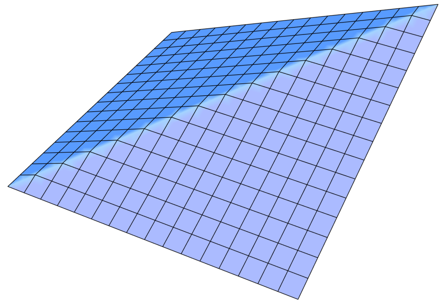

PMATH 352: Complex Analysis

Prof. Frank Zorzitto • Winter 2014 • University of Waterloo
MWF 10:30–11:20AM

Transcribed, LaTeXed, and edited by Rob Zimmerman

Note: These are personal lecture notes, not official course materials. They may contain errors (which by default you should attribute to me, Rob Zimmerman, rather than the course instructor). The permanent home of these — and all of my other lecture notes — is `https://rob-zimmerman.github.io/coursenotes`.

Contents

1. Holomorphic Functions and Complex Derivatives

Lecture 1 — Monday, January 06, 2014

Review of Complex Algebra

We begin with some familiar concepts.

Definition 1.1 Adjoining $i = \sqrt{-1}$ to $\mathbb{R}$, we obtain the field of **complex numbers**

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}.$$

We identify $\mathbb{C}$ with $\mathbb{R}^2$ by mapping $x+iy$ to the vector $\begin{pmatrix} x \\ y \end{pmatrix}$. Given $z = x+iy$ in $\mathbb{C}$, its **conjugate** is $\bar{z} = x - iy$ and its **absolute value** is

$$|z| = \sqrt{x^2+y^2} = \sqrt{z\bar{z}} = \text{distance from } z \text{ to the origin.}$$

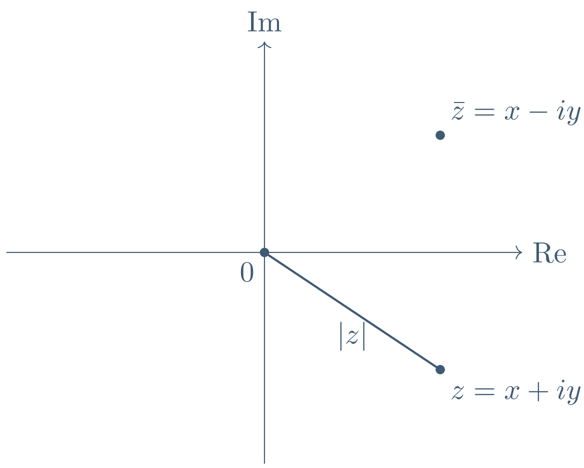

Figure 1

Figure 1 illustrates the setup.

Definition 1.2 If $z = x+iy$ and $p = a+ib$ in $\mathbb{C}$, then the **distance** from z to p is

$$|z-p| = \sqrt{(x-a)^2+(y-b)^2}.$$

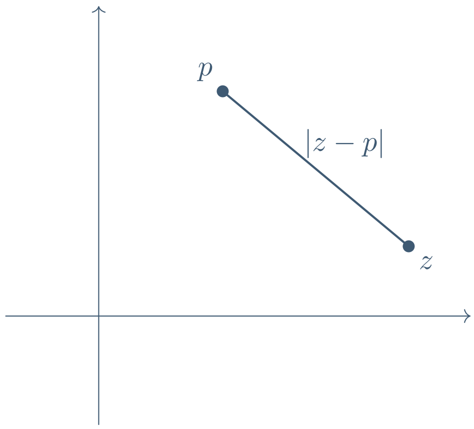

FIGURE 2

The construction is shown in Figure 2.

Definition 1.3 If $p \in \mathbb{C}$ and $r > 0$, the **disk** of center p and radius r is

$$D_p(r) = \{z \in \mathbb{C} : |z - p| < r\}.$$

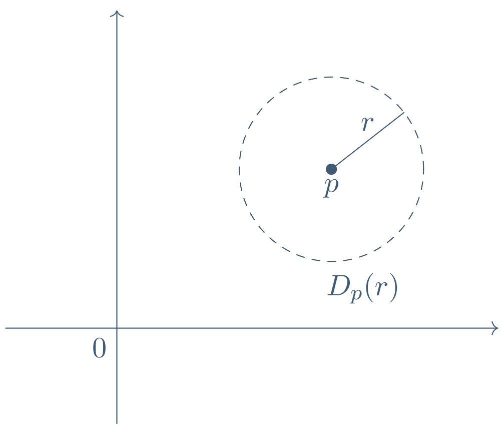

FIGURE 3

Figure 3 records the relevant geometry.

Definition 1.4 A set $\Omega \subseteq \mathbb{C}$ is called **open** provided that for every $p \in \Omega$, there is a radius $r > 0$ such that $D_p(r) \subseteq \Omega$.

Example 1.5 $\Omega = \{z \in \mathbb{C} : |z - i| > 1\}$ is open.

Proof. Let $p \in \Omega$; we seek $r > 0$ with $D_p(r) \subseteq \Omega$. We have $|p - i| > 1$ and we want $r > 0$ such that every $z \in D_p(r)$ also satisfies $|z - i| > 1$. Since $|p - i| > 1$, let us take $r = |p - i| - 1 > 0$.

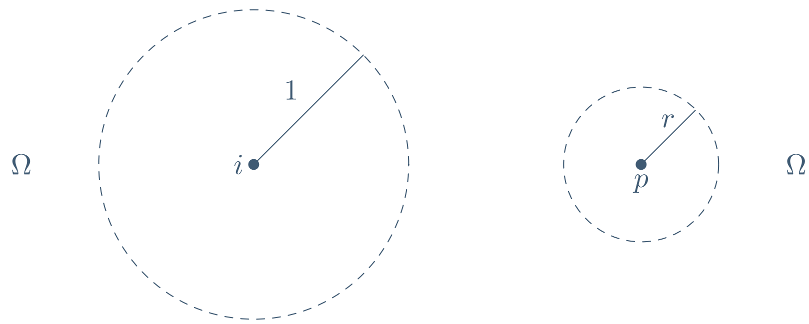

FIGURE 4

The configuration appears in Figure 4.

Then for z in $D_p(r)$, we know $|z - p| < |p - i| - 1$. Thus, $1 < |p - i| - |z - p| \leq |z - i|$. Here we used the triangle inequality for subtraction, which can be written as follows:

$$|u - v| \leq |u - w| + |w - v| \quad \text{by the usual triangle inequality}$$

Thus $|u - v| - |w - v| \leq |u - w|$:

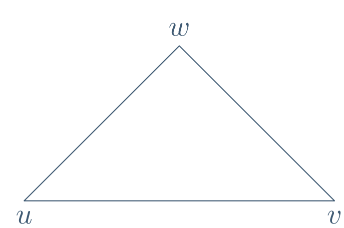

FIGURE 5

Figure 5 illustrates the setup.

□

Example 1.6 The closed disk $\overline{D_0(1)} = \{z : |z| \leq 1\}$ is not open.

Proof. Indeed, take $p = 1$ in $D_0(1)$. For any $r > 0$, the disk $D_1(r)$ contains points z such that $|z| > 1$ (for example, $z = 1 + r/2$):

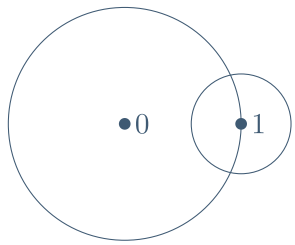

FIGURE 6

The construction is shown in Figure 6. □

Typical open sets include disks $D_p(r)$:

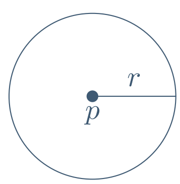

FIGURE 7

Figure 7 records the relevant geometry.

They also include all of $\mathbb{C}$ and the punctured plane $\mathbb{C} \setminus \{0\}$:

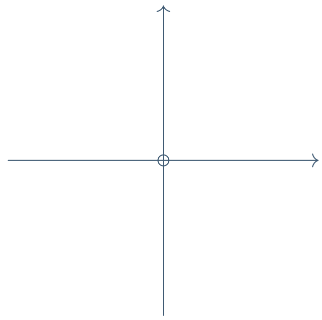

FIGURE 8

The configuration appears in Figure 8.

The **annuli** $\Omega = \{z : r < |z - p| < R\}$ are open:

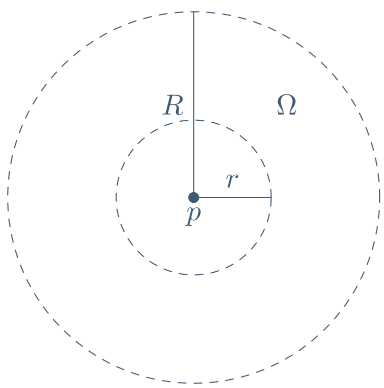

FIGURE 9

Figure 9 illustrates the setup.

We also have open **half-planes** such as $H = \{z : \operatorname{Re} z > 0\}$, where for $z = x + iy$, the real part is $\operatorname{Re}(z) = x$:

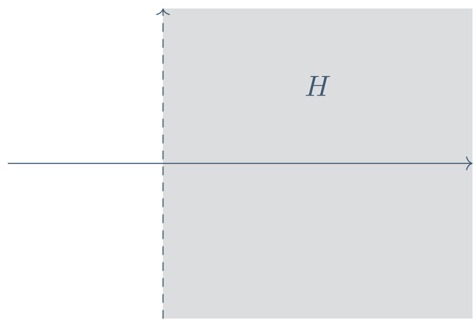

FIGURE 10

The construction is shown in Figure 10.

Finally, the **strips** $S = \{z = x + iy : a < x < b\}$ are open as well:

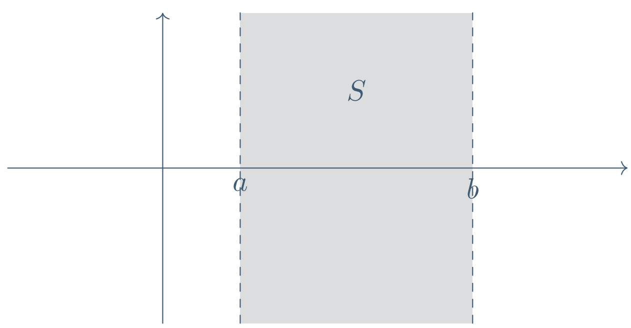

FIGURE 11

Figure 11 records the relevant geometry.

Our goal is to study functions $f : \Omega \to \mathbb{C}$ where Ω is an open set:

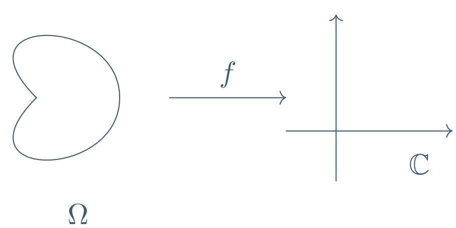

FIGURE 12

The configuration appears in Figure 12.

Definition 1.7 If $f : \Omega \to \mathbb{C}$ and $p \in \Omega$, we say that $f(z) \to w$ as $z \to p$ provided every $\varepsilon > 0$ comes with a $\delta > 0$ such that $|f(z) - w| < \varepsilon$ when $0 < |z - p| < \delta$. If, in addition, $f(p) = w$, we say that f is **continous** at p.

In the language of disks, this says that every ε-disk $D_w(\varepsilon)$ has a corresponding δ-disk $D_p(\delta)$ such that $f(D_p(\delta) \setminus \{p\}) \subseteq D_w(\varepsilon)$.

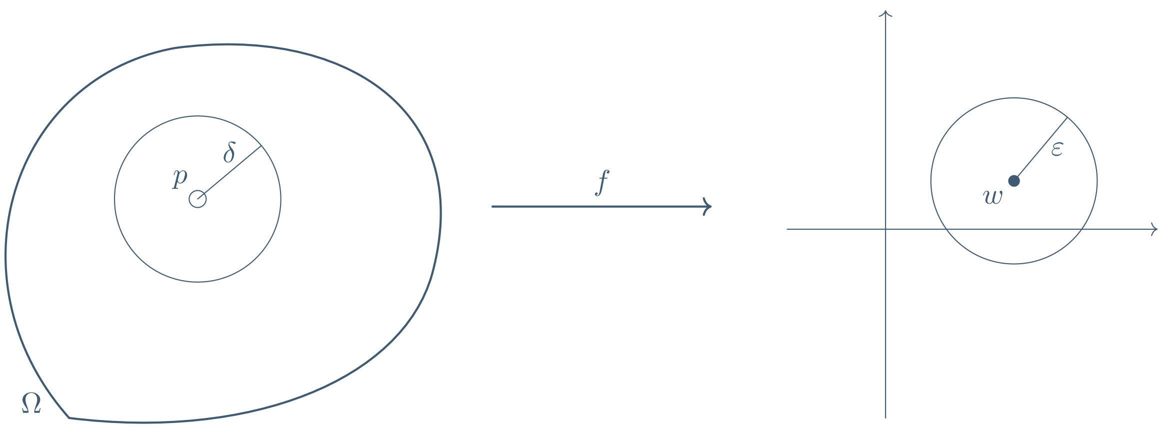

FIGURE 13

Figure 13 illustrates the setup.

Example 1.8 If f is defined on the punctured plane $\mathbb{C} \setminus \{0\}$ by $f(z) = 1/z$, then f is continuous at every $p \in \Omega$ (that is, every $p \neq 0$).

Proof. Fix $\varepsilon > 0$; we will choose $\delta > 0$ such that

$$\left| \frac{1}{z} - \frac{1}{p} \right| < \varepsilon \quad \text{when } |z - p| < \delta.$$

Thus we need $\delta > 0$ such that

$$\frac{1}{|z||p|} |z - p| < \varepsilon \quad \text{when } |z - p| < \delta.$$

To avoid division by zero, we ensure $|z-p| < |p|/2$. We see that if $|z-p| < |p|/2$, then $|z| > |p|/2$, and hence

$$\frac{1}{|z||p|} |z - p| \leq \frac{2}{|p|^2} |z - p|.$$

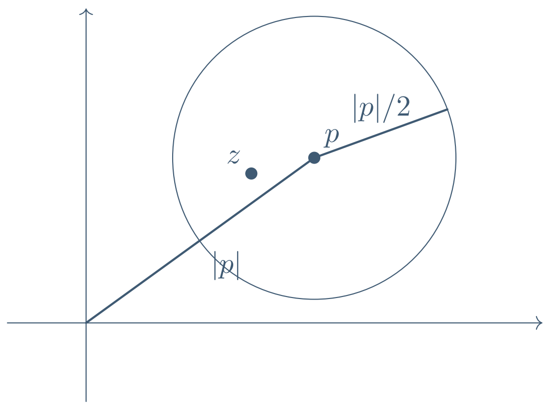

Figure 14

The construction is shown in Figure 14.

Now we can pick a useful δ — namely, $\delta = \min\left\{|p|/2, (|p|^2\varepsilon)/2\right\}$. If $|z-p| < \delta$, then

$$\frac{1}{|z||p|}|z-p| \leq \frac{2}{|p|^2}|z-p| < \frac{2}{|p|^2}\frac{|p|^2\varepsilon}{2} = \varepsilon.$$

□

Lecture 2 — Thursday, January 09, 2014

Complex Derivatives

Let us start with the real case. Recall the standard definition of differentiability in $\mathbb{R}$:

Definition 2.1 If $f : I \to \mathbb{R}$ is defined on an open interval I and $p \in I$, we say that f is **differentiable** at p with derivative m when

$$\frac{f(p+h) - f(p)}{h} \to m \quad \text{as } h \to 0.$$

Proposition 2.2 A function $f : I \to \mathbb{R}$ is differentiable at $p \in I$ with derivative m if and only if there is a function φ defined for all h close to 0 (including $h = 0$) such that for all h close to 0:

(1) $f(p + h) = f(p) + mh + h\varphi(h)$.

(2) $\varphi(h) \to \varphi(0) = 0$ as $h \to 0$ (that is, φ is continuous and vanishes at 0).

Proof. *Forward implication.* If m is the derivative of f at p, let

$$\varphi(h) = \begin{cases} \frac{f(p+h)-f(p)}{h} - m & \text{when } h \neq 0 \\ 0 & \text{when } h = 0 \end{cases}.$$

Then

$$f(p + h) = f(p) + mh + h\varphi(h)$$

for all h close to 0, including $h = 0$. Also, from the definition of φ, $\varphi(h) \to \varphi(0) = 0$ as $h \to 0$.

Reverse implication. Conversely, suppose there is a function φ that satisfies (1) and (2). From (1) we get

$$\frac{f(p + h) - f(p)}{h} - m = \varphi(h)$$

and since $\varphi(h) \to 0$ as $h \to 0$, we see that m is the derivative of f at p. □

The same story goes in the complex case.

Definition 2.3 Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ a complex valued function. We say that f is **complex differentiable** at $p \in \Omega$ with **complex derivative** w when

$$\frac{f(p + h) - f(p)}{h} \to w \quad \text{as } h \to 0.$$

(Here p, h, w are in $\mathbb{C}$, and h is small enough that $p + h \in \Omega$.)

By imitating Proposition 2.2, we obtain the following result:

Proposition 2.4 A function $f : \Omega \to \mathbb{C}$ is complex differentiable at $p \in \Omega$ with derivative w if and only if there is a function $\varphi : \Omega \to \mathbb{C}$ such that:

1. $f(z) = f(p) + w(z-p) + (z-p)\varphi(z)$, and
2. $\varphi(z) \to \varphi(p) = 0$ as $z \to p$.

Proof. Suppose
$$\frac{f(z) - f(p)}{z - p} \to w$$
as $z \to p$. Then
$$\frac{f(z) - f(p) - w(z-p)}{z - p} \to 0$$
as $z \to p$. Put
$$\varphi(z) = \begin{cases} \dfrac{f(z) - f(p) - w(z-p)}{z-p} & \text{for } z \neq p \\ 0 & \text{for } z = p. \end{cases}$$
Clearly $\varphi(z) \to \varphi(p) = 0$ as $z \to p$. And by inspection,
$$f(z) = \varphi(z)(z-p) + f(p) + w(z-p)$$
for all z in Ω. Conversely, suppose $\varphi : \Omega \to \mathbb{C}$ exists to satisfy (1) and (2). Then for $z \neq p$ we get
$$\frac{f(z) - f(p)}{z - p} - w = \varphi(z) \to 0 \quad \text{as } z \to p$$
and so $(f(z) - f(p))/(z-p) \to w$ as $z \to p$. □

Corollary 2.5 If f is differentiable at p, then f is continuous at p.

Proof. By the previous characterization, it is immediate that $f(z) \to f(p)$ as $z \to p$. □

If w is the derivative of f at p, we write $w = f'(p)$. If $f'(p)$ and $g'(p)$ exist, then we have the usual high school differentiation rules:
$$(f \pm g)'(p) = f'(p) \pm g'(p)$$

$$(fg)'(p) = f(p)g'(p) + f'(p)g(p)$$
$$\left(\frac{f}{g}\right)'(p) = \frac{g(p)f'(p) - f(p)g'(p)}{g(p)^2} \quad \text{when } g(p) \neq 0.$$

The proofs are identical to those in real variable calculus (change every x to z), so we skip them.

Also $(z^n)' = nz^{n-1}$ for $n = 1, 2, \ldots,$ while $z^0 = 1$ has derivative 0. The proof again is identical to the one in calculus. Thus, every complex rational function

$$f(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}{b_m z^m + \cdots + b_1 z + b_0}$$

is differentiable at every z where the denominator does not vanish.

Example 2.6 The function
$$f(z) = \frac{z^2 + 4}{z^2 - i}$$
is differentiable at every p except for the two square roots of i, which happen to be $\pm\left(1/\sqrt{2} + i/\sqrt{2}\right)$.

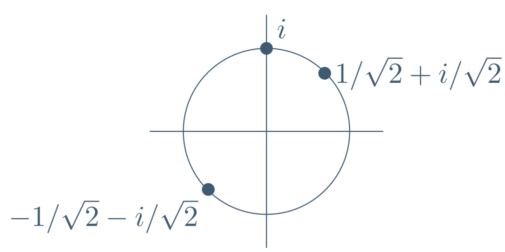

FIGURE 15

Figure 15 records the relevant geometry.

And then
$$f'(z) = \frac{(z^2 - i)(2z) - (z^2 + 4)(2z)}{(z^2 - i)^2} = \frac{(-8 - 2i)z}{(z^2 - i)^2}.$$

Example 2.7 Here is a nice function that is *not* complex differentiable at any p in $\mathbb{C}$:

$$f(z) = f(x+iy) = x - iy = \bar{z}.$$

Suppose for a contradiction that
$$\frac{f(z) - f(p)}{z - p} \to w$$
as $z \to p$. So
$$\frac{\bar{z} - \bar{p}}{z - p} \to w$$
as $z \to p$. Now $z_n = p + 1/n \to p$ as $n \to \infty$, so

$$\frac{\overline{p + \frac{1}{n}} - \bar{p}}{p + 1/n - p} = \frac{\bar{p} + \frac{1}{n} - \bar{p}}{(1/n)} = \frac{1/n}{1/n} = 1 \to 1$$

as $n \to \infty$.

Also $z_n = p + i/n \to p$ as $n \to \infty$, so

$$\frac{\overline{p + i/n} - \bar{p}}{p + i/n - p} = \frac{\bar{p} - i/n - \bar{p}}{i/n} = \frac{-i/n}{i/n} = -1 \to -1$$

as $n \to \infty$. Then $1 = -1$, which happens only in a world that could never be. Thus there is no derivative w.

Some very lovely functions do not have a complex derivative.

Lecture 3 — Friday, January 10, 2014

Jacobians and the Cauchy–Riemann Equations

If a real-valued function $y = f(x)$ has a derivative $f'(p)$ at some p, we have a tangent line

$$T(x) = f(p) + f'(p)(x - p).$$

We often write $f(x) \approx T(x)$ for x close to p. To make this precise we note that

$$\frac{|f(x)-T(x)|}{|x-p|} = \frac{|f(x)-f(p)-f'(p)(x-p)|}{|x-p|} = \left|\frac{f(x)-f(p)}{x-p} - f'(p)\right| \to 0 \text{ as } x \to p.$$

So $f(x) - T(x) \to 0$ even faster than $x - p \to 0$. The function $T : \mathbb{R} \to \mathbb{R}$ equals a linear map $x \mapsto f'(p)x$ plus a constant map $x \mapsto f(p) - f'(p)p$ and is an example of an **affine map**.

In the case of $\mathbb{R}^2$ an affine map looks like

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} k \\ l \end{pmatrix}.$$

Definition 3.1 If $\begin{pmatrix} s \\ t \end{pmatrix} \in \Omega =$ open set in $\mathbb{R}^2$ and $f : \Omega \to \mathbb{R}^2$, we say that f is **real differentiable** at $\begin{pmatrix} s \\ t \end{pmatrix}$ with derivative matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ if and only if

$$f\begin{pmatrix} x \\ y \end{pmatrix} \approx f\begin{pmatrix} s \\ t \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x-s \\ y-t \end{pmatrix}$$

in the following precise sense:

$$\frac{\left\| f\begin{pmatrix} x \\ y \end{pmatrix} - f\begin{pmatrix} s \\ t \end{pmatrix} - \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x-s \\ y-t \end{pmatrix}\right\|}{\left\|\begin{pmatrix} x-s \\ y-t \end{pmatrix}\right\|} \to 0 \text{ as } \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} s \\ t \end{pmatrix}.$$

Also, if $f = \begin{pmatrix} u \\ v \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ is real differentiable at $\begin{pmatrix} s \\ t \end{pmatrix}$, then the partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

exist at $\begin{pmatrix} s \\ t \end{pmatrix}$ *and* the derivative matrix

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

This derivative matrix is known as the **Jacobian** of f evaluated at $\begin{pmatrix} s \\ t \end{pmatrix}$. Note that the mere existence of these partial dervivatives does *not* imply that f is real differentiable.

