \frac{\log(S/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Therefore

$$F(S,t) = S\Phi\left(\frac{\log(S/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}\Phi\left(\frac{\log(S/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right).$$

This is the **Black–Scholes formula** for a European call option.

If we define

$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = \frac{\log(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

then

$$F(S,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2).$$

This completes the derivation of the Black–Scholes pricing formula.

Verification of the Heat Kernel Formula

Finally, we verify the heat kernel formula used above. Consider

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2y\sqrt{\gamma t})e^{-y^2}\, \mathrm{d}y.$$

The initial condition is immediate:

$$u(x,0) = \frac{f(x)}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2}\, \mathrm{d}y = f(x).$$

Now assume that f is twice differentiable and that the boundary terms below vanish. Differentiating under the integral sign gives

$$\begin{aligned} u_t &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f'(x + 2y\sqrt{\gamma t})\frac{y\sqrt{\gamma}}{\sqrt{t}} e^{-y^2}\, \mathrm{d}y \\ &= \frac{\sqrt{\gamma}}{\sqrt{\pi t}} \int_{-\infty}^{\infty} y f'(x + 2y\sqrt{\gamma t})e^{-y^2}\, \mathrm{d}y, \end{aligned}$$

and

$$u_{xx} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f''(x + 2y\sqrt{\gamma t})e^{-y^2}\, \mathrm{d}y.$$

To relate these, integrate by parts in the expression for u_{xx}, taking

$$U = e^{-y^2} \qquad \text{and} \qquad \mathrm{d}V = f''(x + 2y\sqrt{\gamma t})\, \mathrm{d}y.$$

Then

$$\mathrm{d}U = -2ye^{-y^2}\, \mathrm{d}y \qquad \text{and} \qquad V = \frac{1}{2\sqrt{\gamma t}} f'(x + 2y\sqrt{\gamma t}),$$

so

$$u_{xx} = \frac{1}{\sqrt{\pi}} \left[\frac{e^{-y^2}}{2\sqrt{\gamma t}} f'(x + 2y\sqrt{\gamma t}) \right]_{-\infty}^{\infty} + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{y}{\sqrt{\gamma t}} f'(x + 2y\sqrt{\gamma t}) e^{-y^2} \, \mathrm{d}y.$$

The boundary term is zero, for example, if

$$e^{-y^2} f'(x + 2y\sqrt{\gamma t}) \to 0 \quad \text{as } y \to \pm\infty.$$

In that case,

$$\gamma u_{xx} = \frac{\sqrt{\gamma}}{\sqrt{\pi t}} \int_{-\infty}^{\infty} y f'(x + 2y\sqrt{\gamma t}) e^{-y^2} \, \mathrm{d}y = u_t.$$

Thus the formula satisfies the heat equation $u_t = \gamma u_{xx}$ together with the initial condition $u(x, 0) = f(x)$.

Appendix: Lecture and Tutorial Index