How does this result from calculus relate to complex derivatives? We have $f = u + iv : \Omega \to \mathbb{C}$ where $x + iy \mapsto u(x + iy) + iv(x + iy)$. Let $p = s + it \in \Omega$ and let $w = a + ib$. We saw that f is complex differentiable at p with derivative w when

$$\frac{f(z) - f(p)}{z - p} \to w \text{ as } z \to p. \tag{1}$$

This is the same as

$$\frac{f(z) - f(p) - w(z - p)}{z - p} \to 0 \text{ as } z \to p. \tag{2}$$

which is the same as

$$\frac{|f(z) - f(p) - w(z - p)|}{|z - p|} \to 0 \text{ as } z \to p,$$

which is the same as

$$\frac{|f(x + iy) - f(s + it) - (a + ib)((x - s) + i(y - t))|}{|(x - s) + i(y - t)|} \to 0 \text{ as } x + iy \to s + it.$$

With the identification $k + il = \begin{pmatrix} k \\ l \end{pmatrix}$, notice that

$$(m + in)(k + il) = mk - nl + i(ml + nk) = \begin{pmatrix} mk - nl \\ ml + nk \end{pmatrix} = \begin{pmatrix} m & -n \\ n & m \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix}.$$

Back in (2), we saw that f is differentiable at $p = s + it$ with derivative $w = a + ib$

$$\text{if and only if } \frac{\left\| f\begin{pmatrix} x \\ y \end{pmatrix} - f\begin{pmatrix} s \\ t \end{pmatrix} - \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x - s \\ y - t \end{pmatrix} \right\|}{\left\| \begin{pmatrix} x - s \\ y - t \end{pmatrix} \right\|} \to 0 \text{ as } \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} s \\ t \end{pmatrix}. \tag{3}$$

From (3) and (1) we have discovered the following:

Proposition 3.2 If $f : \Omega \to \mathbb{C}$ and $p = s + it \in \Omega$, the function f is complex differentiable at p with derivative $w = a+ib$ if and only if f is real differentiable at $p = \begin{pmatrix} s \\ t \end{pmatrix}$ with Jacobian matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

We also know in this case that

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \text{ evaluated at } \begin{pmatrix} s \\ t \end{pmatrix}.$$

So

$$f'(p) = w = a + ib = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big|_p = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\Big|_p$$

Hence $(\partial u)/(\partial x) = (\partial v)/(\partial y)$ and $(\partial v)/(\partial x) = -(\partial u)/(\partial y)$ evaluated at $p = \begin{pmatrix} s \\ t \end{pmatrix}$. Thus:

Proposition 3.3 (Cauchy–Riemann equations) If $f = u + iv$ is complex differentiable at p, then u, v satisfy the **Cauchy–Riemann equations** at p:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Lecture 4 — Monday, January 13, 2014

Example 4.1 Consider $f(z) = \bar{z}$. That is, $f(x + iy) = x - iy$. We check

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial u}{\partial y} = 0, \qquad \frac{\partial v}{\partial x} = 0, \qquad \text{and} \qquad \frac{\partial v}{\partial y} = -1.$$

So f is not differentiable (which we already saw in Example 2.7).

Example 4.2 Consider the function

$$f(z) = \frac{z}{z+i} = \frac{x+iy}{x+i(1+y)}.$$

First, we massage the function into standard form:

$$\frac{z}{z+i} = \frac{(x+iy)(x-i(1+y))}{x^2+(1+y)^2} = \underbrace{\frac{x^2+y(1+y)}{x^2+(1+y)^2}}_{u} + i\underbrace{\frac{-x}{x^2+(1+y)^2}}_{v}.$$

If $x+iy \neq -i$, we check

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{(x^2+(1+y)^2)2x - (x^2+y(1+y))2x}{(x^2+(1+y)^2)^2} \\ &= \frac{2x(1+y)^2 - 2xy(1+y)}{(x^2+(1+y)^2)^2} = \frac{2x(1+y)}{(x^2+(1+y)^2)^2}.\end{aligned}$$

And

$$\frac{\partial v}{\partial y} = \frac{x \cdot 2(1+y)}{(x^2+(1+y)^2)^2}$$

analogously. Also check $(\partial v)/(\partial x) = -(\partial u)/(\partial y)$. So f is complex differentiable.

Holomorphic Functions and the Chain Rule

Back to $f = \begin{pmatrix} u \\ v \end{pmatrix} : \Omega \to \mathbb{R}^2$ where $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$.

It is possible to have some $\begin{pmatrix} s \\ t \end{pmatrix}$ in Ω where the Jacobian $\begin{pmatrix} (\partial u)/(\partial x) & (\partial u)/(\partial y) \\ (\partial v)/(\partial x) & (\partial v)/(\partial y) \end{pmatrix}$ exists at $\begin{pmatrix} s \\ t \end{pmatrix}$ but f has no real derivative at $\begin{pmatrix} s \\ t \end{pmatrix}$. Mercifully, we do have the following result from calculus:

Proposition 4.3 If $f = \begin{pmatrix} u \\ v \end{pmatrix} : \Omega \to \mathbb{R}^2$ is such that all partials

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

exist and are continuous on all of Ω, then f has a real derivative at every $\begin{pmatrix} s \\ t \end{pmatrix}$ in Ω given by $\begin{pmatrix} (\partial u)/(\partial x) & (\partial u)/(\partial y) \\ (\partial v)/(\partial x) & (\partial v)/(\partial y) \end{pmatrix}$.

For $f = u + iv : \Omega \to \mathbb{R}^2$, this tells us the following:

Proposition 4.4 If $f = u + iv : \Omega \to \mathbb{C}$ is such that the partials

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

exist and are continuous on Ω, *and* if the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \circledast$$

hold on all of Ω, then $f'(z)$ exists for every z in Ω.

Proof. The preceding result from advanced calculus shows that f is real differentiable. The Cauchy–Riemann equations tell us that at any $p \in \Omega$, the derivative matrix

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} \left.\dfrac{\partial u}{\partial x}\right|_p & \left.\dfrac{\partial u}{\partial y}\right|_p \\ \left.\dfrac{\partial v}{\partial x}\right|_p & \left.\dfrac{\partial v}{\partial y}\right|_p \end{bmatrix}$$

satisfies $a = d$ and $c = -b$. That is,

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Then Proposition 3.2 assures us that $f'(p)$ exists. $\square$

Definition 4.5 A function $f : \Omega \to \mathbb{C}$ for which $f'(z)$ exists on all of Ω is called **holomorphic** on Ω.

Such f were so named by students of Cauchy. The term means "fully formed". Every rational function is holomorphic on the open set Ω where its denominator does not vanish.

Example 4.6 $f(z) = f(x+iy) = e^x \cos y + ie^x \sin y$ is such that the partials are continuous and satisfy the Cauchy–Riemann equations over all of $\mathbb{R}^2$. Thus f is holomorphic on $\mathbb{C}$.

Proof. Write
$$u(x,y) = e^x \cos y \qquad \text{and} \qquad v(x,y) = e^x \sin y.$$
Then
$$\frac{\partial u}{\partial x} = e^x \cos y, \qquad \frac{\partial u}{\partial y} = -e^x \sin y, \qquad \frac{\partial v}{\partial x} = e^x \sin y, \qquad \text{and} \qquad \frac{\partial v}{\partial y} = e^x \cos y.$$
Each of these partial derivatives is continuous on $\mathbb{R}^2$. Also,
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$
Therefore, by Proposition 4.4, f is complex differentiable at every point of $\mathbb{C}$. Hence f is holomorphic on $\mathbb{C}$. □

Definition 4.7 A function that is holomorphic over all of $\mathbb{C}$ is called **entire**.

Polynomials and the exponential function are entire.

Proposition 4.8 (Chain rule) Let Ω, Γ be open sets in $\mathbb{C}$, and $f : \Omega \to \Gamma$, $g : \Gamma \to \mathbb{C}$.

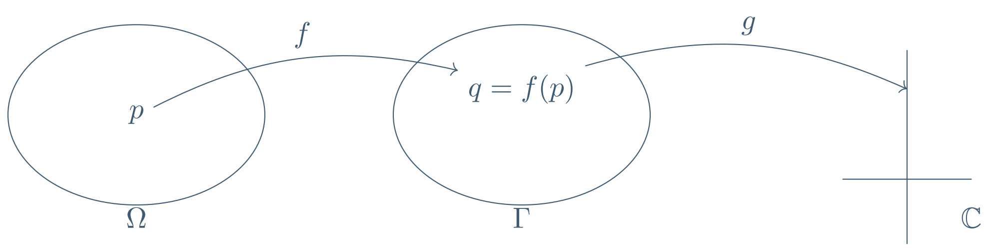

FIGURE 16

The configuration appears in Figure 16.

Let $p \in \Omega$, $q = f(p) \in \Gamma$. If f is complex differentiable at p and g is complex differentiable at $q = f(p)$, then $g \circ f$ is complex differentiable at p and

$$(g \circ f)'(p) = g'(f(p))f'(p).$$

Proof. With $q = f(p)$, the differentiability of g at q tells us there is a function $\varphi : \Gamma \to \mathbb{C}$ such that

1. $g(w) = g(q) + g'(q)(w - q) + (w - q)\varphi(w)$ for all $w \in \Gamma$, and
2. $\varphi(w) \to \varphi(q) = 0$ as $w \to q$.

For $z \in \Omega$ and $z \neq p$ we have

$$\frac{g \circ f(z) - g \circ f(p)}{z - p} = \frac{g(f(z)) - g(f(p))}{z - p} = g'(f(p))\frac{f(z) - f(p)}{z - p} + \frac{f(z) - f(p)}{z - p}\varphi(f(z))$$

by the first item. (Think $w = f(z) \in \Gamma$, $q = f(p)$.) Now as $z \to p$, we have

$$\frac{f(z) - f(p)}{z - p} \to f'(p),$$

and also $f(z) \to f(p)$ since differentiability at p implies continuity at p. Therefore $\varphi(f(z)) \to \varphi(f(p)) = 0$ by continuity of composition, and so

$$\frac{g \circ f(z) - g \circ f(p)}{z - p} \to g'(f(p))f'(p) + f'(p) \cdot 0 = g'(f(p))f'(p).$$

□

Example 4.9 $f(z) = e^{z^2}$ is an entire function since $f'(z) = e^{z^2}(2z)$ for all $z \in \mathbb{C}$.

Proposition 4.8 gives the following result: if Ω, Γ are open sets and $f : \Omega \to \Gamma$, $g : \Gamma \to \mathbb{C}$ are holomorphic on Ω, Γ respectively, then $g \circ f$ is holomorphic on Ω.

Definition 4.10 The functions $z \mapsto \pm iz$ are entire. Therefore, $z \mapsto e^{iz}$ and $z \mapsto e^{-iz}$ are entire. Thus,

$$z \mapsto \frac{e^{iz} + e^{-iz}}{2}$$

is entire. This function is called $\cos z$, or the **complex cosine**. We also have the entire function

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

which is the **complex sine**.

Proposition 4.11 (de Moivre's formula) For any $z \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$(\cos z + i \sin z)^n = \cos nz + i \sin nz.$$

Proof. By the definitions of complex cosine and sine,

$$\cos z + i \sin z = \frac{e^{iz} + e^{-iz}}{2} + i \frac{e^{iz} - e^{-iz}}{2i} = e^{iz}.$$

Therefore,

$$(\cos z + i \sin z)^n = \left(e^{iz}\right)^n = e^{inz}.$$

Applying the same identity with nz in place of z gives

$$e^{inz} = \cos(nz) + i \sin(nz).$$

Combining these equalities yields

$$(\cos z + i \sin z)^n = \cos(nz) + i \sin(nz).$$

$\square$

Lecture 5 — Thursday, January 15, 2015

2. Logarithms and Exponentials

The Complex Exponential and Logarithm

Definition 5.1 The **complex exponential** function is

$$\exp : \mathbb{C} \to \mathbb{C}, \qquad x + iy \mapsto e^x \cos y + ie^x \sin y.$$

Setting $u = e^x \cos y$ and $v = e^x \sin y$, we readily compute that

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = e^x \sin y = -\frac{\partial u}{\partial y}.$$

Since these partials are continuous and satisfy the Cauchy–Riemann equations throughout $\mathbb{C}$, exp is holomorphic on $\mathbb{C}$. In other words, exp is entire. Also,

$$\exp'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y = \exp z.$$

A direct calculation from the definition and trigonometric identities shows that

$$\exp(z)\exp(w) = \exp(z + w).$$

In addition, $\exp(x + i0) = e^x$, the usual exponential on $\mathbb{R}$, and

$$\exp(0 + iy) = \cos y + i \sin y = e^{iy},$$

which is **Euler's formula**. In light of the preceding remarks, it is natural to write $\exp z = e^z$ for all z in $\mathbb{C}$.

Proposition 5.2 If $z \in \mathbb{C}$, then $e^z \neq 0$.

Proof. Suppose $e^z = 0$ at some z. Then for any w in $\mathbb{C}$ we would have

$$e^w = e^{z+w-z} = e^z e^{w-z} = 0 \cdot e^{w-z} = 0.$$

In particular, this would force $1 = e^0 = 0$, which is impossible. Therefore $e^z \neq 0$. Using this fact, we divide by e^z in $e^w = e^z e^{w-z}$ to get $e^w/e^z = e^{w-z}$. $\square$

Unlike the usual exponential function $x \mapsto e^x$ on $\mathbb{R}$, the complex exponential $\exp : \mathbb{C} \to \mathbb{C}$ is not injective. For instance,

$$e^{x+i(y+2\pi)} = e^x(\cos(y+2\pi) + i\sin(y+2\pi)) = e^x(\cos y + i\sin y) = e^{x+iy},$$

while $x + i(y+2\pi) \neq x + iy$. So let us restrict the domain of exp to a region where it is injective. A convenient choice is

$$S = \{x + iy : -\pi < y \leq \pi\}.$$

This S is the strip bounded by the horizontal lines $y = \pm\pi$ and including the line $y = \pi$:

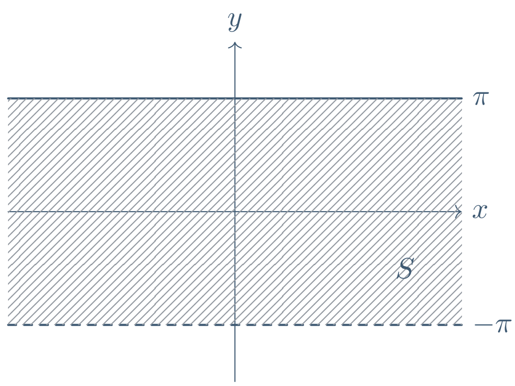

Figure 17

Figure 17 illustrates the setup.

Remember that $\mathbb{C}^*$ denotes the punctured plane $\mathbb{C} \setminus \{0\}$.

Proposition 5.3 The restriction of exp to S implements a bijection between S and $\mathbb{C}^*$.

Proof. We will show that for every w in $\mathbb{C}^*$ (that is, $w \neq 0$), there is a unique z in S such that $e^z = w$. Recall the polar representation of a nonzero complex number w:

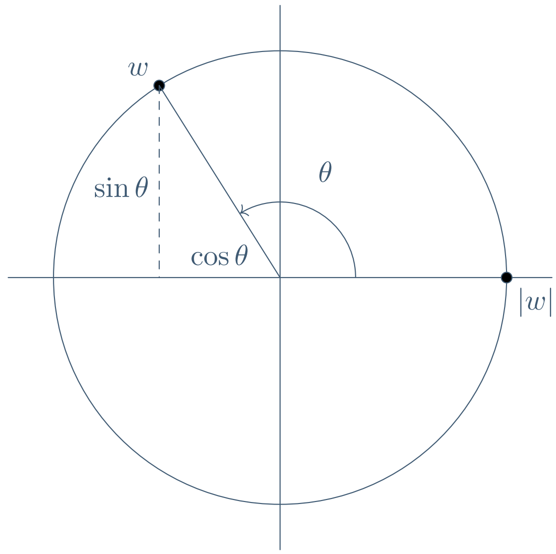

Figure 18

The construction is shown in Figure 18.

From the diagram we obtain

$$w = |w|\cos\theta + i|w|\sin\theta = |w|(\cos\theta + i\sin\theta) = |w|e^{i\theta} = e^{\log|w|}e^{i\theta} = e^{\log|w|+i\theta}$$

Definition 5.4 Any $\theta \in \mathbb{R}$ which satisfies $w = |w|e^{i\theta}$ is called an **argument** of w.

That is, θ is an angle formed by travelling along the circle of radius $|w|$ starting at $|w|$ on the x-axis and finishing at w. Clearly, $\log|w|$ is uniquely determined by w. But θ is *not* uniquely determined by w. Indeed

$$e^{\log|w|+i(\theta+2\pi)} = |w|\cos(\theta+2\pi) + i|w|\sin(\theta+2\pi) = |w|\cos\theta + i|w|\sin\theta = w.$$

So different arguments can give the polar representation of w. However, if we restrict θ to the interval $(-\pi,\pi]$ (that is, $-\pi < \theta \leq \pi$), then there is a unique θ such that

$$w = e^{\log|w|+i\theta}.$$

Definition 5.5 The unique θ in $(-\pi, \pi]$ that gives the polar representation of w is called the **principal argument** of w and we denote it by $\theta = \arg w$.

Thus
$$w = e^{\log|w| + i \arg w}.$$
We see that $\log|w| + i \arg w \in S$ because $\arg w \in (-\pi, \pi]$, and so
$$z = \log|w| + i \arg w$$
is our unique z in S such that $e^z = w$. We have our bijection $\exp : S \longrightarrow \mathbb{C}^*$. □

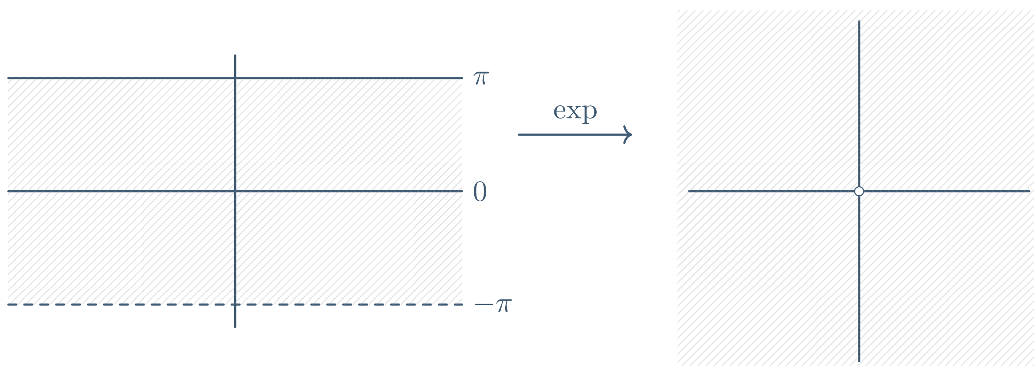

Figure 19

Figure 19 records the relevant geometry.

Definition 5.6 The inverse of the bijection $\exp : S \to \mathbb{C}^*$ is called (naturally) the **complex logarithm** $\log : \mathbb{C}^* \longrightarrow S$.

By definition of inverse functions, we have
$$e^{\log z} = z \quad \text{for all } z \text{ in } \mathbb{C}^*$$
and
$$\log(e^z) = z \quad \text{for all } z \text{ in } S.$$
This function is known as the **principal branch of the logarithm**. Other "branches" of the logarithm could have been chosen by using an alternate interval for the argument θ, instead of

$(-\pi, \pi]$ — say $[0, 2\pi)$, or any $[a, a + 2\pi)$. The formula for log is

$$\log w = \log |w| + i \arg w,$$

where $\log |w|$ denotes the natural logarithm.

If we try to prove that log is holomorphic, we will fail miserably. This is because log comes with a defect: it is *not* continuous at any point on the negative x-axis. Indeed, suppose $p < 0$:

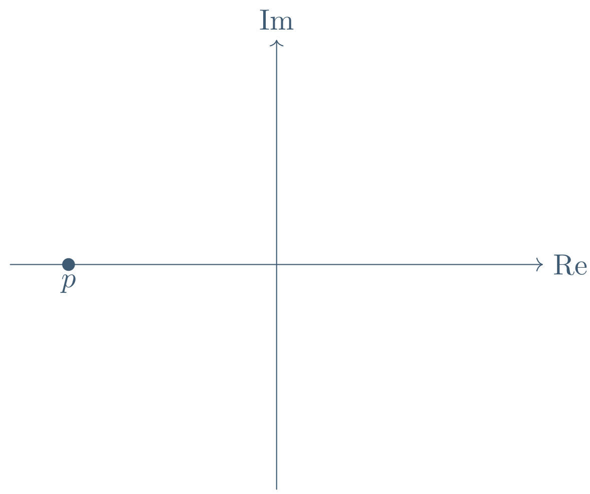

FIGURE 20

The configuration appears in Figure 20.

Now let $u_n = |p|e^{i(\pi+1/n)}$ and $v_n = |p|e^{i(\pi-1/n)}$:

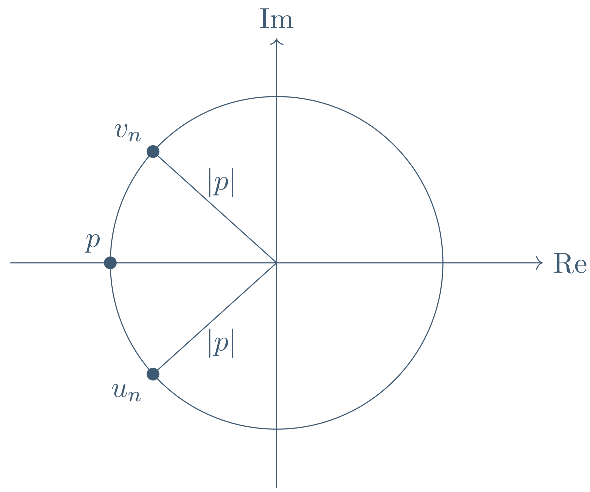

Figure 21

We see that $u_n \to |p|e^{i\pi} = -|p| = p$ and $v_n \to |p|e^{i\pi} = -|p| = p$ as $n \to \infty$. But

$$\log u_n = \log|u_n| + i \arg u_n = \log|p| + i\left(-\pi + \frac{1}{n}\right) \to \log|p| - i\pi$$

while

$$\log v_n = \log|v_n| + i \arg v_n = \log|p| + i\left(\pi - \frac{1}{n}\right) \to \log|p| + i\pi.$$

How awful! Note that $\arg u_n = -\pi + 1/n$ (and *not* $\pi + 1/n$). Figure 21 reveals that $\lim_{z\to p} \log z$ does not exist, which surely makes $\log z$ discontinuous at p.

Lecture 6 — Friday, January 17, 2014

Holomorphicity of Exp and Log

We might ask what the horizontal line $y = \pi$ in S gets mapped to by exp. If $z = x + i\pi$, we get

$$e^z = e^{x+i\pi} = e^x e^{i\pi} = -e^x.$$

As x runs over $\mathbb{R}$, our $-e^x$ covers the negative x-axis where log was discontinuous. So if we *drop* the horizontal line $y = \pi$ from S and *drop* the negative x-axis from $\mathbb{C}^*$ we get two open sets: the open strip $\Sigma = \{x + iy : -\pi < y < \pi\}$ and the cut plane $\Omega = \{x + iy : \text{either } x > 0 \text{ or } y \neq 0\} =$

$\mathbb{C}^* \setminus \{x + i0 : x < 0\}$. And we still have our bijections

$$\Sigma \underset{\log}{\overset{\exp}{\rightleftharpoons}} \Omega.$$

In pictures:

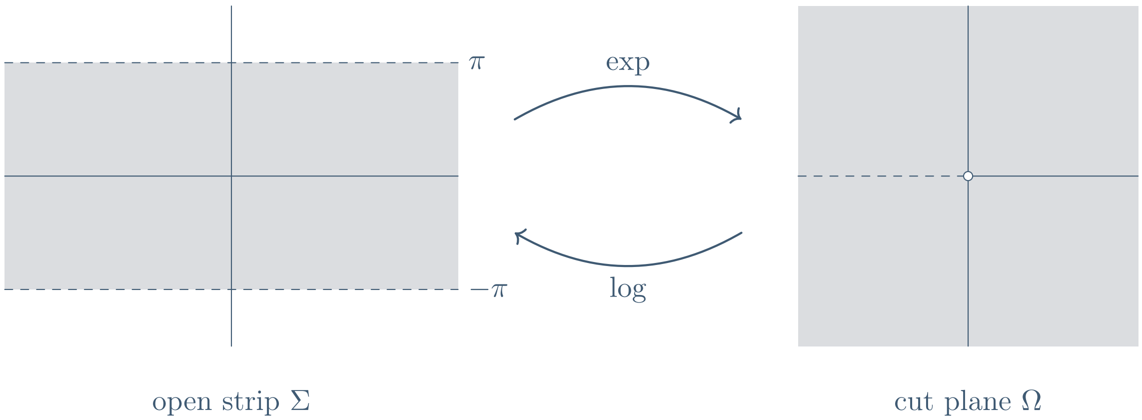

FIGURE 22

The construction is shown in Figure 22.

Although log was discontinuous on the negative x-axis, it turns out to be holomorphic on the cut plane Ω. If $z = x + iy \in \Omega$ (we switched w to z), we have

$$\log z = \log |z| + i \arg z = \log \sqrt{x^2 + y^2} + i \arg(x + iy) = \frac{1}{2} \log\bigl(x^2 + y^2\bigr) + i \arg(x + iy).$$

Proposition 6.1 $\log z : \Omega \to \Sigma$ is holomorphic.

Proof. We need the partial derivatives of

$$\frac{1}{2} \log\bigl(x^2 + y^2\bigr) \qquad \text{and} \qquad \arg(x + iy)$$

so that we can verify the Cauchy–Riemann equations. Thus we need a differentiable formula for $\arg(x + iy)$. No familiar formula seems to cover the whole cut plane Ω, so we study Ω in pieces. Let $H = \{x + iy : x > 0\}$ be the right half-plane:

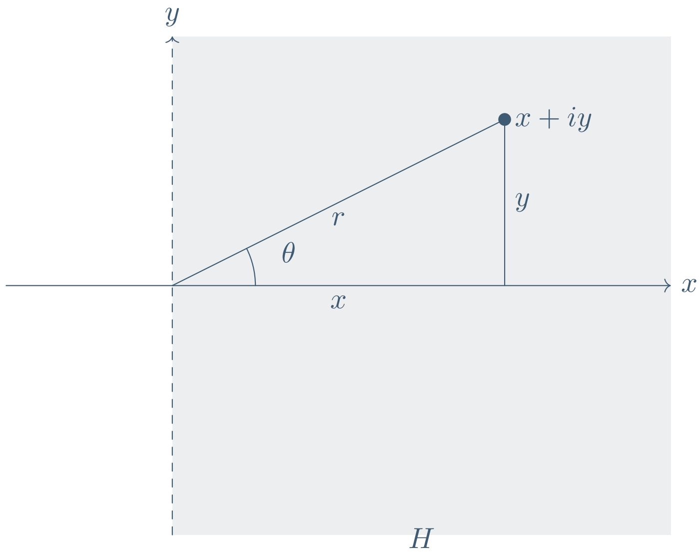

Figure 23

Figure 23 records the relevant geometry.

If $x+iy \in H$, we can see that $\arg(x+iy) = \arctan(y/x)$. Notice that this will not work for $x+iy$ in all of Ω because $\arctan(y/x)$ only gives values in $(-\pi/2, \pi/2)$ while $\arg(x+iy)$ gives values in $(-\pi, \pi)$. But for $x+iy \in H$, we do have

$$\log(x+iy) = \frac{1}{2}\log\left(x^2+y^2\right) + i\arctan\left(\frac{y}{x}\right).$$

Now set $u = 1/2\log\left(x^2+y^2\right)$ and $v = \arctan(y/x)$. Then

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}, \qquad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2},$$
$$\frac{\partial v}{\partial x} = -\frac{y}{x^2+y^2}, \qquad \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}.$$

These partials are continuous on H, and they satisfy the Cauchy–Riemann equations. We conclude that log is holomorphic on H. We are not done, since we want log to be holomorphic on the cut plane Ω. Let us see what we can do with $\arg(x+iy)$ when $x+iy$ is in the half-plane $K = \{x+iy : y > 0\}$ above the x-axis:

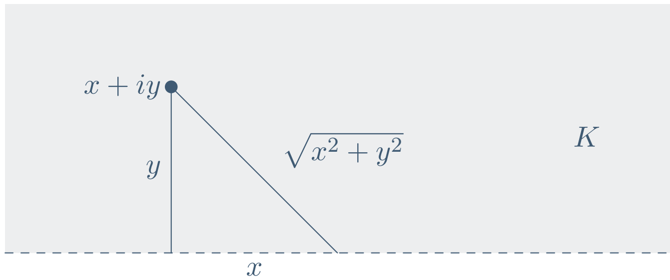

FIGURE 24

The configuration appears in Figure 24.

If $x+iy \in K$, we see that

$$\arg(x+iy) = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right).$$

Hence, for $x+iy \in K$,

$$\log(x+iy) = \frac{1}{2}\log\big(x^2+y^2\big) + i\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right).$$

Here $\arccos\Big(x/\sqrt{x^2+y^2}\Big) \in (0,\pi)$. If we set

$$v_K(x,y) = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right),$$

then a direct computation gives

$$\frac{\partial v_K}{\partial x} = -\frac{y}{x^2+y^2} \qquad \text{and} \qquad \frac{\partial v_K}{\partial y} = \frac{x}{x^2+y^2}.$$

So the Cauchy–Riemann equations hold on K as well. Finally, take L to be the half-plane below the x-axis:

$$L = \{x+iy : y < 0\}.$$

If $x+iy \in L$, we see that

$$\arg(x+iy) = -\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right).$$

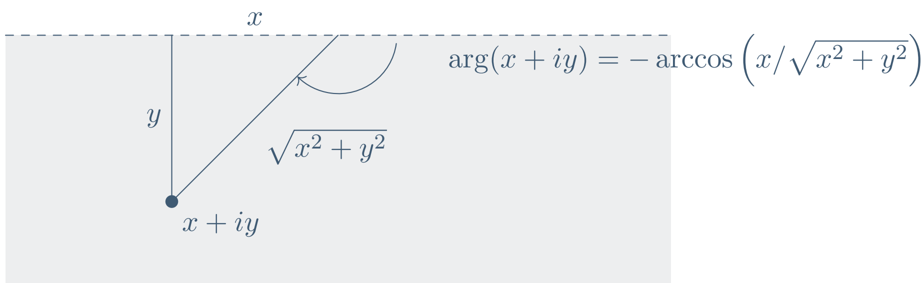

FIGURE 25

Figure 25 illustrates the setup.

If we set

$$v_L(x,y) = -\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right),$$

then direct differentiation gives

$$\frac{\partial v_L}{\partial x} = -\frac{y}{x^2+y^2} \qquad \text{and} \qquad \frac{\partial v_L}{\partial y} = \frac{x}{x^2+y^2}.$$

So the Cauchy–Riemann equations hold on L, and the partial derivatives are continuous there. $\square$

A brief sign comment may reassure those who are worried. On K, we have

$$\log(x+iy) = \frac{1}{2}\log\bigl(x^2+y^2\bigr) + i\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right),$$

while on L

$$\log(x+iy) = \frac{1}{2}\log\bigl(x^2+y^2\bigr) - i\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right).$$

So how can both the "+" and "-" versions satisfy the Cauchy–Riemann equations? The key point is that K and L are disjoint. In differentiating $\arccos\left(x/\sqrt{x^2+y^2}\right)$, the term $\sqrt{y^2}$ appears. On K (where $y>0$), we have $\sqrt{y^2}=y$, while on L (where $y<0$), we have $\sqrt{y^2}=-y$.

Since log is holomorphic on each of H, K, L, our log is holomorphic on their union Ω. Since $z = e^{\log z}$ for each z in Ω, Proposition 4.8 gives

$$1 = z' = (e^{\log z})' = e^{\log z}(\log z)' = z(\log z)'$$

And thus

$$(\log z)' = \frac{1}{z} \text{ for all } z \text{ in the cut plane } \Omega,$$

which is a comforting fact.

Lecture 7 — Monday, January 20, 2014

The holomorphic bijections

$$\Sigma \underset{\log}{\overset{\exp}{\rightleftarrows}} \Omega$$

(where Σ = open strip and Ω = cut plane) are precious functions. Let us use exp and log to define power functions. Take a fixed q in $\mathbb{C}$. Define

$$z^q = e^{q \log z} \text{ for all } z \text{ in } \Omega.$$

By Proposition 4.8,

$$(z^q)' = e^{q\log z} \cdot \frac{q}{z} = q\frac{e^{q\log z}}{e^{\log z}} = qe^{q\log z - \log z} = qe^{(q-1)\log z} = qz^{q-1}, \text{ as might be expected.}$$

Keep in mind that we made a choice for the principal branch of log. Thus we made a choice for z^q. In order for that choice to be holomorphic we had to throw something out — namely the negative x-axis that we cut from $\mathbb{C}^*$ to get the cut plane Ω. A different choice of log, based on a different choice of arg would lead to a different cut and thus a different meaning of z^q. When q is a positive integer we already have a meaning for z^q as $z \cdot z \cdot \cdots \cdot z$ taken q times. It would be disastrous if the definition of z^q as $e^{q\log z}$ gave a different answer. Rest assured: they do give the same answer. Indeed, for a positive q we have

$$e^{q\log z} = e^{\log z + \log z + \cdots + \log z} = e^{\log z} \cdot e^{\log z} \ldots e^{\log z} = z \cdot z \cdot \cdots \cdot z,$$

with the exponent added q times and using the exponent law in the middle step. Also for a positive integer q we have

$$e^{-q\log z} = \frac{1}{e^{q\log z}} = \frac{1}{z \cdot z \cdot \cdots \cdot z} \quad (q \text{ times}).$$

For $q = 0$, both definitions give $z^0 = 1$. Thus the definition of z^q using log and exp agrees with the original meaning of z^q when q is any integer.

Now let us inspect the special case $q = 1/2$. We see that

$$(z^{1/2})' = \frac{1}{2} z^{-1/2},$$

where $z^{1/2}$ means $e^{1/2 \log z}$ and $z^{-1/2}$ means $e^{-1/2 \log z}$, all taken for z in the cut plane Ω. Another thing to notice is that

$$(z^{1/2})^2 = e^{(\frac{1}{2}) \log z} e^{(\frac{1}{2}) \log z} = e^{(\frac{1}{2}) \log z + (\frac{1}{2}) \log z} = e^{\log z} = z \quad \text{for all } z \text{ in } \Omega.$$

And what if $z = x + i0$ where $x > 0$? Does our $z^{1/2}$ agree with our usual positive square root $\sqrt{x}$? Of course:

$$z^{1/2} = e^{(\frac{1}{2}) \log z} = e^{(\frac{1}{2}) \log(x+i0)} = e^{(\frac{1}{2}) \log x} = e^{\log \sqrt{x}} = \sqrt{x}.$$

Where does the cut plane Ω go under the chosen map $z \mapsto z^{1/2}$? To say $z \in \Omega$ means $-\pi < \arg z < \pi$. So

$$z^{1/2} = e^{(\frac{1}{2}) \log z} = e^{(\frac{1}{2})(\log |z| + i \arg z)} = e^{(\frac{1}{2}) \log |z| + i \frac{\arg z}{2}} = \sqrt{|z|} e^{i \frac{\arg z}{2}}.$$

So $\arg(z^{1/2}) = (\arg z)/2$, which covers the interval $(-\pi/2, \pi/2)$. This **square root** function folds the cut plane back into the half-plane H to the right of the y-axis:

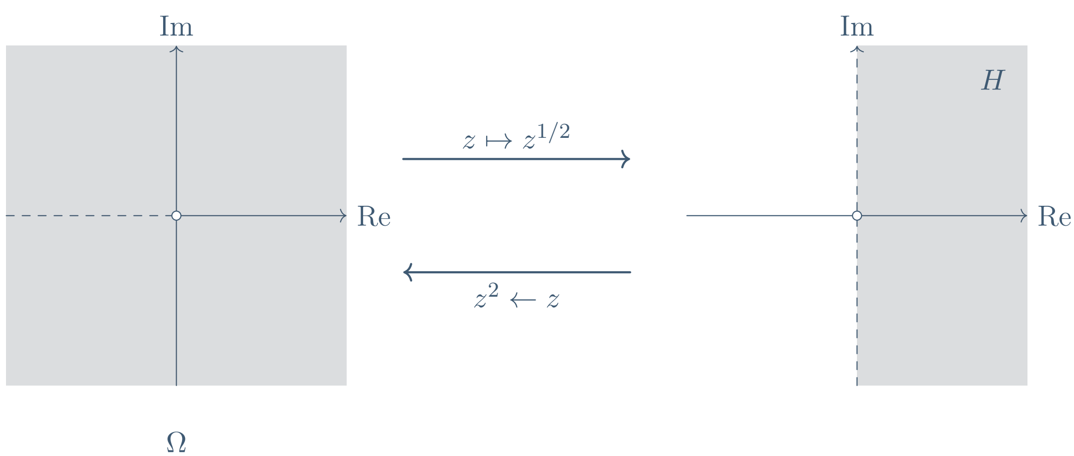

FIGURE 26

Figure 26 displays a bijection between Ω and H whose inverse is the usual squaring function

$z \mapsto z^2$.

3. Paths and Connectedness

Warning Different books use the words *curve*, *path*, *contour*, and *arc* to mean slightly different things.

Suppose we have a real differentiable function $f : I \to \mathbb{R}$ where $I \subseteq \mathbb{R}$ is an interval. If $f' \equiv 0$ on I, then for $x, y \in I$, the mean value theorem guarantees the existence of some $c \in I$ such that

$$f(x) - f(y) = f'(c)(x - y) = 0 \cdot (x - y) = 0.$$

Thus $f' \equiv 0$ implies that f is constant.

The complex analogue of this result is true as well. However, proving it is not quite so simple in $\mathbb{C}$ because the mean value theorem fails.

Example 7.1 Let $f(z) = z^3$. There is no p on the line segment from 1 to i such that

$$f'(p) = \frac{f(i) - f(1)}{i - 1}.$$

Proof. If p is on this line segment, then we can write $p = t + (1 - t)i$ for some $t \in [0, 1]$. If such a p satisfied the proposed equality, then the following would be true:

$$f'(p) = 3(t + (1 - t)i)^2 = \frac{i^3 - 1^3}{i - 1} = i.$$

After expanding the square and simplifying, we find that the only possible candidate is $t = 1/2$, so $p = (1/2) + i(1/2)$. But $f'((1/2) + i(1/2)) = i$ reduces to $3/2 = 1$, and this statement is false. So such a p cannot exist. $\square$

So the mean value theorem is dead in the water, as are its many consequences in the world of $\mathbb{R}$. We must therefore resort to other techniques to deal with complex derivatives.

Definition 7.2 A continuous function $x : [a,b] \to \mathbb{R}$ is called **smooth** on the closed interval $[a,b]$ provided x is continuously differentiable on the entire interval $[a,b]$.

This means that the usual derivative $x'(t)$ exists for all t in the open interval (a,b), the right and left derivatives $x'(a)$ and $x'(b)$ exist in the sense that

$$x'(a) = \lim_{h\to 0^+} \frac{x(a+h)-x(a)}{h} \qquad \text{and} \qquad x'(b) = \lim_{h\to 0^-} \frac{x(b+h)-x(b)}{h},$$

and the derivative function $x' : [a,b] \to \mathbb{R}$ so defined is continuous on $[a,b]$.

Example 7.3 Under this definition of smoothness, the function $x(t) = \sqrt{t}$ is not smooth on $[0,1]$ because $x'(0)$ does not exist. The function given by $x(t) = t^2 \sin(1/t)$ for $t \neq 0$ and by $x(0) = 0$ is also not smooth on $[0,1]$. Here $x'(0)$ exists, but x' is not continuous at 0.

Lecture 8 — Wednesday, January 22, 2014

Smooth Curves and Paths

Definition 8.1 A **curve** in an open subset Ω of $\mathbb{C}$ is a continuous map $\alpha : [a,b] \to \Omega$. If $\alpha = x + iy$ has component functions $x, y : [a,b] \to \mathbb{R}$ that are smooth on the closed interval $[a,b]$, we call the curve α a **smooth curve** in Ω.

The derivative of a smooth curve $\alpha : [a,b] \to \Omega$ is given by

$$\alpha'(t) = x'(t) + iy'(t) \text{ for each } t \in [a,b].$$

It should be clear that the derivative $\alpha'(t)$ represents a vector tangent to the curve at the point $\alpha(t)$:

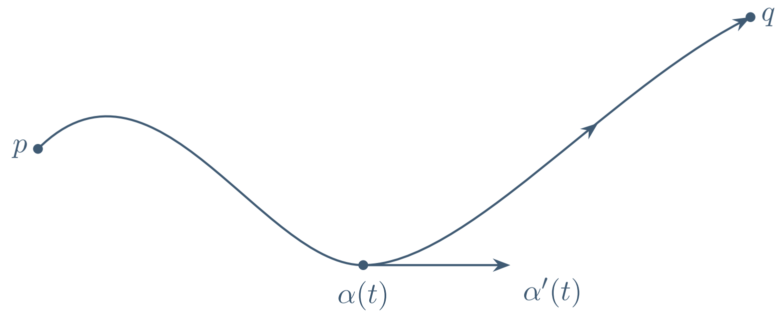

FIGURE 27

Figure 27 records the relevant geometry.

Definition 8.2 If $\alpha(a) = p$ and $\alpha(b) = q$, we say that the curve **runs from *p* to *q***.

Example 8.3 Given two points p and q in $\mathbb{C}$, the straight line

$$\alpha(t) = p + t(q - p) = tq + (1 - t)p \text{ where } t \in [0, 1].$$

is a smooth curve running from p to q, with derivative $\alpha'(t) = q - p$.

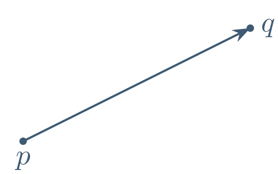

FIGURE 28

The configuration appears in Figure 28.

Example 8.4 Take the curve given by

$$\alpha(t) = e^{it} = \cos t + i \sin t, \text{ where } t \in \left[0, \frac{\pi}{2}\right].$$

This α traces out a quarter-circle running from $p = 1$ to $q = i$. It is smooth with derivative

$\alpha'(t) = ie^{it} = -\sin t + i\cos t$:

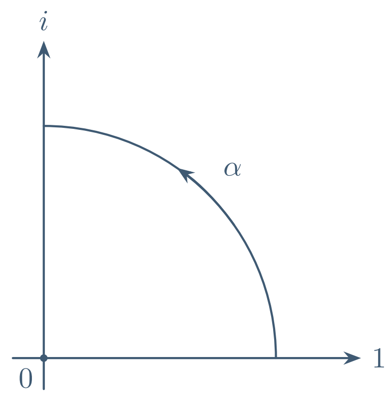

Figure 29

Figure 29 illustrates the setup.

We should not confuse the curve $\alpha : [a, b] \to \Omega$, which is a mapping, with the image set $\alpha([a, b])$.

Definition 8.5 We shall refer to the image set

$$\alpha([a, b]) = \{\alpha(t) : t \in [a, b]\}$$

as the **trajectory** (or **trace**) of the curve α.

When we sketch a curve, all we can really draw is its trajectory.

Now, here is the chain rule for a holomorphic function precomposed with a smooth curve.

Proposition 8.6 If $\alpha : [a, b] \to \Omega$ is a smooth curve and $f : \Omega \to \mathbb{C}$ is holomorphic, then the curve $f \circ \alpha : [a, b] \to \mathbb{C}$ is smooth and

$$(f \circ \alpha)'(t) = f'(\alpha(t))\alpha'(t) \text{ for all } t \in [a, b].$$

The proof of this chain rule is a direct imitation of the rule we gave for the composition of two holomorphic functions in Proposition 4.8. The only adaptation required is that h in that proof should now be restricted to small real numbers instead of small complex numbers. Also, to prove the rule at $t = a$ and $t = b$, we must restrict h to small positive and small negative real numbers, respectively.

The *advantage* of smooth curves α is that, when we need to calculate integrals along them, there are no difficulties involving the continuous derivatives α'. Also, we have Proposition 8.6 available to exploit. The *disadvantage* of smooth curves is that they are not versatile enough to handle all the situations we shall encounter. The following concept will give us what we need. It is both simple and important.

Definition 8.7 A **path** (or **contour**) from p to q in an open set Ω is a finite list of smooth curves

$$\begin{aligned} \alpha_1 &: [a_1, b_1] \to \Omega \\ \alpha_2 &: [a_2, b_2] \to \Omega \\ &\vdots \\ \alpha_n &: [a_n, b_n] \to \Omega, \end{aligned}$$

such that:

$$\begin{aligned} p &= \alpha_1(a_1) \\ \alpha_1(b_1) &= \alpha_2(a_2) \\ \alpha_2(b_2) &= \alpha_3(a_3) \\ &\vdots \\ \alpha_{n-1}(b_{n-1}) &= \alpha_n(a_n) \\ \alpha_n(b_n) &= q. \end{aligned}$$

After a little inspection, we see that a path from p to q is just a concatenation of smooth curves in which each one picks up where the previous one left off. That is, if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are smooth curves in Ω, the list $\alpha_1, \alpha_2, \ldots, \alpha_n$ is called a path running from p to q in Ω when

$$\begin{aligned} &\alpha_1 \text{ runs from } p = p_0 \text{ to some } p_1 \\ &\alpha_2 \text{ runs from } p_1 \text{ to some } p_2 \\ &\alpha_3 \text{ runs from } p_2 \text{ to some } p_3 \\ &\qquad\vdots \\ &\alpha_n \text{ runs from } p_{n-1} \text{ to } q = p_n \end{aligned}$$

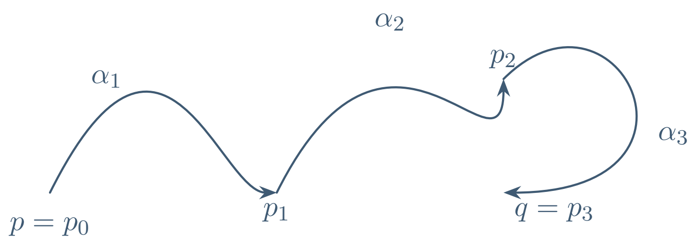

FIGURE 30

The construction is shown in Figure 30.

With some trepidation, we denote the path determined by the smooth curves $\alpha_1, \alpha_2, \ldots, \alpha_n$ by the symbol

$$\alpha = \alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n.$$

Warning The symbols $\vee$ here have no algebraic meaning (such as the formal join operation in a lattice, or the least upper bound in a poset). They merely suggest that the curves join up properly.

Here is a generic picture of a path made up of three smooth curves:

Example 8.8 Here is an example of a path: let

$$\beta : [0, 1] \to \mathbb{C} \text{ where } t \mapsto -i + t(1 + i),$$

and

$$\gamma : [0, \pi] \to \mathbb{C} \text{ where } t \mapsto e^{it}.$$

The path $\beta \vee \gamma$ is shown in Figure 31.

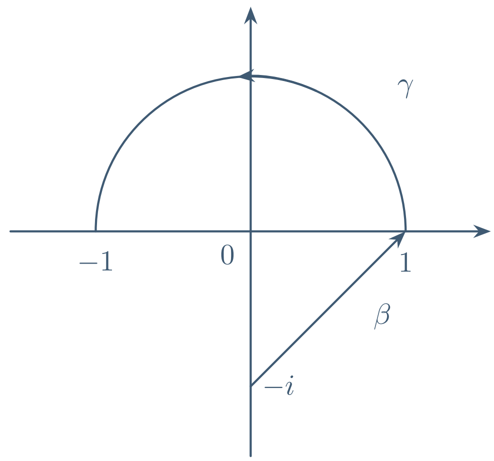

FIGURE 31

Definition 8.9 The **trajectory** of a path is taken to be the union of the trajectories of the smooth curves that make up the path.

Definition 8.10 A **piecewise smooth curve** is a path in which the parametrizing intervals $[a_j, b_j]$ are adjacent; that is, $b_j = a_{j+1}$.

For paths we merely require that $\alpha_j(b_j) = \alpha_{j+1}(a_{j+1})$. However, there is no need to tie our hands with the extra constraint that the parametrizing intervals be adjacent. So we shall speak of paths without worrying about where the parametrizing intervals are located. This removes one headache. All that matters is that the smooth curve α_{j+1} picks up in Ω where the preceding curve α_j left off.

If we should ever need the domain intervals of each α_j to be adjacent, the next result shows how that can be arranged.

Proposition 8.11 If α is a path from p to q in an open set Ω of $\mathbb{C}$, then there exists a bona fide piecewise smooth curve δ defined on some interval such that δ runs from p to q and covers exactly the same trajectory as the path α. That is, the range of δ is the union of the ranges of the smooth curves that make up the path α.

Proof. To simplify notation, let us show the result when the path α is made up of just two smooth curves. Say $\alpha = \beta \vee \gamma$, where

$$\beta : [a, b] \to \Omega \text{ and } \gamma : [c, d] \to \Omega$$

are smooth curves such that

$$p = \beta(a), \beta(b) = \gamma(c) \text{ and } \gamma(d) = q.$$

Now pick $e > b$. Then define the straight line function

$$\ell : [b, e] \to [c, d] \text{ according to } t \mapsto \frac{d-c}{e-b}(t-b) + c.$$

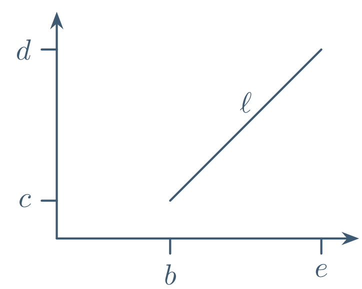

FIGURE 32

The configuration appears in Figure 32.

After that, define

$$\delta : [a, e] \to \Omega \text{ by } t \mapsto \begin{cases} \beta(t) & \text{when } t \in [a, b] \\ \gamma(\ell(t)) & \text{when } t \in [b, e]. \end{cases}$$

A close inspection of the steps should make it clear that δ is a well-defined continuous curve on $[a, e]$, that δ runs from p to q, and that the range of δ equals the union of the ranges of β and γ. All we did was reparametrize γ by means of an interval $[b, e]$ that was adjacent to $[a, b]$. $\square$

Connectedness

We need not fret unduly over Proposition 8.11, but its procedure of splicing the smooth curves that make up a path into one piecewise smooth curve on a single interval has a simple corollary worth mentioning. We should remember that a path is a finite list of smooth curves that hook up in the right way, and is not itself a single curve defined on one interval.

Corollary 8.12 If there is a path from p to q in an open set Ω, then there is a continuous curve from p to q in Ω.

Proof. Suppose there is a path from p to q in Ω. By Proposition 8.11, there exists a piecewise smooth curve

$$\delta : [a,b] \to \Omega$$

that runs from p to q and has the same trajectory as that path. In particular, this δ is a continuous curve from p to q in Ω. $\square$

Definition 8.13 For open sets, the existence of paths within them is related to connectedness. An open set Ω is called **connected** provided Ω is not the union of two disjoint, nonempty, open subsets. If Ω is such a union, we call it **disconnected**. If, for all $p, q \in \Omega$ there exists a path in Ω from p to q, then Ω is **path-connected**.

Theorem 8.14 An open set Ω is connected if and only if for every pair of points p, q in Ω, there is a path from p to q inside Ω.

Proof. *Forward implication.* Suppose first that there are points p and q in Ω for which there is no path in Ω from p to q. We will show that Ω is the union of two disjoint, nonempty, open sets. Let

$$A = \{z \in \Omega : \text{there is a path from } p \text{ to } z\}$$
$$B = \{z \in \Omega : \text{there is not a path from } p \text{ to } z\}.$$

Quite obviously, $p \in A$ and $q \in B$, so these sets are not empty. Just as obviously, $\Omega = A \cup B$, and this union is disjoint. So all we need is that A and B should be open. To get that A is open, take $w \in A$. We have a path α from p to w. Pick $r > 0$ so that the disk $D_w(r) \subseteq \Omega$, as we know for open sets. For each $z \in D_w(r)$, there is a path β from w to z inside the disk $D_w(r)$, and thus inside Ω: namely, the straight line from w to z. The joined path $\alpha \vee \beta$, whereby β is tacked on to the smooth curves that make up α, is a path from p to z inside Ω. Having just shown that $D_w(r) \subseteq A$, we obtain that A is open.

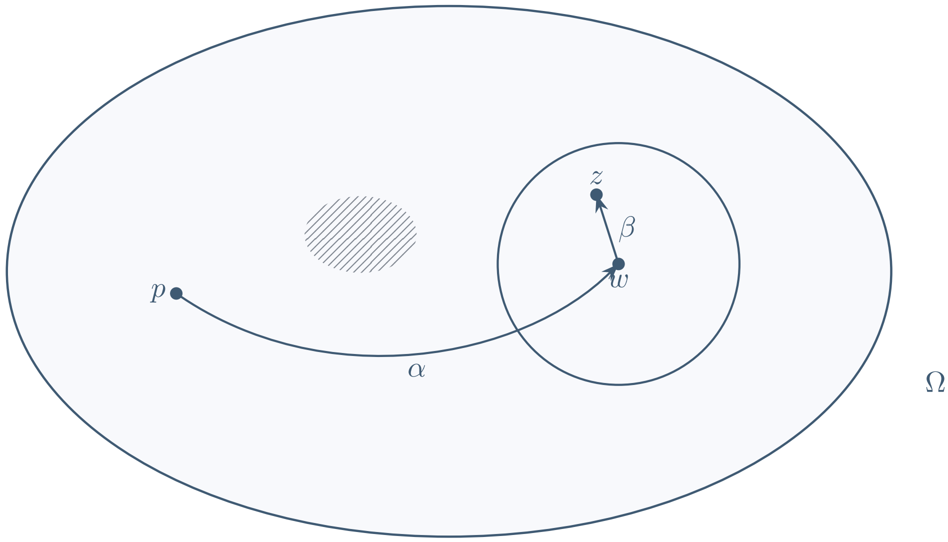

FIGURE 33

Figure 33 illustrates the setup.

A similar argument reveals that B is open. Namely, take w in B and a disk $D_w(r) \subseteq \Omega$. We want to show that $D_w(r) \subseteq B$. Suppose this is not the case, and that some z in $D_w(r)$ lies in A instead. Then we have a path α from p to z in Ω. We also have the straight-line path β from z to w inside $D_w(r)$, and thereby inside Ω. The joined path $\alpha \vee \beta$ is still in Ω and now runs from p to w, contrary to the fact that w was in B.

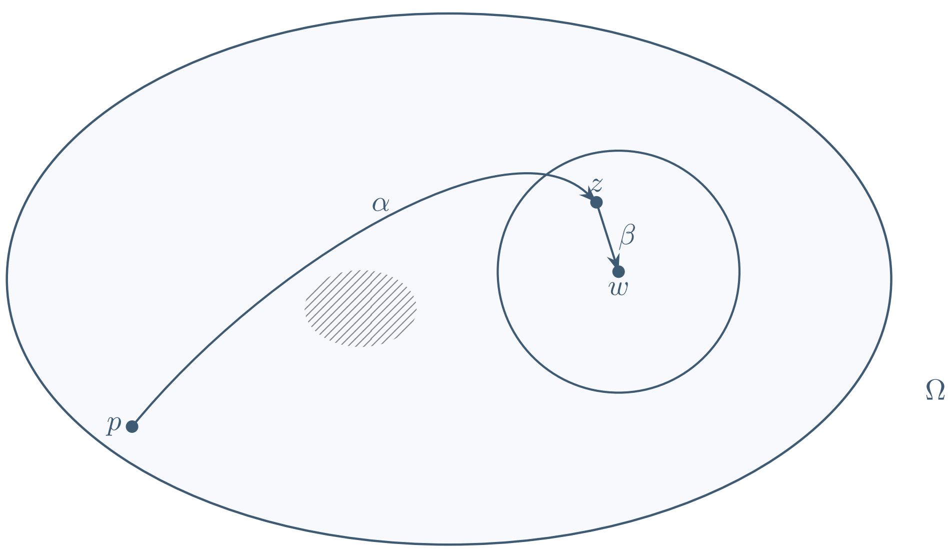

FIGURE 34

The construction is shown in Figure 34.

Reverse implication. Conversely, suppose that Ω is disconnected, so that

$$\Omega = A \cup B \text{ where } A \text{ and } B \text{ are open, nonempty, disjoint sets.}$$

Take p in A and q in B. We will show there is no path from p to q inside Ω. If, on the contrary, there were a path in Ω running from p to q, Corollary 8.12 tells us that there would be an actual curve (continuous)

$$\alpha : [a, b] \to \Omega \text{ such that } \alpha(a) = p \text{ and } \alpha(b) = q.$$

For those who know some topology, here is one way to wrap up the proof. Since α is continuous and the domain interval $[a, b]$ is connected, the image set

$$\alpha([a, b]) = \{\alpha(t) : t \in [a, b]\}$$

is also connected. This cannot be, however, because the sets $A \cap \alpha([a, b])$ and $B \cap \alpha([a, b])$ provide a disconnection of $\alpha([a, b])$. For those who don't know topology, look at the inverse image sets

$$\alpha^{-1}(A) = \{t \in [a, b] : \alpha(t) \in A\} \text{ and } \alpha^{-1}(B) = \{t \in [a, b] : \alpha(t) \in B\}.$$

These inverse images yield a partition of the interval $[a, b]$ into two disjoint, nonempty sets, open relative to $[a, b]$. Note that these inverse image sets are not empty because a is in the first set while b is in the second set. This is a contradiction to a well-known fact of real analysis: an interval cannot be partitioned into two disjoint nonempty sets that are open relative to the interval. $\square$

Definition 8.15 A connected open set will henceforth be called a **region**.

We know lots of regions: punctured planes, half-planes, disks, annuli, cut planes, and $\mathbb{C}$ itself.

Lecture 9 — Friday, January 24, 2014

Here comes a seemingly innocuous but important theorem.

Theorem 9.1 If $f : \Omega \to \mathbb{C}$ is holomorphic on Ω and $f' = 0$ on a region Ω, then f is constant on Ω.

Proof. Take $p, q \in \Omega$. Since Ω is connected, there is a path in Ω running from p to q. Let

$$\alpha = \alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n$$

be that path made up of smooth curves $\alpha_j : [a_j, b_j] \to \Omega$, defined on suitable intervals $[a_j, b_j]$. By means of Proposition 8.6, for each t in $[a_j, b_j]$, we get

$$(f \circ \alpha_j)'(t) = f'(\alpha_j(t))\alpha_j'(t) = 0\alpha_j'(t) = 0.$$

The real component functions of each $f \circ \alpha_j$ are real-valued functions of a real variable, and we just showed that they have vanishing derivative on $[a_j, b_j]$. From elementary calculus we conclude that these component functions are constant on each $[a_j, b_j]$. Therefore each $f \circ \alpha_j$ is constant on each interval $[a_j, b_j]$. Thus f is constant on each trajectory $\alpha_j[a_j, b_j]$. Since adjacent trajectories overlap, that is, $\alpha_j(b_j) = \alpha_{j+1}(a_{j+1})$, the function f is constant on the total trajectory of the path α. Since p and q are in the trajectory of α, we see that $f(p) = f(q)$. □

More importantly, Theorem 9.1 implies something about what are known as primitives.

Definition 9.2 A function f is called a **primitive** of a function g on a region Ω provided $f' = g$ on Ω.

There is no guarantee that a given function g will have a primitive, but a simple consideration based on Theorem 9.1 shows that any two primitives for g on a region Ω (should they exist) can only differ by a constant:

Corollary 9.3 If $f_1, f_2 : \Omega \to \mathbb{C}$ satisfy $f_1' = f_2'$ on a region Ω, then $f_1 - f_2 = c$ for some constant $c \in \mathbb{C}$.

Proof. Define $h = f_1 - f_2$. Then $h : \Omega \to \mathbb{C}$ is holomorphic and

$$h' = f_1' - f_2' = 0$$

on Ω. By Theorem 9.1, h is constant on Ω. Therefore there exists $c \in \mathbb{C}$ such that

$$f_1 - f_2 = h = c$$

on Ω. □

Example 9.4 Take the region

$$\Omega = \{x + iy : \text{either } y \neq 0 \text{ or } x > 0\},$$

which is the cut plane. We know that on the cut plane the derivative of the principal branch of the logarithm $f(z) = \log z = \log |z| + i \arg z$ is $1/z$. Therefore any primitive for $w = 1/z$ on the cut plane Ω can only be $\log z + c$, where c is any complex constant.

4. Complex Power Series

Convergence of Power Series and Hadarmard's Formula

Definition 9.5 A sequence $\{z_n\}$ of complex numbers **converges** to p if for every $\varepsilon > 0$, we have $|z_n - p| < \varepsilon$ eventually. The sequence is **Cauchy** if for every $\varepsilon > 0$, there is an N such that $|z_n - z_m| < \varepsilon$ whenever $m, n \geq N$.

Theorem 9.6 A sequence $\{z_n\}$ is Cauchy if and only if $\{z_n\}$ converges.

Proof. If $z_n \to z$ in $\mathbb{C}$, then $\{z_n\}$ is Cauchy by the triangle inequality, exactly as in real analysis.

Conversely, suppose $\{z_n\}$ is Cauchy in $\mathbb{C}$, and write $z_n = x_n + iy_n$. Then

$$|x_n - x_m| \leq |z_n - z_m| \qquad \text{and} \qquad |y_n - y_m| \leq |z_n - z_m|.$$

Hence $\{x_n\}$ and $\{y_n\}$ are Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exist $x, y \in \mathbb{R}$ such that $x_n \to x$ and $y_n \to y$. Let $z = x + iy$. Then

$$|z_n - z| = \sqrt{(x_n - x)^2 + (y_n - y)^2} \to 0,$$

so $z_n \to z$ in $\mathbb{C}$. Therefore every Cauchy sequence in $\mathbb{C}$ converges. □

Thus $\mathbb{C}$ is itself complete.

When $\{z_n\}$ converges to p, we write $z_n \to p$, or sometimes $p = \lim_{n\to\infty} z_n$. Now, for a complex sequence $\{z_n\}$, the sequence of sums $S_n = z_1 + z_2 + \cdots + z_n$ is called a series and is denoted by $\sum_{n=1}^{\infty} z_n$. The z_n are called the terms of the series.

Definition 9.7 The series $\sum_{n=1}^{\infty} z_n$ **converges** when $S_n \to s$ for some $s \in \mathbb{C}$. In this case, s is called the sum of the series, and we also write $s = \sum_{n=1}^{\infty} z_n$. If $\{S_n\}$ has no limit, the series **diverges**.

Lecture 10 — Monday, January 27, 2014

Here are two baby convergence tests:

Proposition 10.1

(1) If $z_n \not\to 0$, then $\sum_{n=1}^{\infty} z_n$ diverges.

(2) If $\sum_{n=1}^{\infty} |z_n|$ converges, then $\sum_{n=1}^{\infty} z_n$ converges. We say in this case that $\sum_{n=1}^{\infty} z_n$ **converges absolutely**.

Proof. (1) If $\sum_{n=1}^{\infty} z_n$ converged to s, say, then $z_n = s_n - s_{n-1} \to s - s = 0$. This is the first test.

(2) Let $s_n = z_1 + \cdots + z_n$. For $n > m$ we use the triangle inequality to get

$$|s_n - s_m| = |z_{m+1} + z_{m+2} + \cdots + z_n| \leq |z_{m+1}| + |z_{m+2}| + \cdots + |z_n|.$$

Since $\sum_{n=1}^{\infty} |z_n|$ converges, its partial sums are Cauchy. And so $|z_{m+1}| + \cdots + |z_n| \to 0$ as $m, n \to \infty$. Thus $|s_n - s_m| \to 0$ as $n, m \to \infty$. And so $\sum_{n=1}^{\infty} z_n$ converges. $\square$

Let $a_0, a_1, a_2, \dots \in \mathbb{C}$ and let $z \in \mathbb{C}$. We can thus form the **power series** $\sum_{n=0}^{\infty} a_n z^n$. If $z = 0$, the series converges to a_0 since all

$$s_n = a_0 + a_1 0^1 + a_2 0^2 + \cdots + a_n 0^n = a_0.$$

The interesting question is whether $\sum_{n=0}^{\infty} a_n z^n$ converges for z other than 0.

Example 10.2 When all $a_n = 1$ we get the **geometric series** $\sum_{n=0}^{\infty} z^n$. In this case

$$s_n = 1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} = \frac{1}{1 - z} - \frac{z^{n+1}}{1 - z}$$

as long as $z \neq 1$. If $|z| < 1$, we see that

$$\left| \frac{z^{n+1}}{1 - z} \right| = \frac{|z|^{n+1}}{(|1 - z|)} \to 0,$$

which tells us that $\sum_{n=0}^{\infty} z^n$ converges to $1/(1 - z)$. If $|z| \geq 1$, we have $|z^n| = |z|^n \not\to 0$, which tells us that $\sum_{n=0}^{\infty} z^n$ diverges.

Definition 10.3 Recall that if t_n is a bounded sequence, then the biggest **cluster point** of t_n is the number $p = \limsup t_n$ with the property that $t_n < p + \varepsilon$ eventually and $p - \varepsilon < t_n$ infinitely often. If t_n is not bounded above, we declare $\limsup t_n = \infty$.

Proposition 10.4 (Hadamard's formula) Suppose $\sum_{n=0}^{\infty} a_n z^n$ is a complex power series. Let

$$R = \left(\limsup_{n \to \infty} |a_n|^{1/n} \right)^{-1}.$$

By convention, we take $1/0 = \infty$ and $1/\infty = 0$.

1. If $z \in \mathbb{C}$ and $R < |z|$, then $a_n z^n \not\to 0$ and the series $\sum_{n=0}^{\infty} a_n z^n$ does not converge.
2. If $z \in \mathbb{C}$ and $|z| < R$, then $\sum_{n=0}^{\infty} |a_n z^n|$ converges, and thus $\sum_{n=0}^{\infty} a_n z^n$ converges.

Proof. We argue in two parts.

$$\begin{aligned}
\frac{1}{\limsup_{n \to \infty} |a_n|^{1/n}} < |z| &\implies \frac{1}{|z|} < \limsup_{n \to \infty} |a_n|^{1/n} \\
&\implies \frac{1}{|z|} < |a_n|^{1/n} \text{ infinitely often} \\
&\implies 1 < |a_n|^{1/n} |z| \text{ infinitely often} \\
&\implies 1 < |a_n z^n|^{1/n} \text{ infinitely often} \\
&\implies 1 < |a_n z^n| \text{ infinitely often} \\
&\implies a_n z^n \not\to 0
\end{aligned}$$

$$\implies \sum_{n=0}^{\infty} a_n z^n \text{ diverges.}$$

Lecture 11 — Wednesday, January 29, 2014

Proof continued. Now suppose $|z| < (\limsup_{n\to\infty} |a_n|^{1/n})^{-1}$. Pick r such that $|z| < r < (\limsup_{n\to\infty} |a_n|^{1/n})^{-1}$. Then $|z|/r < 1$ and $\limsup_{n\to\infty} |a_n|^{1/n} < 1/r$.

$$\begin{aligned}
&\implies |a_n|^{1/n} < \frac{1}{r} \text{ eventually} \\
&\implies |a_n|^{1/n}|z| < \frac{|z|}{r} \text{ eventually} \\
&\implies |a_n z^n|^{1/n} \le \left|\frac{z}{r}\right| \text{ eventually} \\
&\implies |a_n z^n| \le \left|\frac{z}{r}\right|^n \text{ eventually.}
\end{aligned}$$

Since $|z/r| < 1$, the geometric series $\sum_{n=0}^{\infty} |z/r|^n$ converges. By comparison, $\sum_{n=0}^{\infty} |a_n z^n|$ converges, so $\sum_{n=0}^{\infty} a_n z^n$ converges. ☐

Definition 11.1 The quantity $R \in [0,\infty]$ is called the **radius of convergence** (or **radius** for short) of the series $\sum_{n=0}^{\infty} a_n z^n$. The formula

$$R = \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1}$$

is called **Hadamard's formula** for the radius.

Example 11.2 Consider the power series

$$\sum_{n=1}^{\infty} \frac{(3-4i)^n}{n} z^n.$$

We have

$$|a_n|^{1/n} = \left|\frac{(3-4i)^n}{n}\right|^{1/n} = \left(\frac{5^n}{n}\right)^{1/n} = \frac{5}{n^{1/n}} \to 5$$

as $n \to \infty$. So $R = 1/5$.

Here is an observation about $\limsup$. Suppose s_n, t_n are bounded sequences and $\limsup s_n < \limsup t_n$. Pick b such that $\limsup s_n < b < \limsup t_n$. Then $s_n < b$ eventually and $b < t_n$ infinitely often. So $s_n < t_n$ infinitely often. And so, if $t_n \le s_n$ eventually, then $\limsup t_n \le \limsup s_n$.

Example 11.3 Now, say $a_n \in \mathbb{C}$ and, for $n \ge 1$,

$$\frac{1}{n^2} \le |a_n| \le n^2$$

Then

$$\left(\frac{1}{n^2}\right)^{1/n} \le |a_n|^{1/n} \le (n^2)^{1/n}.$$

So

$$\limsup \left(\frac{1}{n^2}\right)^{1/n} = \limsup \left(\frac{1}{n^{1/n}}\right)^2 = \lim_{n\to\infty} \left(\frac{1}{n^{1/n}}\right)^2 = 1.$$

Also, $\limsup (n^2)^{1/n} = 1$. Therefore $\limsup |a_n|^{1/n} = 1$. By Proposition 10.4, the radius of $\sum_{n=0}^{\infty} a_n z^n$ is $1/1 = 1$.

Example 11.4 (Bonus problem) If p_n is the nth prime, that is, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, $\ldots$, what is the radius of $\sum_{n=1}^{\infty} p_n z^n$? We need $\limsup p_n^{1/n}$, so some estimates on the size of the nth prime are needed...

By the prime number theorem, $p_n \sim n \log n$, so $\limsup_{n\to\infty} p_n^{1/n} = 1$. Therefore the radius of the series is 1.

Now let us look at the radius from another perspective. Again, start with $\sum_{n=0}^{\infty} a_n z^n$. Let $B = \{r \ge 0 : |a_n| r^n \text{ stays bounded as } n \to \infty\}$. Put $R = \sup B$ if B is bounded; otherwise, put $R = \infty$.

Example 11.5 Here are some quick-and-dirty radius calculations.

1. If $a_n = i^n$, then $B = \{r \geq 0 : r^n \text{ is bounded}\} = [0, 1]$, so $R = 1$.
2. If $a_n = i^n n$, then $B = \{r \geq 0 : nr^n \text{ is bounded}\} = [0, 1)$, so $R = 1$.
3. If $a_n = 2^n(1 + i)$, then $B = \{r \geq 0 : \sqrt{2}2^n r^n \text{ is bounded}\} = [0, 1/2]$, so $R = 1/2$.
4. If $a_n = ((1 + i)^n)/n!$, then $B = \{r \geq 0 : (\sqrt{2}^n r^n)/n! \text{ is bounded}\} = [0, \infty)$, so $R = \infty$.
5. If $a_n = n!$, then $B = \{r \geq 0 : n!r^n \text{ is bounded}\} = \{0\}$, so $R = 0$.

Proposition 11.6 If $z \in \mathbb{C}$ and $R < |z|$, then $a_n z^n \not\to 0$ and so $\sum_{n=0}^{\infty} a_n z^n$ diverges. If $z \in \mathbb{C}$ and $|z| < R$, then $\sum_{n=0}^{\infty} |a_n z^n|$ converges and so $\sum_{n=0}^{\infty} a_n z^n$ converges.

Proof. Say $R < |z|$. Certainly $|a_n||z|^n = |a_n z^n|$ is not bounded, and so $a_n z^n \not\to 0$. Say $|z| < R$. Pick r such that $|z| < r \in B$. Then $|z|/r < 1$. Let M be an upper bound for all $|a_n|r^n$. Then

$$|a_n z^n| = |a_n|r^n \left|\frac{z}{r}\right|^n \leq M \left|\frac{z}{r}\right|^n.$$

Since $|z/r| < 1$, the geometric series $\sum_{n=0}^{\infty} M\,|z/r|^n$ converges. By comparison, $\sum_{n=0}^{\infty} |a_n z^n|$ converges. ☐

Lecture 12 — Friday, January 31, 2014

So $R = \sup B$ is an alternate formula for the radius of $\sum_{n=0}^{\infty} a_n z^n$. We can use this alternate formula to get a handy result about the radius of a power series:

Proposition 12.1 If $\sum_{n=0}^{\infty} a_n z^n$ has radius R, then $\sum_{n=1}^{\infty} n a_n z^n$ still has radius R.

Proof. We have $B = \{r \geq 0 : |a_n|r^n \text{ stays bounded}\}$ and $R = \sup B$. If $0 \leq s < R$, pick an r from B such that $s < r$. Then

$$|n a_n s^n| = |a_n r^n| \cdot n \left|\frac{s}{r}\right|^n.$$

Now $|a_n r^n|$ stays bounded. And $|s/r| < 1$, which forces $n\,|s/r|^n$ to stay bounded. Thereby $|n a_n s^n|$ stays bounded. (In fact, $|n a_n s^n| \to 0$.) Let $C = \{s \geq 0 : |n a_n s^n| \text{ stays bounded}\}$. We just saw that $C \supseteq [0, R)$, so $\sup C \geq R$. Clearly $C \subseteq B$ and so $\sup C \leq R$. Hence $\sup C = R$. ☐

Derived Series and the Differentiation Theorem

Definition 12.2 We suppose $\sum_{n=0}^{\infty} a_n z^n$ has positive radius R. The **derived series** is defined to be $\sum_{n=1}^{\infty} n a_n z^{n-1}$. By Proposition 12.1, $\sum_{n=1}^{\infty} n a_n z^n$ has radius R; away from $z = 0$, the derived series is its quotient by z, while at $z = 0$ it takes the finite value a_1. Hence the derived series also has radius R.

It would be desirable to say that

$$\left(\sum_{n=0}^{\infty} a_n z^n\right)' = \sum_{n=0}^{\infty} (a_n z^n)' = \sum_{n=1}^{\infty} n a_n z^{n-1} \tag{$\star$}$$

for all z in $D_0(R)$. However, the limit process of differentiation becomes interchanged with the limit process of infinite summation. That should raise a red flag, as the following example illustrates.

Consider the double sequence x_{mn}:

$$\begin{array}{ccccccccc}
 & n \rightarrow & & & & & & & \\
m \downarrow & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & \rightarrow 1 \\
 & 0 & 1 & 1 & 1 & 1 & 1 & \cdots & \rightarrow 1 \\
 & 0 & 0 & 1 & 1 & 1 & 1 & \cdots & \rightarrow 1 \\
 & 0 & 0 & 0 & 1 & 1 & 1 & \cdots & \rightarrow 1 \\
 & 0 & 0 & 0 & 0 & 1 & 1 & \cdots & \\
 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\
 & \vdots & \vdots & \vdots & \vdots & & \vdots & & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & & \\
 & 0 & \underbrace{0}_{\text{as } m\to\infty} & 0 & 0 & \cdots & 0 & \cdots &
\end{array}$$

$$\lim_{m\to\infty} \lim_{n\to\infty} x_{mn} = \lim_{m\to\infty} 1 = 1$$

$$\lim_{n\to\infty} \lim_{m\to\infty} x_{mn} = \lim_{n\to\infty} 0 = 0$$

These are not the same.

To obtain the important result ($\star$), we have to try harder. A preliminary algebraic identity will help. For $n \geq 2$ and $z, p \in \mathbb{C}$ with $z \neq p$, we have

$$\begin{aligned}\frac{z^n - p^n}{z-p} - np^{n-1} &= z^{n-1} + z^{n-2}p + z^{n-3}p^2 + \cdots + zp^{n-2} + p^{n-1} - np^{n-1} \\ &= z^{n-1} + z^{n-2}p + z^{n-3}p^2 + z^{n-4}p^3 + \cdots + zp^{n-2} - (n-1)p^{n-1} \\ &= (z-p)\Big(z^{n-2} + 2z^{n-3}p + 3z^{n-4}p^2 + 4z^{n-5}p^3 + \dots \\ &\qquad + (n-2)zp^{n-3} + (n-1)p^{n-2}\Big).\end{aligned}$$

A direct calculation verifies this identity.

Theorem 12.3 (The differentiation theorem) Suppose $\sum_{n=0}^{\infty} a_n z^n$ has radius $R > 0$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ when $z \in D_0(R)$ (that is, $|z| < R$). Then:

1. $\sum_{n=1}^{\infty} n a_n z^{n-1}$ also has radius R (by Proposition 12.1).
2. f is holomorphic on $D_0(R)$.
3. $f'(p) = \sum_{n=1}^{\infty} n a_n p^{n-1}$ for all $p \in D_0(R)$.

Proof. For $p \in D_0(R)$ let $g(p) = \sum_{n=1}^{\infty} n a_n p^{n-1}$. We need to prove that

$$\frac{f(z) - f(p)}{z - p} - g(p) \to 0 \quad \text{as } z \to p.$$

Pick r such that $|p| < r < R$:

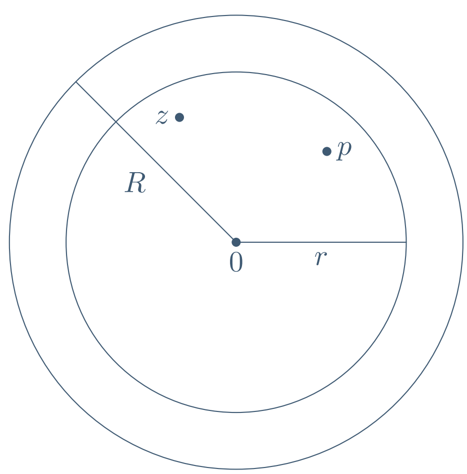

Figure 35

Figure 35 records the relevant geometry.

For $z \in D_0(r)$ we get:

$$\begin{aligned}
&\frac{f(z)-f(p)}{z-p} - g(p) \\
&= \frac{1}{z-p}\sum_{n=0}^{\infty} a_n(z^n - p^n) - \sum_{n=1}^{\infty} n a_n p^{n-1} && \text{by the usual limit rules} \\
&= \frac{1}{z-p}\sum_{n=1}^{\infty} a_n(z^n - p^n) - \sum_{n=1}^{\infty} n a_n p^{n-1} \\
&= \sum_{n=1}^{\infty} a_n \left(\frac{z^n - p^n}{z-p} - np^{n-1}\right) \\
&= \sum_{n=2}^{\infty} a_n \left(\frac{z^n - p^n}{z-p} - np^{n-1}\right) && \text{the term for } n=1 \text{ was } 0 \\
&= (z-p)\sum_{n=2}^{\infty} a_n \Big(z^{n-2} + 2z^{n-3}p + 3z^{n-4}p^2 + \cdots + (n-1)p^{n-2}\Big) && \text{by our algebraic identity.}
\end{aligned}$$

Since $|z| < r$ and $|p| < r$, the triangle inequality yields:

$$\begin{aligned}
&\left|z^{n-2} + 2z^{n-3}p + 3z^{n-4}p^2 + \cdots + (n-1)p^{n-2}\right| \\
&\le r^{n-2}(1 + 2 + 3 + \cdots + n - 1) \\
&= r^{n-2}\frac{n(n-1)}{2} \\
&\le n^2 r^{n-2}.
\end{aligned}$$

Choose ρ with $r < \rho < R$. Since $\sum_{n=0}^{\infty} |a_n|\rho^n$ converges, its terms are bounded, say $|a_n|\rho^n \le C$. Therefore

$$n^2|a_n|r^{n-2} \le \frac{C}{r^2}n^2\left(\frac{r}{\rho}\right)^n,$$

and the series on the right converges. Thus $\sum_{n=2}^{\infty} n^2|a_n|r^{n-2} = M < \infty$. Then for z in $D_0(r)$, we have:

$$\begin{aligned}
\left|\frac{f(z)-f(p)}{z-p} - g(p)\right| &\le |z-p|\sum_{n=2}^{\infty} |a_n||z^{n-2} + 2z^{n-3}p + \cdots + (n-1)p^{n-2}| \\
&\le |z-p|\sum_{n=2}^{\infty} |a_n|n^2 r^{n-2}
\end{aligned}$$

$$\leq |z-p|M.$$

Now let $z \to p$ and apply the squeeze theorem. □

Lecture 13 — Monday, February 03, 2014

Theorem 12.3 shows that power series induce holomorphic functions on disks. Here is our first interesting application of Theorem 12.3:

Example 13.1 The power series $\sum_{n=0}^{\infty} z^n/n!$ converges for all $z \in \mathbb{C}$. This holds because the set

$$B = \left\{ r \geq 0 : \frac{r^n}{n!} \text{ stays bounded} \right\} = [0, \infty).$$

So the radius for this series is ∞. The function $f(z) = \sum_{n=0}^{\infty} z^n/n!$ is entire and

$$f'(z) = \sum_{n=1}^{\infty} n \frac{1}{n!} z^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = f(z).$$

Here is an interesting non-coincidence. We have $f' = f$ on $\mathbb{C}$. We also know that $(e^z)' = e^z \neq 0$ on $\mathbb{C}$. So

$$\left(\frac{f(z)}{e^z} \right)' = \frac{e^z f'(z) - f(z)(e^z)'}{e^{2z}} = \frac{e^z f(z) - f(z) e^z}{e^{2z}} = 0.$$

It follows from Theorem 9.1 that $f(z)/e^z = C$ is constant on $\mathbb{C}$. That is, $f(z) = Ce^z$ on $\mathbb{C}$. Then $1 = f(0) = Ce^0 = C$ gives $f(z) = e^z$.

To summarize:

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \quad \text{for all } z \in \mathbb{C}.$$

Definition 13.2 Given $\sum_{n=0}^{\infty} a_n z^n$, we can readily form the **integrated series**

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} = a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + \dots$$

The integrated series has the same radius R as the original series: in Hadamard's formula, the extra factor $(n+1)^{-1/(n+1)}$ tends to 1, and shifting the coefficient index does not change the limsup. If $g(z) = \sum_{n=0}^{\infty} a_n/(n+1) z^{n+1}$ on $D_0(R)$, then Theorem 12.3 gives $g'(z) = \sum_{n=0}^{\infty} a_n z^n$ on $D_0(R)$. So if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $D_0(R)$, this holomorphic f has a primitive on $D_0(R)$ given by the integrated series.

Example 13.3 We know

$$\frac{1}{(1-z)} = 1 + z + z^2 + \dots + z^n + \dots$$

on $D_0(1)$. So

$$\frac{1}{(1+z)} = \frac{1}{(1-(-z))} = 1 - z + z^2 - z^3 + \dots$$

when $|z| < 1$.

The integrated series for this is

$$g(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots \quad \text{still on } D_0(1)$$

and $g'(z) = 1/(1+z)$.

We also have $\log z$, holomorphic on the cut plane Ω shown in Figure 36.

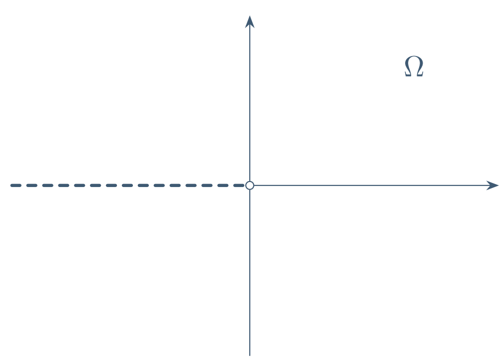

FIGURE 36

So $\log(1+z)$ is holomorphic on $\Pi = \{z : 1+z \in \Omega\}$, which is another cut plane.

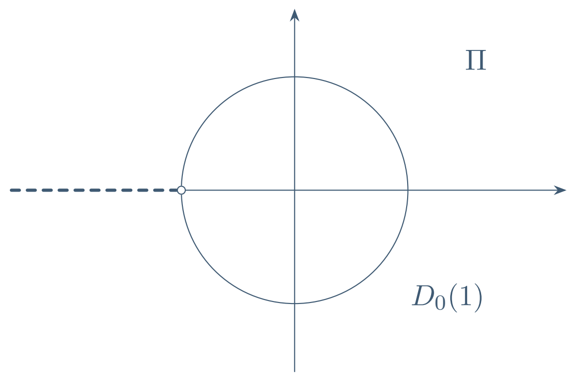

Figure 37

Figure 37 illustrates the setup. Note that $D_0(1) \subseteq \Pi$ and since $\log(1+z)' = 1/(1+z)$ on $D_0(1)$, we obtain

$$\log(1+z) = g(z) + C = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots + C \quad \text{on } D_0(1),$$

where C is constant. Put $z = 0$ to get $0 = 0 + C$, which gives $C = 0$. So we have a power series representation

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots \quad \text{for } z \in D_0(1).$$

Are there power series representations for $\log(1+z)$ elsewhere on Π?

5. Path Integrals and Primitives

Path Integrals

It is widely understood that the cornerstone of complex analysis is Cauchy's theorem. After we go through its proof, we can reap the benefits of its various applications. However, the statement and proof of the theorem, if done right, are complicated. We will have to proceed with care.

First, we have to build up the notion of integration along a path. Start with a continuous curve

$$\alpha = x + iy : [a,b] \to \mathbb{C}$$

$$t \mapsto x(t) + iy(t).$$

Definition 13.4 The **integral** of the continuous curve α is defined to be

$$\int_a^b \alpha = \int_a^b \alpha(t)\,\mathrm{d}t = \int_a^b x(t)\,\mathrm{d}t + i\int_a^b y(t)\,\mathrm{d}t.$$

Here, the integrals $\int_a^b x(t)\,\mathrm{d}t$ and $\int_a^b y(t)\,\mathrm{d}t$ are the usual Riemann integrals of continuous functions on $[a, b]$.

Example 13.5 Take $\alpha(t) = e^{it} = \cos t + i\sin t$ over $[0, \pi/2]$. Then

$$\int_0^{\pi/2} \alpha = \int_0^{\pi/2} \cos t\,\mathrm{d}t + i\int_0^{\pi/2} \sin t\,\mathrm{d}t = 1 + i.$$

This integral has properties we might expect. For example, complex linearity holds:

$$\int_a^b \alpha \pm \beta = \int_a^b \alpha \pm \int_a^b \beta$$

and

$$\int_a^b c\alpha = c\int_a^b \alpha$$

where $c \in \mathbb{C}$. The second equality is not quite as trivial as the first. To verify it, write $c = u + iv$ and $\alpha = x + iy$, where $u, v \in \mathbb{R}$ and $x, y : [a, b] \to \mathbb{R}$ are continuous. Then

$$c\alpha = (u + iv)(x + iy) = (ux - vy) + i(vx + uy).$$

Hence, by the definition of the integral and the usual linearity of real Riemann integrals,

$$\begin{aligned}
\int_a^b c\alpha &= \int_a^b (ux - vy) + i\int_a^b (vx + uy) \\
&= u\int_a^b x - v\int_a^b y + i\left(v\int_a^b x + u\int_a^b y\right) \\
&= (u + iv)\left(\int_a^b x + i\int_a^b y\right) \\
&= c\int_a^b \alpha.
\end{aligned}$$

We also have the following important bound:

Proposition 13.6

$$\left|\int_a^b \alpha\right| \leq \int_a^b |\alpha|,$$

where the term on the right is the usual Riemann integral of $|\alpha| : [a,b] \to \mathbb{R}$ defined by $t \mapsto \sqrt{x^2(t) + y^2(t)}$.

Proof. This can be done by going back to Riemann sums, but here is a sneaky alternative. We can think of multiplication by a complex number of modulus 1 as a rotation of $\mathbb{C}$. Hence, pick $c \in \mathbb{C}$ such that $|c| = 1$ and $c \displaystyle\int_a^b \alpha$ is a nonnegative real number. Then

$$\begin{aligned}
\left|\int_a^b \alpha\right| &= \left|c\int_a^b \alpha\right| && \text{since } |c| = 1 \\
&= c\int_a^b \alpha && \text{since } c\int_a^b \alpha \in \mathbb{R} \text{ and } \geq 0 \\
&= \int_a^b c\alpha && \text{linearity} \\
&= \int_a^b \operatorname{Re}(c\alpha) + i\int_a^b \operatorname{Im}(c\alpha) && \text{definition of the integral} \\
&= \int_a^b \operatorname{Re}(c\alpha) && \text{since these integrals are all real} \\
&\leq \int_a^b |\operatorname{Re}(c\alpha)| && \text{usual triangle inequality for real-valued functions} \\
&\leq \int_a^b |c\alpha| && \text{since } |\operatorname{Re}(c\alpha)| \leq |c\alpha| \\
&= \int_a^b |\alpha| && \text{since } |c| = 1.
\end{aligned}$$

$\square$

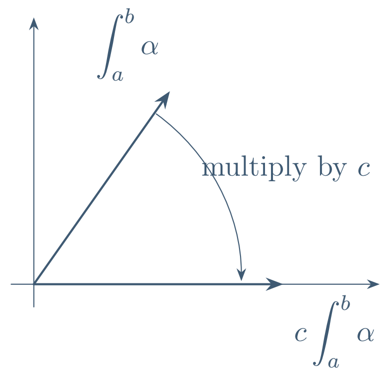

FIGURE 38

The construction is shown in Figure 38.

Change of Variables

If $x : [a,b] \to \mathbb{R}$ is continuous and $h : [c,d] \to [a,b]$ has continuous derivative h' on $[c,d]$ and $h(c) = a$, $h(d) = b$, then we know from calculus that

$$\int_c^d x(h(t))h'(t)\,\mathrm{d}t = \int_a^b x(s)\,\mathrm{d}s.$$

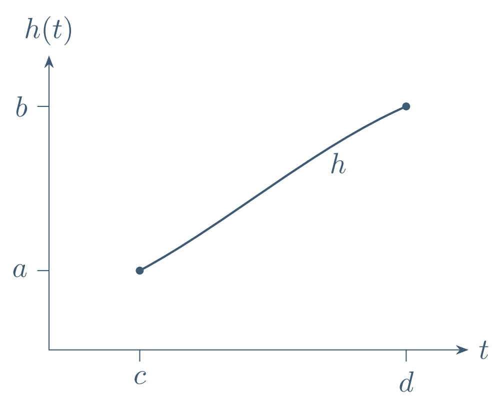

FIGURE 39

Figure 39 records the relevant geometry.

If, on the other hand, we have $h(c) = b$ and $h(d) = a$, then we also know from calculus that

$$\int_c^d x(h(t))h'(t)\,\mathrm{d}t = -\int_a^b x(s)\,\mathrm{d}s.$$

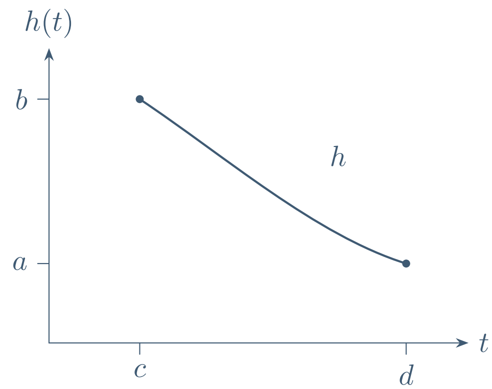

FIGURE 40

The configuration appears in Figure 40.

When we apply this change of variables to the real and imaginary parts of the curve $\alpha = x + iy : [a, b] \to \mathbb{C}$, we get the same change of variables result for curves.

We thus have the following summary:

Proposition 13.7 If $\alpha : [a, b] \to \mathbb{C}$ is a curve and $h : [c, d] \to [a, b]$ has continuous derivative h' on $[c, d]$ with $h(c) = a$ and $h(d) = b$, then

$$\int_c^d \alpha(h(t))h'(t)\,\mathrm{d}t = \int_a^b \alpha(s)\,\mathrm{d}s.$$

On the other hand, if $h(c) = b$ and $h(d) = a$, then

$$\int_c^d \alpha(h(t))h'(t)\,\mathrm{d}t = -\int_a^b \alpha(s)\,\mathrm{d}s.$$

Lecture 14 — Wednesday, February 05, 2014

What we really want to integrate are functions $f : \Omega \to \mathbb{C}$ along curves in Ω. We shall be working with the following integrals a lot!

Definition 14.1 If $f : \Omega \to \mathbb{C}$ is a continuous function and if $\gamma : [a, b] \to \Omega$ is a smooth curve then the **integral of f along γ** is defined by

$$\int_\gamma f = \int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

We might observe that the function $t \mapsto f(\gamma(t))\gamma'(t)$ is a continuous map from $[a, b]$ into $\mathbb{C}$, and thereby its integral in the definition above fits the preceding discussion of integrals.

Here is a brief motivation for the definition. Let $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ be a fine partition of $[a, b]$ and let $p_j = \gamma(t_j)$ on the trajectory of γ.

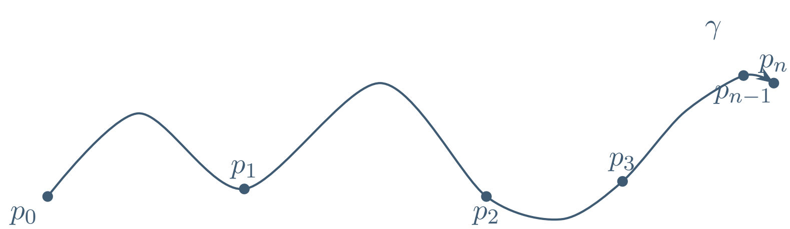

FIGURE 41

Figure 41 illustrates the setup.

From our impression of Riemann sums, we might expect that

$$\begin{aligned}\int_\gamma f &\approx \sum_{j=1}^n f(p_j)(p_j - p_{j-1})\\ &= \sum_{j=1}^n f(\gamma(t_j))(\gamma(t_j) - \gamma(t_{j-1}))\\ &= \sum_{j=1}^n f(\gamma(t_j))\frac{\gamma(t_j) - \gamma(t_{j-1})}{t_j - t_{j-1}}(t_j - t_{j-1})\end{aligned}$$

$$\approx \sum_{j=1}^{n} f(\gamma(t_j))\gamma'(t_j)(t_j - t_{j-1})$$
$$\approx \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$$

since for a fine partition and continuous γ', we might hope

$$\gamma'(t_j) \approx \frac{(\gamma(t_j) - \gamma(t_{j-1}))}{(t_j - t_{j-1})}.$$

Those who know line integrals from vector calculus will have a better motivation for the definition.

Example 14.2 Pick any $p \in \mathbb{C}$. The smooth curve $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by $t \mapsto p + e^{it}$ describes a circle going counterclockwise once around p.

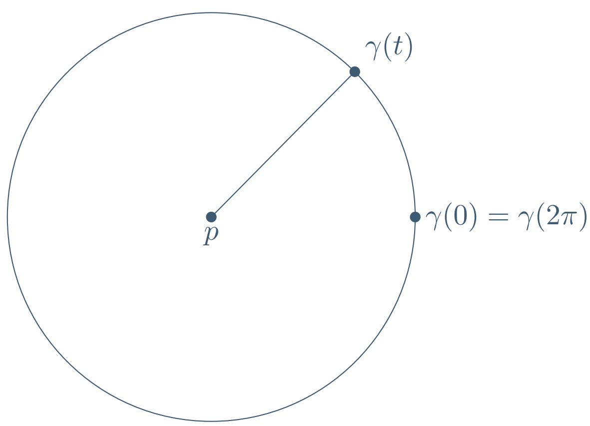

FIGURE 42

The orientation is shown in Figure 42.

Let us calculate $\displaystyle\int_\gamma 1/(z-p)\,\mathrm{d}z$ and $\displaystyle\int_\gamma 1/(z-p)^2\,\mathrm{d}z$. The first integral is

$$\int_\gamma \frac{1}{z-p}\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{(p+e^{it})-p} \cdot ie^{it}\,\mathrm{d}t$$
$$= \int_0^{2\pi} \frac{1}{e^{it}} \cdot ie^{it}\,\mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^{2\pi} i \, \mathrm{d}t \\
&= 2\pi i.
\end{aligned}
$$

The second is

$$
\begin{aligned}
\int_\gamma \frac{1}{(z-p)^2} \, \mathrm{d}z &= \int_0^{2\pi} \frac{1}{(p+e^{it}-p)^2} i e^{it} \, \mathrm{d}t \\
&= \int_0^{2\pi} i e^{-it} \, \mathrm{d}t \\
&= \int_0^{2\pi} i(\cos t - i \sin t) \, \mathrm{d}t \\
&= \int_0^{2\pi} \sin t \, \mathrm{d}t + i \int_0^{2\pi} \cos t \, \mathrm{d}t \\
&= 0.
\end{aligned}
$$

If $\gamma : [a,b] \to \mathbb{C}$ is a smooth curve and $h : [c,d] \to [a,b]$ has a continuous derivative with $h(c) = a$ and $h(d) = b$, we call $\gamma \circ h : [c,d] \to \mathbb{C}$ a reparametrization of γ. Clearly, γ and $\gamma \circ h$ traverse the same trajectory in $\mathbb{C}$.

Proposition 14.3 (Reparameterization) If $\gamma : [a,b] \to \Omega$ and $h : [c,d] \to [a,b]$ gives the reparametrization $\gamma \circ h$, and $f : \Omega \to \mathbb{C}$ is continuous, then

$$
\int_{\gamma \circ h} f = \int_\gamma f.
$$

Reparametrization does not change the integral of f.

Proof. This goes back to Proposition 13.7, applied to the curve $\alpha(t) = f(\gamma(t))\gamma'(t)$ defined on $[a,b]$. We have

$$
\begin{aligned}
\int_{\gamma \circ h} f &= \int_c^d f((\gamma \circ h)(t))(\gamma \circ h)'(t) \, \mathrm{d}t \\
&= \int_c^d f(\gamma(h(t)))\gamma'(h(t))h'(t) \, \mathrm{d}t \quad \text{(by the chain rule on } \gamma \circ h) \\
&= \int_c^d \alpha(h(t))h'(t) \, \mathrm{d}t
\end{aligned}
$$

$$= \int_a^b \alpha(s)\,\mathrm{d}s \quad \text{(by Proposition 13.7)}$$
$$= \int_a^b f(\gamma(s))\gamma'(s)\,\mathrm{d}s$$
$$= \int_\gamma f. \qquad \square$$

Lecture 15 — Friday, February 07, 2014

Definition 15.1 A special case of Proposition 15.2 needs to be singled out. If $\gamma : [a, b] \to \mathbb{C}$ is a smooth curve, its **opposite curve** is defined by

$$\begin{aligned}[a, b] &\to \mathbb{C}\\ t &\mapsto \gamma(a + b - t)\end{aligned}$$

We denote the opposite curve of γ by $\tilde{\gamma}$ (with a "$\sim$" on top).

In other words, we take the function

$$\begin{aligned}h : [a, b] &\to [a, b]\\ t &\mapsto a + b - t.\end{aligned}$$

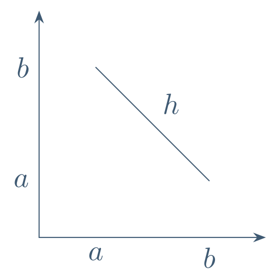

Figure 43

Figure 43 records the relevant geometry.

The opposite curve is $\gamma \circ h$. If γ runs from p to q, the opposite curve goes in reverse from q to p.

By Proposition 15.2 we see that

$$\int_{\tilde{\gamma}} f = -\int_{\gamma} f.$$

If $\gamma = x + iy$, we have $\gamma' = x' + iy'$ and $|\gamma'| = \sqrt{(x')^2 + (y')^2}$. Thus,

$$\int_a^b |\gamma'(t)|\,\mathrm{d}t = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t.$$

But this is the well-known formula for the length of γ, that is, the distance covered by γ. The path-integral estimate tells us that

$$\left|\int_{\gamma} f\right| \leq \|f\|_{\gamma} \cdot \mathrm{length}(\gamma),$$

which is a useful way to remember the estimate.

When the continuously differentiable map $h : [c,d] \to [a,b]$ reverses the endpoints, that is, $h(c) = b$ and $h(d) = a$, we get the same result as in Proposition 14.3, except with a sign change.

Proposition 15.2 If $\gamma : [a,b] \to \Omega$ and $h : [c,d] \to [a,b]$ is continuously differentiable and such that $h(c) = b$, $h(d) = a$, then for any continuous $f : \Omega \to \mathbb{C}$ we have

$$\int_{\gamma \circ h} f = -\int_{\gamma} f.$$

Proof. This proof is identical to that of Proposition 14.3, except that we use the second part of Proposition 13.7. It again comes from the change-of-variables formula in calculus. □

Primitives

In order to have some flexibility, it is useful to involve consider integrals over paths.

If $f : \Omega \to \mathbb{C}$ is continuous and $\gamma = \gamma_1 \vee \gamma_2 \vee \cdots \vee \gamma_n$ is a path in Ω, we naturally define

$$\int_{\gamma} f = \int_{\gamma_1} f + \int_{\gamma_2} f + \cdots + \int_{\gamma_n} f.$$

The following properties are easy to see:

$$\int_{\gamma_1 \vee \gamma_2 \vee \cdots \vee \gamma_n} (f \pm g) = \int_{\gamma_1 \vee \gamma_2 \vee \cdots \vee \gamma_n} f \pm \int_{\gamma_1 \vee \gamma_2 \vee \cdots \vee \gamma_n} g,$$

$$\int_{\gamma_1 \vee \gamma_2 \vee \cdots \vee \gamma_n} cf = c \int_{\gamma_1 \vee \gamma_2 \vee \cdots \vee \gamma_n} f \quad \text{for any } c \in \mathbb{C},$$

and

$$\begin{aligned} \left| \int_{\gamma_1 \vee \gamma_2 \vee \cdots \vee \gamma_n} f \right| &= \left| \sum_{j=1}^{n} \int_{\gamma_j} f \right| \leq \sum_{j=1}^{n} \left| \int_{\gamma_j} f \right| \leq \sum_{j} \|f\|_{\gamma_j} \text{length}(\gamma_j) \\ &\leq \sum_{j=1}^{n} \|f\|_{\gamma} \text{length}(\gamma_j) = \|f\|_{\gamma} \sum_{j=1}^{n} \text{length}(\gamma_j) = \|f\|_{\gamma} \text{length}(\gamma). \end{aligned}$$

Note that $\|f\|_{\gamma}$ is the maximum of $|f|$ over all the trajectories of the γ_j, and length(γ) is just defined to be $\sum_{j=1}^{n} \text{length}(\gamma_j)$.

Definition 15.3 We also say that a path γ from p to q is a **closed path** when $p = q$.

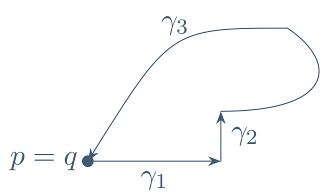

Figure 44

The configuration appears in Figure 44.

It is important to know whether $\displaystyle\int_{\gamma} f = 0$ when the path γ is closed. This relates to the notion of primitives.

Definition 15.4 A continuous function $f : \Omega \to \mathbb{C}$ has a **primitive** on Ω when there is a holomorphic function $g : \Omega \to \mathbb{C}$ such that $f = g'$ on Ω.

Note that this is different from Definition 9.2. There, a primitive described a relationship between two functions f and g, assuming differentiability from the start. Here, having a primitive is a *property* of a given function f.

If Ω is a region, primitives can only differ by a constant, as in Corollary 9.3.

Back in calculus, every continuous $f : I \to \mathbb{R}$ had a primitive, which was called an antiderivative. Namely, pick $a \in I$, and then the integral function

$$g(x) = \int_a^x f(t)\,\mathrm{d}t$$

creates a primitive for f. But in $\mathbb{C}$, the story is more complex (pardon the pun).

Example 15.5 The function $f(z) = 1/z$ has no primitive on $\mathbb{C} \setminus \{0\}$, but it does have one on the cut plane $\Omega = \{x + iy : \text{either } x > 0 \text{ or } y \neq 0\}$.

Proof. For $\mathbb{C} \setminus \{0\}$, assume toward contradiction that a primitive g exists with $g' = 1/z$. Take the unit circle $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$. Then

$$\int_\gamma \frac{\mathrm{d}z}{z} = \int_0^{2\pi} g'(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_0^{2\pi} [g(\gamma(t))]'\,\mathrm{d}t = g(\gamma(2\pi)) - g(\gamma(0)) = 0,$$

because $\gamma(0) = \gamma(2\pi) = 1$. But direct computation gives

$$\int_\gamma \frac{\mathrm{d}z}{z} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,\mathrm{d}t = i\int_0^{2\pi} 1\,\mathrm{d}t = 2\pi i \neq 0,$$

a contradiction. Hence $1/z$ has no primitive on $\mathbb{C} \setminus \{0\}$.

On Ω, the principal branch log is holomorphic, and from computations done long ago, we have

$$(\log z)' = \frac{1}{z} \quad \text{for all } z \in \Omega.$$

Therefore log is a primitive of $1/z$ on Ω. $\square$

The following proposition should remind us of the fundamental theorem of calculus:

Proposition 15.6 If $f : \Omega \to \mathbb{C}$ is continuous and has a primitive $g : \Omega \to \mathbb{C}$, then for any path γ in Ω from p to q we have:

$$\int_\gamma f = g(q) - g(p).$$

In particular, $\int_\gamma f$ does not depend on γ, but only on its endpoints; moreover, $\int_\gamma f = 0$ whenever the path is closed.

Proof. First we handle the case where γ is a single smooth curve. We have:

$$\begin{aligned}
\int_\gamma f &= \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t && \text{(definition)} \\
&= \int_a^b g'(\gamma(t))\gamma'(t)\,\mathrm{d}t && \text{(g is a primitive)} \\
&= \int_a^b [g(\gamma(t))]'\,\mathrm{d}t && \text{(by Proposition 8.6)} \\
&= g(\gamma(b)) - g(\gamma(a)) && \text{(fundamental theorem of calculus on the components of $g \circ \gamma$)} \\
&= g(q) - g(p).
\end{aligned}$$

More generally, let $\gamma = \gamma_1 \vee \gamma_2 \vee \cdots \vee \gamma_n$ with smooth curves γ_j that run from p_{j-1} to p_j, where $p = p_0$ and $q = p_n$. Then by definition

$$\begin{aligned}
\int_\gamma f &= \sum_{j=1}^n \int_{\gamma_j} f \\
&= \sum_{j=1}^n (g(p_j) - g(p_{j-1})) && \text{(by the part just done)} \\
&= g(p_n) - g(p_0) && \text{(since the series telescopes)} \\
&= g(q) - g(p). && \square
\end{aligned}$$

Example 15.7 $f(z) = 1/z$ has the primitive $g(z) = \log(z)$ on $\mathbb{C} \setminus (-\infty, 0]$. Thus, for the straight line γ going from 1 to $1+i$ we get

$$\int_\gamma f = \log(1+i) - \log(1) = \log|1+i| + i\arg(1+i) - \log(1) = \log\left(\sqrt{2}\right) + i\frac{\pi}{4}.$$

Example 15.8 $f(z) = 1/z^3$ has a primitive on $\mathbb{C} \setminus \{0\}$. The primitive is $g(z) = -1/(2z^2)$. Thus, for any path γ from 3 to i that does not go through 0, we get

$$\int_\gamma f = \frac{-1}{2\cdot i^2} - \frac{-1}{2\cdot 3^2} = \frac{1}{2} - \frac{1}{18} = \frac{4}{9}.$$

Example 15.9 $f(z) = \cos z$ has a primitive $g(z) = \sin z$ on $\mathbb{C}$. Thus, for any closed path γ we know:

$$\int_\gamma \cos z \, \mathrm{d}z = 0.$$

Path Independence

Definition 15.10 We say that the integrals of a continuous $f : \Omega \to \mathbb{C}$ are **path independent** provided that for any two points p, q in Ω, and any two paths β, γ running from p to q, we have

$$\int_\gamma f = \int_\beta f.$$

We just saw that any continuous f with a primitive has path-independent integrals.

Proposition 15.11 If f has path independent integrals, and γ is any closed path in Ω, then

$$\int_\gamma f = 0.$$

Proof. Suppose γ runs from p to p. Let $\beta : [0,1] \to \Omega$ be the constant path at p, $t \mapsto p$. Using the assumption we see that

$$\int_\gamma f = \int_\beta f = \int_0^1 f(\beta(t))\beta'(t)\,\mathrm{d}t = \int_0^1 f(p)\cdot 0\,\mathrm{d}t = 0.$$

$\square$

Proposition 15.12 Conversely, suppose that a continuous $f : \Omega \to \mathbb{C}$ is such that $\displaystyle\int_\gamma f = 0$ for every closed path in Ω. Then the integrals for f are path independent.

Proof. Let α, β be paths from p to q in Ω. We have the opposite path $\tilde{\alpha}$ from q to p.

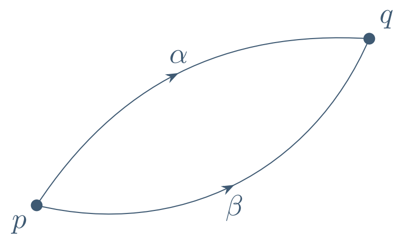

FIGURE 45

Figure 45 illustrates the setup.

Then we join β to $\tilde{\alpha}$ and make the closed path $\gamma = \beta \vee \tilde{\alpha}$ from p to p. By assumption, $\displaystyle\int_\gamma f = 0$.

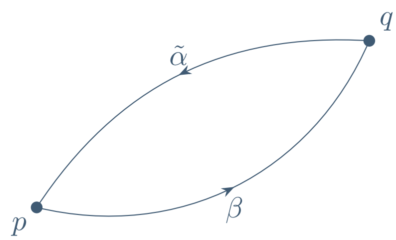

FIGURE 46

The construction is shown in Figure 46.

But then

$$0 = \int_\gamma f = \int_{\beta \vee \tilde{\alpha}} f = \int_\beta f + \int_{\tilde{\alpha}} f = \int_\beta f - \int_\alpha f.$$

Therefore $\displaystyle\int_\beta f = \int_\alpha f$. $\square$

Lecture 16 — Monday, February 10, 2014

The FTC for Complex Analysis

It turns out that Proposition 15.6 has a converse.

Proposition 16.1 (FTC for complex analysis) If Ω is a region and a continuous $f : \Omega \to \mathbb{C}$ is such that

$$\int_\gamma f = 0$$

for all closed paths in Ω, then f has a primitive in Ω.

Proof. The integrals for our f are path-independent. Pick $p \in \Omega$. If $z \in \Omega$ and γ is any path in Ω from p to z, the integral $\displaystyle\int_\gamma f$ depends only on z, not on the path chosen from p to z. Let $g(z)$ be the common value of all such $\displaystyle\int_\gamma f$. We verify that g is a primitive of f on Ω. For $w \in \Omega$, we want $g'(w) = f(w)$. We will prove that for z close to w there is a function $\varphi(z)$ such that

$$|g(z) - g(w) - f(w)(z - w)| \leq \varphi(z)|z - w|$$

and $\varphi(z) \to 0$ as $z \to w$.

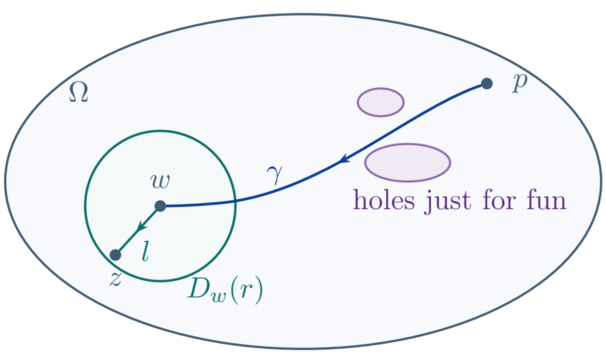

Figure 47

Figure 47 records the relevant geometry.

Let $D_w(r)$ be chosen so that $D_w(r) \subseteq \Omega$. For $z \in D_w(r)$ let l be the straight line from w to z, explicitly $l : [0,1] \to D_w(r)$, $t \mapsto w + t(z-w)$. Take any path γ from p to w in Ω. Then $\gamma \vee l$ is a path from p to z in Ω, so

$$\begin{aligned} g(z) - g(w) - f(w)(z-w) &= \int_{\gamma \vee l} f - \int_\gamma f - f(w)(z-w) = \int_l f - f(w)(z-w) \\ &= \int_l f - f(w) \int_l 1 = \int_l (f - f(w)), \end{aligned}$$

since $\displaystyle\int_l 1 = \int_0^1 (w + t(z-w))' \, \mathrm{d}t = z - w$. Taking absolute values and estimating gives

$$|g(z) - g(w) - f(w)(z-w)| \le \|f - f(w)\|_l \cdot \text{length}(l) = \|f - f(w)\|_l \cdot |z-w|.$$

So we only need to show $\|f - f(w)\|_l \to 0$ as $z \to w$. Take $\varepsilon > 0$. We need $\delta > 0$ such that $\|f - f(w)\|_l < \varepsilon$ when $|z - w| < \delta$. This means that for all $t \in [0,1]$, we have $|f(w + t(z-w)) - f(w)| < \varepsilon$ when $|z-w| < \delta$. By continuity of f at w, there is $\delta > 0$ such that $|f(\zeta) - f(w)| < \varepsilon$ whenever $|\zeta - w| < \delta$. If $|z - w| < \delta$, then for every $t \in [0,1]$ we have

$$|w + t(z-w) - w| = t|z-w| < \delta,$$

so with $\zeta = w + t(z-w)$ we get $|f(w + t(z-w)) - f(w)| < \varepsilon$. Hence $\|f - f(w)\|_l < \varepsilon$ when $|z - w| < \delta$, and therefore $g'(w) = f(w)$. □

Propositions 15.6 and 16.1 taken together may be viewed as the complex version of the fundamental theorem of calculus. While the results are quite nice, neither the condition

(1) f has a primitive on Ω

nor the condition

(2) $\displaystyle\int_\gamma f = 0$ for every closed path in Ω

is particularly easy to check in practice. The most challenging part is seeing how (2) can be used to obtain (1). That is where Cauchy's theorem will come in. For now, here are a few examples showing how (1) $\implies$ (2), and how $\neg$(2) $\implies$ $\neg$(1).

Example 16.2 Take $f(z) = 1/z$ on the punctured plane $\mathbb{C} \setminus \{0\}$. The curve $\gamma : [0, 2\pi] \to \mathbb{C} \setminus \{0\}$, $t \mapsto e^{it}$, describes the unit circle once in a counterclockwise direction.

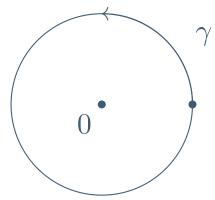

Figure 48

The configuration appears in Figure 48.

Easily we find that

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it}\,\mathrm{d}t = 2\pi i \neq 0.$$

Thus $1/z$ has no primitive on $\mathbb{C}\setminus\{0\}$, which is something we already knew from Example 15.5.

A useful strategy for showing that a given f has no primitive on some Ω is to find a closed path γ in Ω for which $\displaystyle\int_\gamma f \neq 0$.

Example 16.3 The function $1/z$ *does* have a primitive on the cut plane $\Omega = \{x + iy : y \neq 0 \text{ or } x > 0\}$ — namely the principal branch of $\log z$, as we also showed in Example 15.5. Thus for any closed path γ in Ω we get

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = 0.$$

Example 16.4 If

$$f(z) = \begin{cases} \dfrac{(e^z - 1)}{z} & \text{for } z \neq 0 \\ 1 & \text{for } z = 0 \end{cases},$$

let us show that f has a primitive on $\mathbb{C}$ and thereby that $\displaystyle\int_\gamma f = 0$ for every closed path γ in $\mathbb{C}$.

For $z \neq 0$ we see that

$$\begin{aligned} f(z) = \frac{e^z - 1}{z} &= \frac{(1 + z + z^2/2! + z^3/3! + \dots) - 1}{z} \\ &= 1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \dots + \frac{z^{n-1}}{n!} + \dots . \end{aligned}$$

A small inspection shows that this expansion also gives $f(0)$.

According to the differentiation theorem (Theorem 12.3), a primitive for f is given by the integrated series

$$g(z) = z + \frac{z^2}{2 \cdot 2!} + \frac{z^3}{3 \cdot 3!} + \frac{z^4}{4 \cdot 4!} + \dots + \frac{z^n}{n \cdot n!} + \dots$$

Thus $\displaystyle\int_\gamma (e^z - 1)/z \, \mathrm{d}z = 0$ for all closed paths γ.

We need some better ways to tell if $\displaystyle\int_\gamma f = 0$ when the paths γ are closed.

6. Cauchy's Theorem (Baby Version)

The Cauchy–Goursat Theorem

Cauchy's theorem is widely considered to be the cornerstone of complex analysis. In full generality, the theorem takes time to develop. However, much can be done with a simpler version that applies to triangles. The bisection technique we adopt is due to E. Goursat from roughly 130 years ago.

First we need a little notation. If $p, q \in \mathbb{C}$, the straight-line path from p to q will be denoted by $\overline{pq}$. More explicitly, $\overline{pq}$ is the curve

$$\overline{pq} : [0, 1] \longrightarrow \mathbb{C} \text{ where } t \mapsto p + t(q - p)$$

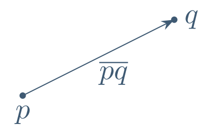

FIGURE 49

Figure 49 illustrates the setup.

A small observation comes next.

Proposition 16.5 If s is a point on the line segment $\overline{pq}$, and f is a continuous function, then

$$\int_{\overline{ps}} f + \int_{\overline{sq}} f = \int_{\overline{pq}} f$$

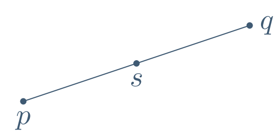

FIGURE 50

The subdivision at s is shown in Figure 50.

Proof. Let $\gamma = \overline{pq} : [0,1] \to \mathbb{C}$ where $t \mapsto p + t(q-p)$ and say $s = \gamma(c)$ for some $c \in [0,1]$. Then

$$\int_\gamma f = \int_0^1 f(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_0^c f(\gamma(t))\gamma'(t)\,\mathrm{d}t + \int_c^1 f(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_{\gamma_1} f + \int_{\gamma_2} f,$$

for the usual calculus reasons, where $\gamma_1 : [0,c] \to \mathbb{C}$ ($t \mapsto p + t(q-p)$) and $\gamma_2 : [c,1] \to \mathbb{C}$ ($t \mapsto p + t(q-p)$) are the restrictions of γ. But $\overline{ps}$ is a reparametrization of γ_1. Indeed, $\overline{ps} = \gamma_1 \circ h$ where h is the function $t \mapsto ct$.

Here are the details. For $t \in [0,1]$ we have

$$\begin{aligned}\overline{ps}(t) &= p + t(s-p) \\ &= p + t(\gamma(c) - p) \\ &= p + t(p + c(q-p) - p)\end{aligned}$$

$$\begin{aligned} &= p + ct(q-p) \\ &= p + h(t)(q-p) \\ &= \gamma_1(h(t)) \\ &= \gamma_1 \circ h(t) \end{aligned}$$

Thus

$$\int_{\gamma_1} f = \int_{\overline{ps}} f.$$

For similar reasons,

$$\int_{\gamma_2} f = \int_{\overline{sq}} f.$$

Put it all together to get

$$\int_{\overline{pq}} f = \int_{\overline{ps}} f + \int_{\overline{sq}} f.$$

$\square$

Lecture 17 — Wednesday, February 12, 2014

If p, q, r are three points in $\mathbb{C}$, the triangle they form, including the sides and the interior, is denoted by $\Delta(p, q, r)$, or just Δ when there is no ambiguity.

The path $\overline{pq} + \overline{qr} + \overline{rp}$ that travels around the edges of $\Delta = \Delta(p, q, r)$ will be denoted by $\partial\Delta(p, q, r)$ or $\partial\Delta$ if no confusion arises.

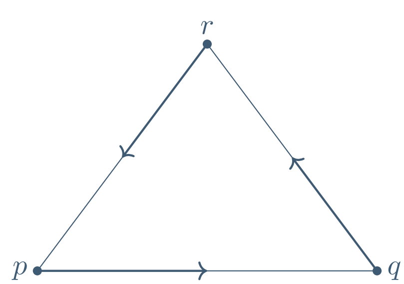

Figure 51

Figure 51 records the relevant geometry. The solid triangle is $\Delta(p, q, r)$. The arrows indicate $\partial\Delta(p, q, r)$.

Theorem 17.1 (Cauchy–Goursat theorem) If f is holomorphic on a region Ω and Δ is a triangle *entirely* inside Ω, then

$$\int_{\partial\Delta} f = 0.$$

Proof. Take any $\varepsilon > 0$. The strategy will be to prove that

$$\left|\int_{\partial\Delta} f\right| \leq \varepsilon(\text{length }\partial\Delta)^2.$$

Let p, q, r be the vertices of Δ, and let s, t, u be the midpoints of the sides pq, qr, and rp, respectively.

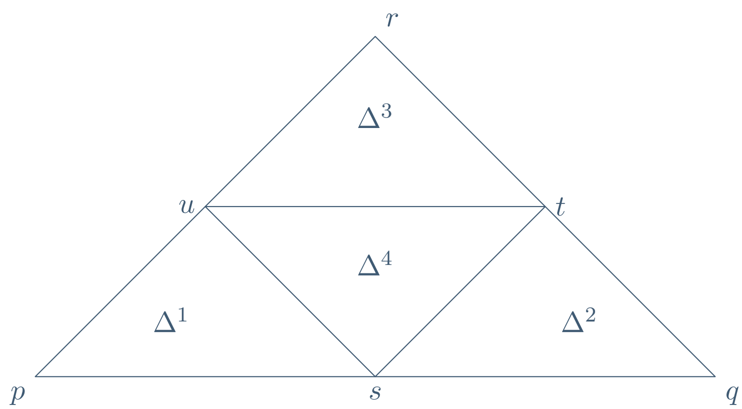

FIGURE 52

The configuration appears in Figure 52.

Then $\Delta = \Delta(p, q, r)$ breaks into four triangles: $\Delta^1 = \Delta(p, s, u)$, $\Delta^2 = \Delta(s, q, t)$, $\Delta^3 = \Delta(t, r, u)$, and $\Delta^4 = \Delta(s, t, u)$.

Then

$$\int_{\partial\Delta} f = \int_{\overline{pq}} f + \int_{\overline{qr}} f + \int_{\overline{rp}} f = \int_{\overline{ps}} f + \int_{\overline{sq}} f + \int_{\overline{qt}} f + \int_{\overline{tr}} f + \int_{\overline{ru}} f + \int_{\overline{up}} f,$$

and

$$\left(\int_{\overline{ps}} f + \int_{\overline{su}} f + \int_{\overline{up}} f\right) + \left(\int_{\overline{sq}} f + \int_{\overline{qt}} f + \int_{\overline{ts}} f\right)$$

$$+\left(\int_{\overline{tr}} f + \int_{\overline{ru}} f + \int_{\overline{ut}} f\right) + \left(\int_{\overline{us}} f + \int_{\overline{st}} f + \int_{\overline{tu}} f\right)$$

equals the same quantity, since $\overline{su}$ is the opposite path of $\overline{us}$, $\overline{ts}$ is the opposite path of $\overline{st}$, and $\overline{ut}$ is the opposite path of $\overline{tu}$, so the inserted edge integrals cancel. Hence

$$\int_{\partial\Delta} f = \int_{\partial\Delta^1} f + \int_{\partial\Delta^2} f + \int_{\partial\Delta^3} f + \int_{\partial\Delta^4} f,$$

and therefore

$$\left|\int_{\partial\Delta} f\right| \le \sum_{j=1}^{4} \left|\int_{\partial\Delta^j} f\right|.$$

For one of these triangles, call it Δ_1, we have

$$\left|\int_{\partial\Delta} f\right| \le 4\left|\int_{\partial\Delta_1} f\right|,$$

and

$$\text{length } \partial\Delta_1 = \frac{1}{2}\text{length } \partial\Delta.$$

Since $\Delta \subseteq \Omega$, we also have $\Delta_1 \subseteq \Omega$. Now apply the same argument to Δ_1. We obtain $\Delta_2 \subseteq \Delta_1$ such that

$$\left|\int_{\partial\Delta_1} f\right| \le 4\left|\int_{\partial\Delta_2} f\right| \qquad \text{and} \qquad \text{length } \partial\Delta_2 = \frac{1}{2}\text{length } \partial\Delta_1.$$

Thus

$$\left|\int_{\partial\Delta} f\right| \le 4^2\left|\int_{\partial\Delta_2} f\right| \qquad \text{and} \qquad \text{length } \partial\Delta_2 = \frac{1}{2^2}\text{length } \partial\Delta.$$

Repeating this bisection process gives a nested sequence of triangles

$$\Omega \supseteq \Delta \supseteq \Delta_1 \supseteq \Delta_2 \supseteq \cdots \supseteq \Delta_n \supseteq \dots$$

such that

$$\left|\int_{\partial\Delta} f\right| \le 4^n\left|\int_{\partial\Delta_n} f\right| \qquad \text{and} \qquad \text{length } \partial\Delta_n = \frac{1}{2^n}\text{length } \partial\Delta.$$

Since the Δ_n are nonempty and compact, there is a point p lying in all of them by the Cantor intersection theorem. Because f is holomorphic on Ω, $f'(p)$ exists. Hence, for our fixed $\varepsilon > 0$, there is $\delta > 0$ such that

$$|f(z) - f(p) - f'(p)(z-p)| \leq \varepsilon |z-p| \quad \text{when } |z-p| < \delta.$$

For large enough n we have $\Delta_n \subseteq D_p(\delta)$, for instance whenever length $\partial\Delta_n < \delta$.

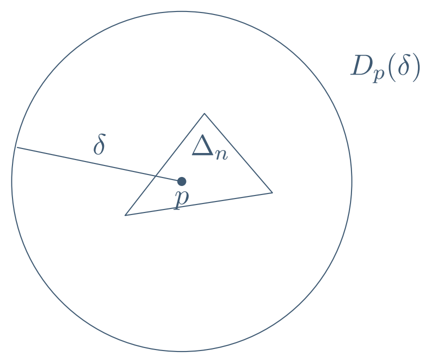

Figure 53

Figure 53 illustrates the setup.

Also, the function $z \mapsto f(p) + f'(p)(z-p)$ has a primitive on $\mathbb{C}$, so

$$\int_{\partial\Delta_n} (f(p) + f'(p)(z-p))\,\mathrm{d}z = 0.$$

Now we estimate:

$$\begin{aligned}
\left|\int_{\partial\Delta} f\right| &\leq 4^n \left|\int_{\partial\Delta_n} f(z)\,\mathrm{d}z\right| \\
&= 4^n \left|\int_{\partial\Delta_n} (f(z) - f(p) - f'(p)(z-p))\,\mathrm{d}z\right| \\
&\leq 4^n \left(\sup_{z\in\Delta_n} |f(z) - f(p) - f'(p)(z-p)|\right) \text{length } \partial\Delta_n \\
&\leq 4^n \left(\sup_{z\in\Delta_n} \varepsilon|z-p|\right) \text{length } \partial\Delta_n \\
&\leq 4^n \varepsilon (\text{length } \partial\Delta_n)^2 \\
&= 4^n \varepsilon \left(\frac{1}{2^n} \text{length } \partial\Delta\right)^2
\end{aligned}$$

$$= \varepsilon(\text{length } \partial\Delta)^2.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\int_{\partial\Delta} f = 0.$$

□

Next we need a slight improvement to Theorem 17.1 for reasons that will be revealed in due course.

Proposition 17.2 If f is holomorphic on Ω, except possibly at one point $c \in \Omega$, but f remains continuous at c, and $\Delta = \Delta(p, q, r)$ is any triangle inside Ω, then

$$\int_{\partial\Delta} f = 0.$$

Proof. If $c \notin \Delta$, we are in the situation of Theorem 17.1 and

$$\int_{\partial\Delta} f = 0.$$

If c is a vertex of Δ, say $c = p$, pick points u, v on sides pq, pr, respectively, and close to $p = c$. As argued in Theorem 17.1, we see that

$$\int_{\partial\Delta} f = \int_{\partial\Delta(p,u,v)} f + \int_{\partial\Delta(v,u,r)} f + \int_{\partial\Delta(u,q,r)} f.$$

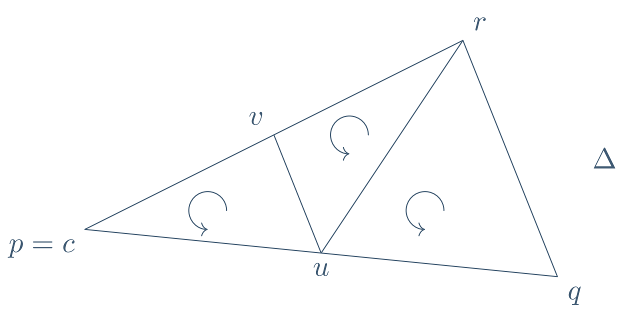

FIGURE 54

The construction is shown in Figure 54.

By Theorem 17.1, we have

$$\int_{\partial\Delta(u,q,r)} f = \int_{\partial\Delta(v,u,r)} f = 0.$$

Hence

$$\begin{aligned}\left|\int_{\partial\Delta} f\right| &= \left|\int_{\partial\Delta(p,u,v)} f\right| \\ &\leq \|f\|_{\Delta(p,u,v)} \cdot \text{length } \partial\Delta(p,u,v) \\ &\leq \|f\|_{\Delta} \cdot \text{length } \partial\Delta(p,u,v) \to 0 \quad \text{as } u, v \to p.\end{aligned}$$

If c is on an edge of Δ, say on the segment pq, argue as in Theorem 17.1 to get

$$\int_{\partial\Delta} f = \int_{\partial\Delta(p,c,r)} f + \int_{\partial\Delta(c,q,r)} f.$$

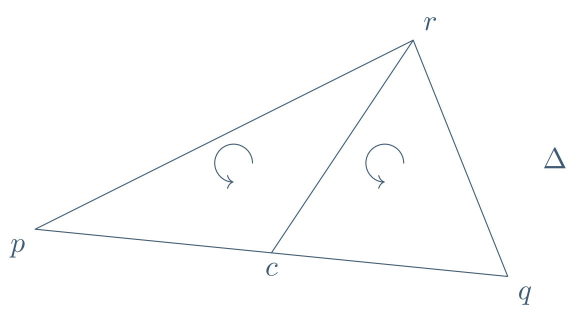

Figure 55

Figure 55 records the relevant geometry.

By the part just done applied to each $\Delta(p,c,r)$ and $\Delta(c,q,r)$, we can see that

$$\int_{\partial\Delta} f = 0 + 0 = 0.$$

Finally, if c is in the interior of Δ, we can argue again as in Theorem 17.1 to obtain

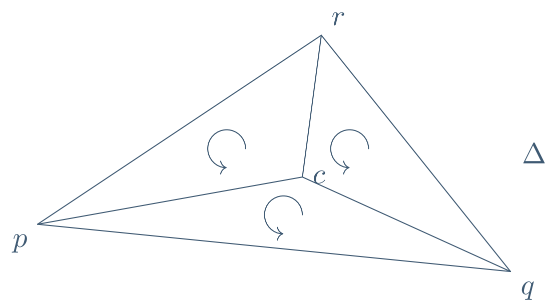

FIGURE 56

The configuration appears in Figure 56.

$$\int_{\partial\Delta} f = \int_{\partial\Delta(p,c,r)} f + \int_{\partial\Delta(c,p,q)} f + \int_{\partial\Delta(c,q,r)} f = 0 + 0 + 0 = 0$$

using the earlier cases. □

Lecture 18 — Friday, February 14, 2014

Cauchy's Theorem (Baby Version)

The next result will already yield some neat applications even though it deals with a limited type of region.

Definition 18.1 We say that the region Ω is **convex from a point $p \in \Omega$** provided for every z in Ω the line $\overline{zp}$ lies in Ω.

Examples include disks, eggs, the cut plane, stars, and strips. These are convex from a point, unlike regions with holes such as the punctured plane $\mathbb{C} \setminus \{0\}$ or annuli. The fact that we can handle holes is what makes the next result "basic".

Theorem 18.2 (Cauchy's theorem, baby version) If Ω is convex from a point and f is holomorphic on Ω except possibly at one point where it remains continuous, then f has a primitive in Ω; equivalently,

$$\int_\gamma f = 0$$

for every closed path in Ω.

Proof. This proof has much in common with the proof of Proposition 16.1 in the previous section.

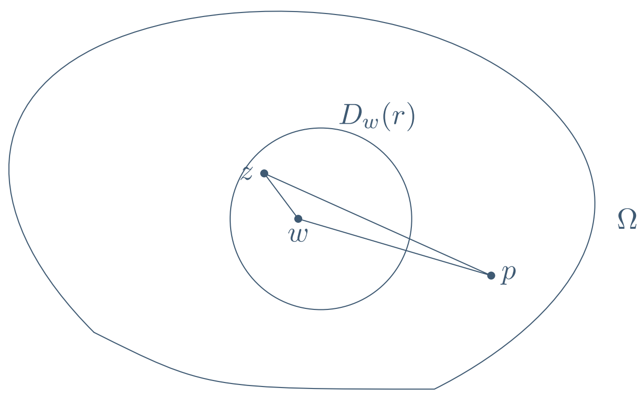

Figure 57

Figure 57 illustrates the setup.

If p is a point that Ω is convex from, define $g(z) = \displaystyle\int_{\overline{pz}} f$ for every $z \in \Omega$. For $w \in \Omega$, we'll show that $g'(w) = f(w)$. To that end, we will prove there is a function $\varphi(z)$, defined for z close to w and such that

$$|g(z) - g(w) - f(w)(z-w)| \le \varphi(z)|z-w|$$

and $\varphi(z) \to 0$ as $z \to w$. First take a $D_w(r) \subseteq \Omega$. For every z in $D_w(r)$, the triangle $\Delta(p, w, z)$ lies in Ω because Ω is convex from p and the line segment wz is inside Ω. By Proposition 17.2, we know that

$$0 = \int_{\partial\Delta(p,w,z)} f = \int_{\overline{pz}} f + \int_{\overline{zw}} f + \int_{\overline{wp}} f.$$

By using the opposite paths $\overline{wz}$ and $\overline{pw}$, we see that

$$\int_{\overline{pz}} f - \int_{\overline{pw}} f = \int_{\overline{wz}} f.$$

Now for every z in $D_w(r)$ we get

$$\begin{aligned}|g(z)-g(w)-f(w)(z-w)| &= \left|\int_{\overline{pz}} f - \int_{\overline{pw}} f - f(w)(z-w)\right| \\ &= \left|\int_{\overline{wz}} f(\zeta)\,\mathrm{d}\zeta - f(w)(z-w)\right| \\ &= \left|\int_{\overline{wz}} (f(\zeta)-f(w))\,\mathrm{d}\zeta\right|\end{aligned}$$

because

$$\int_{\overline{wz}} f(w)\,\mathrm{d}\zeta = f(w)\int_{\overline{wz}} 1\,\mathrm{d}\zeta = f(w)\int_0^1 (w+t(z-w))'\,\mathrm{d}t = f(w)(z-w).$$

In particular,

$$\left|\int_{\overline{wz}} (f(\zeta)-f(w))\,\mathrm{d}\zeta\right| \le \|f-f(w)\|_{\overline{wz}} \cdot \text{length } \overline{wz} = \|f-f(w)\|_{\overline{wz}} \cdot |z-w|.$$

We just need to show that $\|f-f(w)\|_{\overline{wz}} \to 0$ as $z \to w$. To this end, let $\varepsilon > 0$ and observe that $\|f-f(w)\|_{\overline{wz}} = \max_{t\in[0,1]} |f(w+t(z-w)) - f(w)|$. Take a $\delta > 0$ that works for the continuity of f at w. So whenever $|\zeta - w| < \delta$ we get $|f(\zeta) - f(w)| < \varepsilon$. Now if $|z-w| < \delta$ we also get $|w + t(z-w) - w| = t|z-w| < \delta$ for every $t \in [0,1]$. So $|f(w+t(z-w)) - f(w)| < \varepsilon$ over all $t \in [0,1]$. That is $\|f-f(w)\|_{\overline{wz}} < \varepsilon$, and we're done. □

Here is a cheap example:

Example 18.3 Consider

$$\int_\gamma \frac{\mathrm{d}z}{z^2+1}$$

where $\gamma : [0, 2\pi] \to \mathbb{C}$ is defined by $t \mapsto i + e^{it}$.

To solve this, first note that

$$\frac{1}{z^2+1} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right),$$

so

$$\int_\gamma \frac{\mathrm{d}z}{z^2+1} = \frac{1}{2i}\int_\gamma \frac{\mathrm{d}z}{z-i} - \frac{1}{2i}\int_\gamma \frac{\mathrm{d}z}{z+i}.$$

Also note that $f(z) = 1/(z+i)$ is holomorphic over $D_i(1.5)$, which is brutally convex. By

Theorem 18.2 applied to $\Omega = D_i(1.5)$, we have

$$\int_\gamma f = 0.$$

Hence

$$\int_\gamma \frac{\mathrm{d}z}{z^2+1} = \frac{1}{2i}\int_\gamma \frac{\mathrm{d}z}{z-i} = \frac{1}{2i}2\pi i = \pi.$$

And here is a cool fact that is worth memorizing:

Example 18.4 For any $r > 0$,

$$\int_\gamma \frac{\mathrm{d}z}{z-p} = 2\pi i$$

for $\gamma(t) = p + re^{it}$ and $t \in [0, 2\pi]$.

Proof. With $\gamma(t) = p + re^{it}$, we have

$$\gamma'(t) = ire^{it} \qquad \text{and} \qquad \gamma(t) - p = re^{it}.$$

Hence

$$\int_\gamma \frac{\mathrm{d}z}{z-p} = \int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)-p}\,\mathrm{d}t = \int_0^{2\pi} \frac{ire^{it}}{re^{it}}\,\mathrm{d}t = i\int_0^{2\pi} 1\,\mathrm{d}t = 2\pi i.$$

□

Lecture 19 — Monday, February 24, 2014

7. The Cauchy Integral Formula

Cacuhy's Integral Formula (Early Version)

We have just learned that on a region Ω that is convex from a point, a function f that is continuous on Ω and holomorphic at all but perhaps one point of Ω must have a primitive, and thereby satisfies

$\displaystyle\int_\gamma f = 0$ for all closed paths in Ω.

Here comes a neat application of Theorem 18.2.

Theorem 19.1 (Cauchy's integral formula, early version) If Ω is convex from a point, f is holomorphic on Ω, γ is a closed path in Ω, and $w \in \Omega \setminus \gamma^*$ (where γ^* is the image or trajectory of the path), then

$$\int_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z = f(w)\int_\gamma \frac{\mathrm{d}z}{z-w}.$$

Proof. By linearity, this amounts to showing that

$$\int_\gamma \frac{f(z)-f(w)}{z-w}\,\mathrm{d}z = 0. \tag{$\star$}$$

Put

$$g(z) = \begin{cases} \dfrac{f(z)-f(w)}{z-w}, & z \in \Omega \setminus \{w\}, \\ f'(w), & z = w. \end{cases}$$

The function g is continuous on Ω, including at w. Also, g is holomorphic except perhaps at w. Theorem 18.2 applies to g on Ω. Thus ($\star$) holds. $\square$

A special case of this formula should be singled out — namely the case where γ is a circle.

Proposition 19.2 Let Ω be any region, f holomorphic on Ω, $w \in \Omega$, $D_w(R)$ any disk inside Ω (where R could be as big as possible), and $0 < r < R$. Let $\gamma : [0, 2\pi] \to D_w(R)$ be the path $t \mapsto w + re^{it}$, which is the circle of radius r that goes once around the center w counterclockwise. Then

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z.$$

Proof. Theorem 19.1 applies to f on $D_w(R)$. Hence

$$\int_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z = f(w)\int_\gamma \frac{\mathrm{d}z}{z-w} = f(w)\int_0^{2\pi} \frac{ire^{it}}{w+re^{it}-w}\,\mathrm{d}t = f(w)\int_0^{2\pi} i\,\mathrm{d}t = f(w)2\pi i.$$

Now divide by $2\pi i$. $\square$

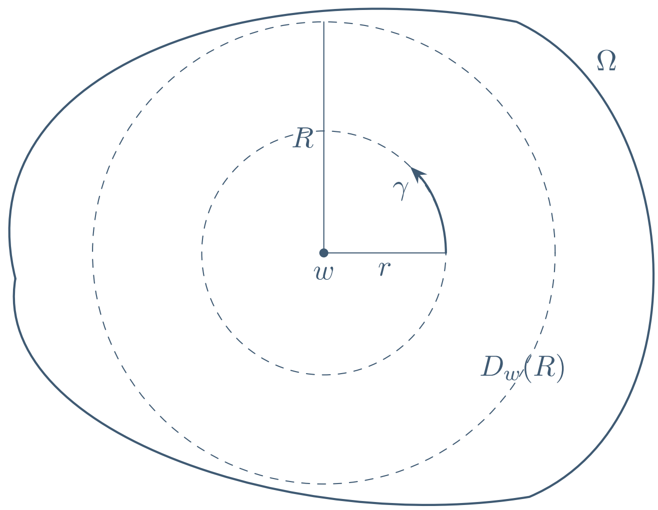

FIGURE 58

The construction is shown in Figure 58.

Thus we learn that the values of f on the boundary of $D_w(r)$ determine the value at the center w. Shortly, we shall learn that the values of f on the boundary of $D_w(r)$ determine the values of f at *all* points inside $D_w(r)$. Maybe this is why f is called *holomorphic* (that is, "fully shaped"). You cannot change f in one zone of its domain without having an effect far away. The values of f seem interdependent as a kind of organic whole.

Contrast this with real differentiable functions. For example:

$$f(x) = x^2 \qquad \text{and} \qquad g(x) = \begin{cases} x^2, & \text{for } x \geq 0 \\ 0, & \text{for } x < 0 \end{cases}.$$

FIGURE 59

Figure 59 records the relevant geometry.

Even though $f = g$ on the right of 0, they have a life of their own on the left. Yet both f and g are differentiable.

Here is a small example to illustrate Proposition 19.2.

Example 19.3 Let

$$\gamma : [0, 2\pi] \to \mathbb{C}, \qquad t \mapsto i + e^{it},$$

Let us find

$$\int_\gamma \frac{1}{z^2+1}\,\mathrm{d}z.$$

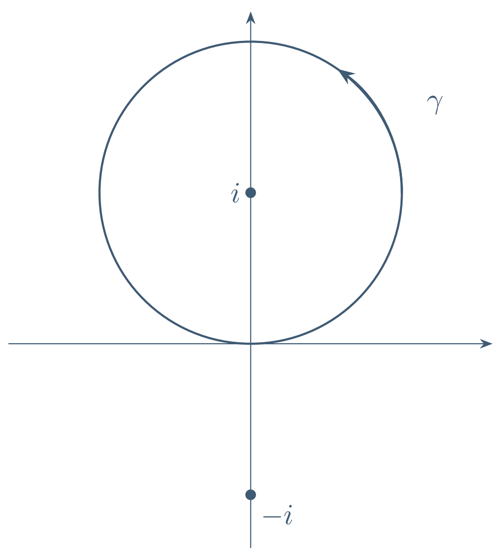

Figure 60

The configuration appears in Figure 60.

We have

$$\int_\gamma \frac{1}{z^2+1}\,\mathrm{d}z = \int_\gamma \frac{1/(z+i)}{z-i}\,\mathrm{d}z.$$

Since $1/(z+i)$ is holomorphic on $\mathbb{C} \setminus \{-i\}$ and the circle stays away from $-i$, we use Proposition 19.2 and get
$$\int_\gamma \frac{1}{z^2+1}\,\mathrm{d}z = 2\pi i \cdot \frac{1}{i+i} = \pi.$$

Here is another way to look at Proposition 19.2 using the parametrization of the circle γ:
$$f(w) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(w+re^{it})ire^{it}}{w+re^{it}-w}\,\mathrm{d}t = \frac{1}{2\pi i}\int_0^{2\pi} if(w+re^{it})\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} f(w+re^{it})\,\mathrm{d}t.$$

This integral is the average of f along the circle γ. Thus the value at the center equals the average of f around the circle.

Winding Numbers

Looking back at Theorem 19.1, we had that
$$\int_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z = f(w)\int_\gamma \frac{\mathrm{d}z}{z-w}$$
provided that Ω is convex from a point, γ is a closed path in Ω, $w \in \Omega \setminus \gamma^*$ (where γ^* is the trajectory or image of γ), and f is holomorphic on Ω.

In order to exploit Theorem 19.1 more effectively, we need to take a closer look at $\displaystyle\int_\gamma (\mathrm{d}z)/(z-w)$.

Suppose, outrageously, that we had smooth functions $r : [0,1] \to (0,\infty)$ and $\theta : [0,1] \to \mathbb{R}$ and that $\gamma : [0,1] \to \mathbb{C}$ was given by
$$\gamma(t) = w + r(t)e^{i\theta(t)}.$$

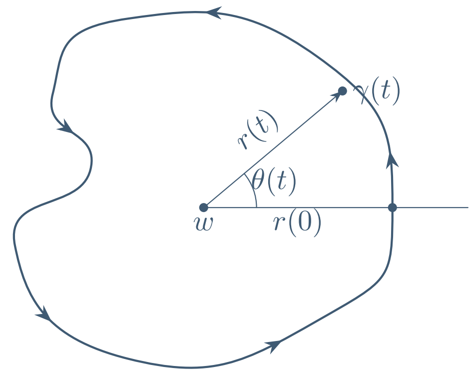

Figure 61

Figure 61 illustrates the setup.

For such a special γ, we get

$$\int_\gamma \frac{\mathrm{d}z}{z-w} = \int_0^1 \frac{(w+r(t)e^{i\theta(t)})'}{w+r(t)e^{i\theta(t)}-w}\,\mathrm{d}t = \int_0^1 \frac{r'(t)e^{i\theta(t)}+r(t)ie^{i\theta(t)}\theta'(t)}{r(t)e^{i\theta(t)}}\,\mathrm{d}t.$$

This equals

$$\log r(1) - \log r(0) + i(\theta(1)-\theta(0)).$$

But $\gamma(0)=\gamma(1)$ since γ is closed. Thus

$$r(0)e^{i\theta(0)} = r(1)e^{i\theta(1)},$$

which gives $r(0)=r(1)$ and $\theta(1)-\theta(0)=2\pi n$ for some integer n.

Hence

$$\int_\gamma \frac{\mathrm{d}z}{z-w} = 2\pi i n \text{ for some integer } n.$$

What is the geometric meaning of n? Looking at the equation above, we see that n is the number of revolutions that $\gamma(t)=w+r(t)e^{i\theta(t)}$ has taken around the point w.

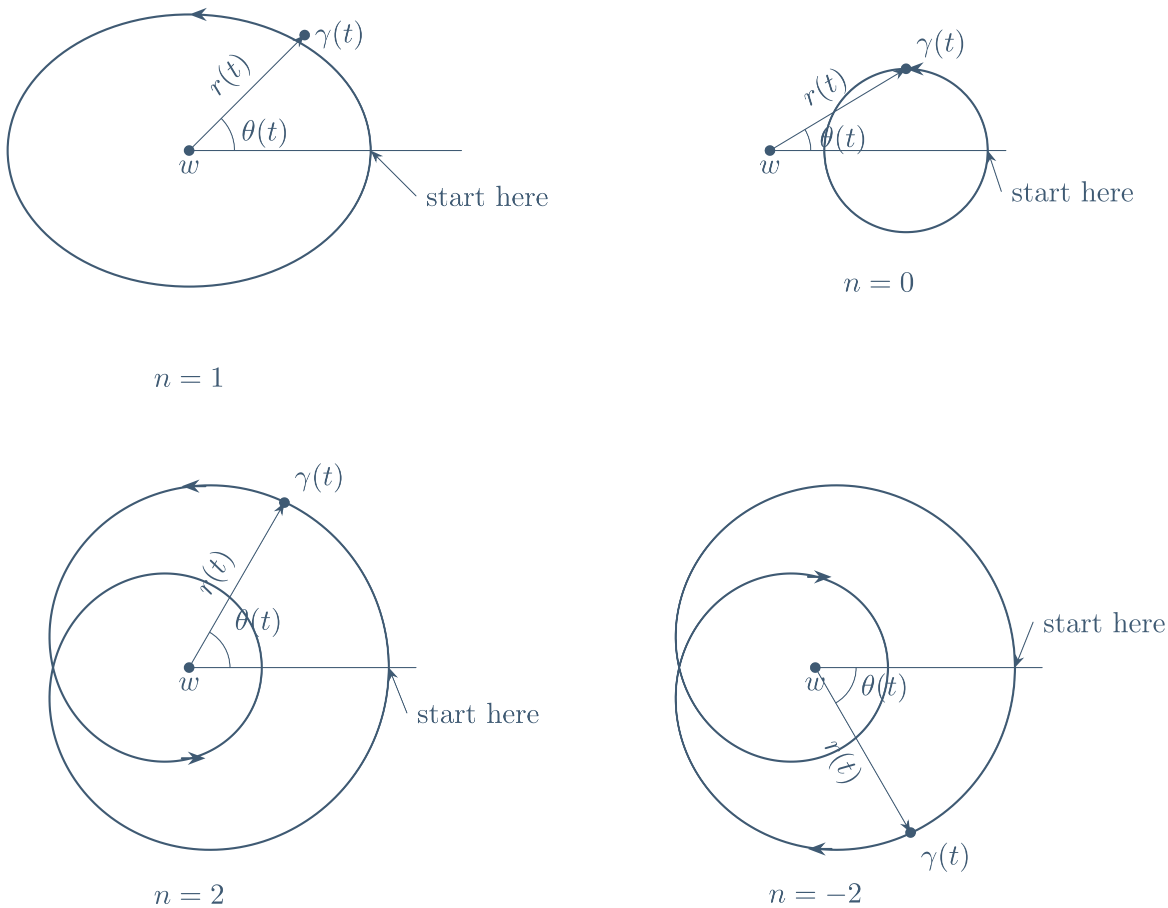

Figure 62

The construction is shown in Figure 62.

The trouble with the approach we just took is this. For a smooth $\gamma : [0, 1] \to \mathbb{C}$ with $w \notin \gamma^*$, we put $\gamma(t) - w$ in polar form as

$$\gamma(t) - w = |\gamma(t) - w| e^{i\theta(t)},$$

and thereby wrote

$$\gamma(t) = w + r(t) e^{i\theta(t)}.$$

But how did we make a continuous choice for $\theta(t) = \arg(\gamma(t) - w)$? We need a proof that skirts this issue.

Lecture 20 — Wednesday, February 26, 2014

Theorem 20.1 If γ is a closed path in $\mathbb{C}$ and $w \in \mathbb{C} \setminus \gamma^*$ (where $\gamma^* = \gamma([a,b])$), then

$$\frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z-w} \text{ is an integer.}$$

Proof. For any complex number $z = x + i\theta$, we see that

$$\begin{aligned} e^z = 1 &\iff e^x(\cos\theta + i\sin\theta) = 1 \\ &\iff x = 0 \text{ and } \theta = 2\pi k \text{ for some integer } k \\ &\iff z = 2\pi i k \\ &\iff \frac{z}{2\pi i} \text{ is an integer.} \end{aligned}$$

Thus we reduce to showing that

$$\exp\left(\int_\gamma \frac{\mathrm{d}z}{z-w}\right) = 1.$$

We can write $\gamma = \gamma_1 \vee \gamma_2 \vee \cdots \vee \gamma_N$, where $\gamma_j : [0,1] \to \mathbb{C}$ are smooth curves that run from p_{j-1} to p_j such that $p_0 = p_N$.

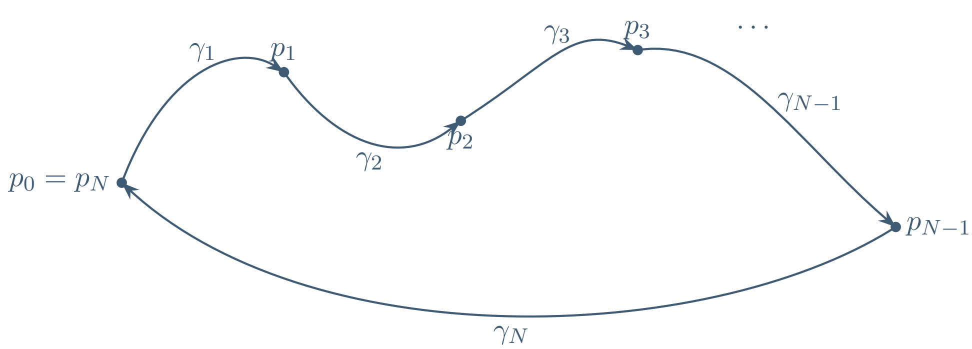

FIGURE 63

Figure 63 records the relevant geometry.

If $\beta : [0,1] \to \mathbb{C}$ is any smooth curve that misses w (for example, β can be one of the γ_j), define another smooth (not necessarily closed) curve $\varphi : [0,1] \to \mathbb{C}$ by

$$\varphi(x) = \exp\left(\int_0^x \frac{\beta'(t)}{\beta(t) - w}\,\mathrm{d}t\right).$$

By the fundamental theorem of calculus applied to the real and imaginary parts of φ, and by Proposition 8.6, we get

$$\varphi'(x) = \varphi(x)\frac{\beta'(x)}{\beta(x) - w},$$

and so

$$\frac{\varphi'(x)}{\varphi(x)} = \frac{\beta'(x)}{\beta(x) - w} \qquad \text{or} \qquad \varphi'(x)(\beta(x) - w) = \varphi(x)\beta'(x).$$

This suggests that $\varphi(x)$ and $\beta(x) - w$ have the same logarithmic derivative and therefore ought to be proportional. To see this, use the quotient rule (it works for smooth curves in $\mathbb{C}$) and obtain

$$\left(\frac{\varphi(x)}{\beta(x) - w}\right)' = \frac{(\beta(x) - w)\varphi'(x) - \beta'(x)\varphi(x)}{(\beta(x) - w)^2} = 0.$$

Thus $\varphi(x) = c(\beta(x) - w)$ for some constant c and all $x \in [0,1]$. After we note that $\varphi(0) = 1$, we get $c = 1/(\beta(0) - w)$. Hence

$$\exp\int_0^1 \frac{\beta'(t)}{\beta(t) - w}\,\mathrm{d}t = \frac{\beta(1) - w}{\beta(0) - w}.$$

Now apply this to each γ_j to get

$$\exp\int_{\gamma_j} \frac{\mathrm{d}z}{z - w} = \exp\int_0^1 \frac{\gamma_j'(t)}{\gamma_j(t) - w}\,\mathrm{d}t = \frac{\gamma_j(1) - w}{\gamma_j(0) - w} = \frac{p_j - w}{p_{j-1} - w}.$$

Thus

$$\exp\int_\gamma \frac{\mathrm{d}z}{z - w} = \exp\left(\sum_{j=1}^N \int_{\gamma_j} \frac{\mathrm{d}z}{z - w}\right) = \prod_{j=1}^N \frac{p_j - w}{p_{j-1} - w} = \frac{p_N - w}{p_0 - w} = 1,$$

since $p_N = p_0$ (remember that γ is closed). Therefore

$$\frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z - w}$$

is an integer. □

Definition 20.2 For a closed path γ in $\mathbb{C}$, the integer $1/(2\pi i)\displaystyle\int_\gamma (\mathrm{d}z)/(z-w)$ defined for every $w \notin \gamma^*$ is called the **index** or **winding number** of γ around w. We write

$$\operatorname{ind}_\gamma(w) = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z-w}.$$

Here is an unsettling fact:

Proposition 20.3 If γ is a closed path in $\mathbb{C}$ and $\Omega = \mathbb{C} \setminus \gamma^*$ is the complement of the trajectory of γ, then

$$\operatorname{ind}_\gamma : \Omega \to \mathbb{Z}, \qquad w \mapsto \operatorname{ind}_\gamma(w)$$

is a *continuous* function.

For psychological reasons, it is okay to replace the codomain $\mathbb{Z}$ with $\mathbb{C}$.

This fact is less troubling after we learn in real analysis that a function f which maps into a discrete topological space is continuous if and only if it is locally constant (i.e., for every $x \in X =: \operatorname{Dom}(f)$, there exists an open neighborhood $U \subseteq X$ such that $f(u) = f(x)$ for all $u \in U$.

Proof of Proposition 20.3. To prove that $\operatorname{ind}_\gamma$ is continuous at $p \in \Omega$, we will find a radius $r > 0$ and a constant K such that

$$|\operatorname{ind}_\gamma(w) - \operatorname{ind}_\gamma(p)| \le K|w-p| \quad \text{when } w \in D_p(r).$$

In other words, we will show that $\operatorname{ind}_\gamma$ is K-Lipschitz on $D_p(r)$. That will definitely do the job. Pick $r > 0$ so that $D_p(2r) \subseteq \Omega$. Thus for every $w \in D_p(r)$ and every $z \in \gamma^*$, we have $|w - z| \ge r$.

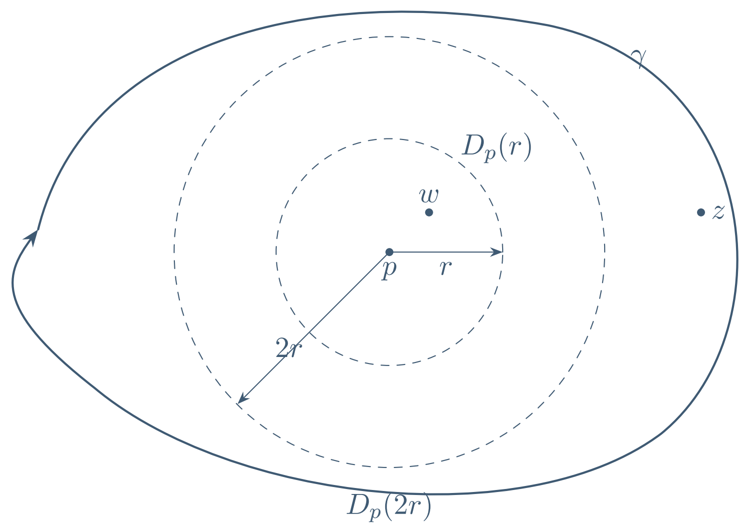

Figure 64

The configuration appears in Figure 64.

For $w \in D_p(r)$ we have

$$\begin{aligned}
|2\pi i(\mathrm{ind}_\gamma(w) - \mathrm{ind}_\gamma(p))| &= \left| \int_\gamma \left(\frac{1}{z-w} - \frac{1}{z-p} \right) \mathrm{d}z \right| \\
&= \left| \int_\gamma \frac{w-p}{(z-w)(z-p)} \,\mathrm{d}z \right| \\
&= \left| \int_\gamma \frac{\mathrm{d}z}{(z-w)(z-p)} \right| |w-p| \\
&\le \left\| \frac{1}{(z-w)(z-p)} \right\|_{\gamma^*} \mathrm{length}(\gamma)|w-p| \\
&\le \frac{\mathrm{length}(\gamma)}{r^2}|w-p|, \quad \text{since } |z-w||z-p| \ge r^2.
\end{aligned}$$

Thus

$$|\mathrm{ind}_\gamma(w) - \mathrm{ind}_\gamma(p)| \le \frac{\mathrm{length}(\gamma)}{2\pi r^2}|w-p| \quad \text{for all } w \in D_p(r),$$

which proves continuity at p. □

Thus, if Ω is a connected open supset of $\mathbb{C} \setminus \gamma^*$, then ind_γ is constant on Ω. If Ω is a disjoint union of connected open subsets, then ind_γ has to be constant on each connected component. For example, Figure 65 shows the five connected components that make up $\mathbb{C} \setminus \gamma^*$. Only the fifth (the "outside one") is unbounded.

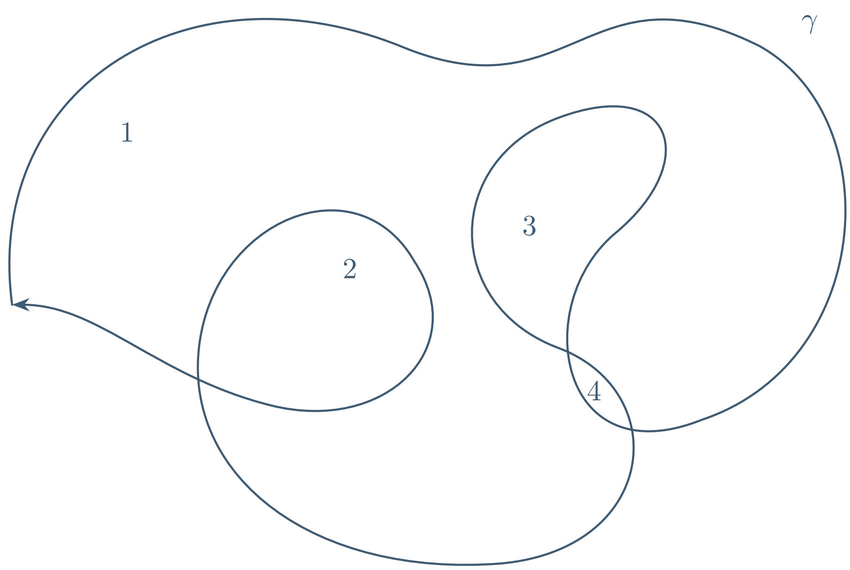

Figure 65

Lecture 21 — Friday, February 28, 2014

The constant integer values that ind_γ takes on the components of $\mathbb{C} \setminus \gamma^*$ depend on γ. However, on the unbounded component of $\mathbb{C} \setminus \gamma^*$, the value of ind_γ is known.

Proposition 21.1 The index function ind_γ is zero on the unbounded component of $\mathbb{C} \setminus \gamma^*$.

This matches the geometric intuition that far away from γ^* there should be no winding.

Proof. For w in the unbounded component of $\mathbb{C} \setminus \gamma^*$, we have

$$|\mathrm{ind}_\gamma(w)| = \frac{1}{2\pi} \left| \int_\gamma \frac{\mathrm{d}z}{z-w} \right| \leq \frac{1}{2\pi} \left\| \frac{1}{z-w} \right\|_{\gamma^*} \mathrm{length}(\gamma).$$

To estimate the norm, observe that for $z \in \gamma^*$ and $w \in \mathbb{C} \setminus \gamma^*$,

$$|w - z| \geq |w| - |z| \geq |w| - \max_{z \in \gamma^*} |z| = |w| - \|z\|_{\gamma^*}.$$

So

$$\left|\frac{1}{w - z}\right| \leq \frac{1}{|w| - \|z\|_{\gamma^*}}.$$

As $|w| \to \infty$, the right-hand side tends to 0, so $\|1/(z - w)\|_{\gamma^*} \to 0$. Hence $|\mathrm{ind}_\gamma(w)| \to 0$ as $|w| \to \infty$. Since ind_γ is a constant integer on the unbounded component, that constant must be 0. $\square$

Cauchy's Integral Formula (Common Version)

Now let us put these facts about the index function to work, together with Proposition 19.2.

Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the circle given by $\gamma(t) = p + re^{it}$ of center p and radius r.

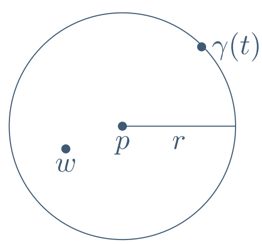

FIGURE 66

The construction is shown in Figure 66.

Here $\mathbb{C} \setminus \gamma^*$ has two components: the inside and the outside of the circle. We saw that $\mathrm{ind}_\gamma(w) = 0$ when w is on the outside of the circle.

If w is on the inside of the circle, then

$$\mathrm{ind}_\gamma(w) = \mathrm{ind}_\gamma(p) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{rie^{it}}{p + re^{it} - p} \,\mathrm{d}t = \frac{1}{2\pi i} \int_0^{2\pi} i \,\mathrm{d}t = \frac{2\pi i}{2\pi i} = 1.$$

(The equality $\mathrm{ind}_\gamma(w) = \mathrm{ind}_\gamma(p)$ holds by the constancy of ind_γ on components of $\mathbb{C} \setminus \gamma^*$.)

Now we get something more down to earth.

Theorem 21.2 (Cauchy's integral formula, common version) If Ω is any region, $D_p(R)$ is any disk contained in Ω (where R could be as large as possible), $0 < r < R$, $\gamma : [0, 2\pi] \to D_p(R)$ is the usual circle path $t \mapsto p + re^{it}$, and f is holomorphic on Ω, then for every $w \in D_p(r)$ we get

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w} \, \mathrm{d}z.$$

Proof. Theorem 19.1 applied in $D_p(R)$ tells us:

$$f(w) \int_\gamma \frac{\mathrm{d}z}{z-w} = \int_\gamma \frac{f(z)}{z-w} \, \mathrm{d}z.$$

That is,

$$f(w) \cdot 2\pi i \cdot \mathrm{ind}_\gamma(w) = \int_\gamma \frac{f(z)}{z-w} \, \mathrm{d}z,$$

and we saw that $\mathrm{ind}_\gamma(w) = 1$, regardless of w inside the circle γ^*. Thus

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w} \, \mathrm{d}z. \qquad \square$$

This is a stunning property of holomorphic functions. It tells us how to find the values of f inside a disk by simply knowing its values on the border.

Example 21.3 Let us find

$$\int_\gamma \frac{\sin(z)}{z^2+1} \, \mathrm{d}z$$

where $\gamma(t) = 2e^{it}$ for $t \in [0, 2\pi]$.

We want to apply Theorem 21.2, but we need to do some work first. As we've seen before,

$$\frac{1}{z^2+1} = \frac{1}{2i} \left(\frac{1}{z-i} - \frac{1}{z+i} \right),$$

so

$$\int_\gamma \frac{\sin(z)}{z^2+1} \, \mathrm{d}z = \frac{1}{2i} \left(\int_\gamma \frac{\sin(z)}{z-i} \, \mathrm{d}z - \int_\gamma \frac{\sin(z)}{z+i} \, \mathrm{d}z \right) = \frac{2\pi i}{2i} \left(\sin(i) - \sin(-i) \right) = \frac{\pi}{i} \left(-e + \frac{1}{e} \right).$$

What about if, instead of γ, we consider the path β that traces out a square centered at the origin with side length 4 in the positive orientation? The integral is the same, because both contours wind once around i and $-i$ and the integrand is holomorphic away from those two points:

$$\int_\beta \frac{\sin(z)}{z^2+1}\,\mathrm{d}z = \frac{\pi}{i}\left(-e+\frac{1}{e}\right).$$

Lecture 22 — Monday, March 03, 2014

8. Applications of Cauchy's Formula

Uniform Convergence and the Weierstrass M-Test

The first major payoff of Theorem 21.2 is that every holomorphic function over any region is analytic. For that we need a refresher on uniform convergence.

Definition 22.1 If A is a subset of $\mathbb{C}$ and $f : A \to \mathbb{C}$ is a function, the **norm** of f over A is

$$\|f\|_A = \sup_{z\in A} |f(z)|.$$

If f is unbounded on A, we set $\|f\|_A = \infty$.

Definition 22.2 A sequence of functions $f_n : A \to \mathbb{C}$ converges to a function $f : A \to \mathbb{C}$ **uniformly** on A when $\|f_n - f\|_A \to 0$ as $n \to \infty$.

This means that for every $\varepsilon > 0$ there is some K such that $\|f_n - f\|_A < \varepsilon$ when $n > K$. In other words, there is K such that for all z in A, $|f_n(z) - f(z)| < \varepsilon$ when $n > K$.

Here are some facts on uniform convergence.

Proposition 22.3 If f_n are continuous on A and $f_n \to f$ uniformly, then f is continuous on A.

Proof. Fix $p \in A$ and let $\varepsilon > 0$. Since $f_n \to f$ uniformly on A, there is N such that

$$\|f - f_N\|_A < \frac{\varepsilon}{3}.$$

Because f_N is continuous at p, there is $\delta > 0$ such that whenever $z \in A$ and $|z - p| < \delta$, we have

$$|f_N(z) - f_N(p)| < \frac{\varepsilon}{3}.$$

For such z, the triangle inequality gives

$$\begin{aligned} |f(z) - f(p)| &\leq |f(z) - f_N(z)| + |f_N(z) - f_N(p)| + |f_N(p) - f(p)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

So f is continuous at p. Since p was arbitrary, f is continuous on A. ☐

Proposition 22.4 If γ is a path in Ω and f_n is a sequence of continuous functions that converges uniformly on γ^* to a function f, then

$$\int_\gamma f_n \to \int_\gamma f.$$

Proof.

$$\left| \int_\gamma f_n - \int_\gamma f \right| = \left| \int_\gamma (f_n - f) \right| \leq \|f_n - f\|_{\gamma^*} \cdot \text{length}(\gamma) \to 0.$$

☐

If f_n is a sequence of functions on a set A, we can form the series $\displaystyle\sum_{n=1}^{\infty} f_n$.

Definition 22.5 If the sequence of partial sums $s_n = \displaystyle\sum_{k=1}^{n} f_k$ converges uniformly on A to a function s, we say that the series **converges uniformly** to s.

A most useful test of uniform convergence of series is due to Weierstrass:

Proposition 22.6 (Weierstrass M-test) If M_n are constants such that $\|f_n\|_A \leq M_n$

and $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

Proof. Let
$$s_n = \sum_{k=1}^{n} f_k$$
be the sequence of partial sums. We first show that $\{s_n\}$ is uniformly Cauchy on A. Take $m > n$ and $z \in A$. Then
$$|s_m(z) - s_n(z)| = \left|\sum_{k=n+1}^{m} f_k(z)\right| \leq \sum_{k=n+1}^{m} |f_k(z)| \leq \sum_{k=n+1}^{m} M_k.$$
Since
$$\sum_{k=1}^{\infty} M_k$$
converges, its tails go to 0. Therefore, for any $\varepsilon > 0$, there is N such that for all $n \geq N$,
$$\sum_{k=n+1}^{\infty} M_k < \varepsilon.$$
Hence, whenever $m > n \geq N$ and $z \in A$,
$$|s_m(z) - s_n(z)| \leq \sum_{k=n+1}^{m} M_k \leq \sum_{k=n+1}^{\infty} M_k < \varepsilon.$$
So $\{s_n\}$ is uniformly Cauchy on A. Now fix $z \in A$. Then $\{s_n(z)\}$ is Cauchy in $\mathbb{C}$, so it converges by completeness. Define
$$s(z) = \lim_{n\to\infty} s_n(z)$$
and let $n \geq N$. For all $m > n$ and all $z \in A$ we have $|s_m(z) - s_n(z)| < \varepsilon$, and letting $m \to \infty$ gives
$$|s(z) - s_n(z)| \leq \varepsilon.$$
Therefore $\|s - s_n\|_A \leq \varepsilon$ for all $n \geq N$, so $s_n \to s$ uniformly on A. This means that
$$\sum_{n=1}^{\infty} f_n$$
converges uniformly on A. $\square$

A simple application of this test is to power series. A power series

$$\sum_{n=0}^{\infty} a_n(z-p)^n$$

with radius R converges absolutely on $D_p(R)$; that is,

$$\sum_{n=0}^{\infty} |a_n(z-p)^n|$$

converges for all $z \in D_p(R)$. However, it need not converge uniformly on $D_p(R)$.

For example, take the geometric series

$$\sum_{n=0}^{\infty} z^n$$

, which converges on $D_0(1)$. Here the partial sums are

$$s_n(z) = 1 + z + \cdots + z^n = \frac{1-z^{n+1}}{1-z} = \frac{1}{1-z} - \frac{z^{n+1}}{1-z}.$$

Now $z^{n+1}/(1-z) \to 0$ on $D_0(1)$, but not uniformly. Indeed,

$$\left\| \frac{z^{n+1}}{1-z} \right\|_{D_0(1)} = \infty$$

for all n, because when z is real and close to 1 we have a vertical asymptote.

However, we do have the following redemption for power series.

Proposition 22.7 If

$$\sum_{n=0}^{\infty} a_n(z-p)^n$$

converges with radius R and if $0 < r < R$, then on the closed disk $\overline{D_p(r)} = \{z : |z-p| \le r\}$, the series converges uniformly.

Proof. For $z \in \overline{D_p(r)}$, we see that

$$|a_n(z-p)^n| \le |a_n| r^n.$$

Since $0 < r < R$, the series

$$\sum_{n=0}^{\infty} |a_n| r^n$$

converges. By the Weierstrass M-test,

$$\sum_{n=0}^{\infty} a_n (z-p)^n$$

converges uniformly on $\overline{D_p(r)}$. □

Equivalence of Holomorphicity and Analyticity

Here comes a very important result.

Theorem 22.8 (Holomorphicity implies analyticity) If f is holomorphic on a region Ω and $D_p(R)$ is any disk inside Ω, then there is a power series $\sum_{n=0}^{\infty} a_n(z-p)^n$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-p)^n \quad \text{for all } z \text{ in } D_p(R).$$

The proof combines Theorem 21.2 with the geometric series.

Proof. Pick a radius r where $0 < r < R$ and define the usual circle $\gamma : [0, 2\pi] \to D_p(R)$ by $t \mapsto p + re^{it}$. For every z in $D_p(r)$, Theorem 21.2 furnishes us with

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta.$$

Now with $\zeta \in \gamma^*$ and $z \in D_p(r)$ we obtain

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta - p - (z-p)} = \frac{f(\zeta)}{\zeta - p} \left(\frac{1}{1 - \dfrac{z-p}{\zeta - p}} \right) = \frac{f(\zeta)}{\zeta - p} \sum_{n=0}^{\infty} \left(\frac{z-p}{\zeta - p} \right)^n = \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - p)^{n+1}} (z-p)^n,$$

since $|(z-p)/(\zeta-p)| = (|z-p|)/r < 1$. This is a series of functions defined for $\zeta \in \gamma^*$. Also,

$$\left\| \frac{f(\zeta)}{(\zeta-p)^{n+1}}(z-p)^n \right\|_{\gamma^*} = \|f\|_{\gamma^*} \frac{|z-p|^n}{r^{n+1}}.$$

Since $|z-p|/r < 1$, the series

$$\sum_{n=0}^{\infty} \|f\|_{\gamma^*} \frac{|z-p|^n}{r^{n+1}}$$

converges. By the Weierstrass M-test,

$$\sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta-p)^{n+1}}(z-p)^n$$

converges uniformly on γ^*. Thus we can integrate this series along γ term by term to get

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-p)^{n+1}}\,\mathrm{d}\zeta\right)(z-p)^n.$$

The numbers

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-p)^{n+1}}\,\mathrm{d}\zeta$$

have nothing to do with z in $D_p(r)$. Thus

$$f(z) = \sum_{n=0}^{\infty} a_n(z-p)^n \quad \text{for all } z \text{ in } D_p(r).$$

Lecture 23 — Wednesday, March 05, 2014

Proof continued. One consideration remains: do the a_n depend on the choice of $0 < r < R$? No! We also know from the convergence of this power series that $a_n = f^{(n)}(p)/n!$. Since every z in $D_p(R)$ has an r such that $|z-p| < r < R$, we also see that the series

$$\sum_{n=0}^{\infty} a_n(z-p)^n$$

converges on $D_p(R)$. In other words, the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!}(z-p)^n$$

converges to $f(z)$ on $D_p(R)$. $\square$

We have now proven that all holomorphic functions on a region Ω are analytic on Ω. The converse holds because of the differentiation theorem (Theorem 12.3). Henceforth, we can refer to functions being analytic or holomorphic on Ω and know we are talking about the same space of functions.

Definition 23.1 As in calculus, the coefficients $a_n = f^{(n)}(p)/n!$ are called the **Taylor coefficients** of f at p.

Cauchy's Integral Formula for Taylor Coefficients

Before the proof of Theorem 22.8 slips from our consciousness, we should observe a nice corollary.

Corollary 23.2 (Cauchy integral formula for Taylor coefficients) If f is holomorphic on a region Ω, $D_p(R) \subseteq \Omega$, and $\gamma : [0, 2\pi] \to D_p(R)$ is the circle path of center p and radius $0 < r < R$, then

$$\frac{f^{(n)}(p)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - p)^{n+1}} \, \mathrm{d}\zeta.$$

Proof. In the proof of Theorem 22.8, we obtained

$$f(z) = \sum_{n=0}^{\infty} a_n (z-p)^n$$

on $D_p(r)$ with

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - p)^{n+1}} \, \mathrm{d}\zeta.$$

For a power series expansion about p, the differentiation theorem (Theorem 12.3) gives

$$a_n = \frac{f^{(n)}(p)}{n!}.$$

Combining these two identities yields

$$\frac{f^{(n)}(p)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - p)^{n+1}}\,\mathrm{d}\zeta.$$

□

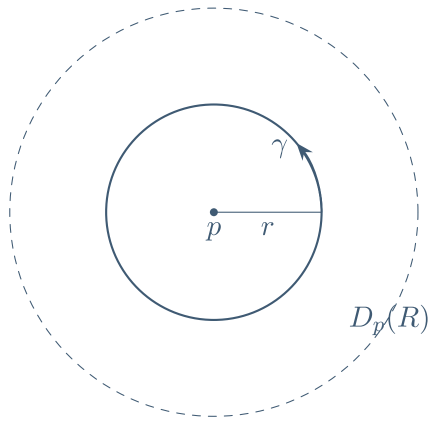

FIGURE 67

Figure 67 records the relevant geometry.

Here are more corollaries.

Corollary 23.3 Every holomorphic function on Ω has derivatives $f^{(n)}(z)$ of all orders at every point of Ω.

Proof. Holomorphic functions are analytic, and the differentiation theorem (Theorem 12.3) for power series handles this immediately. □

Corollary 23.4 Every holomorphic function f on a region Ω has a local primitive.

Proof. More specifically, for any $p \in \Omega$, let $D_p(R)$ be the biggest possible disk inside Ω. We know

$$f(z) = \sum_{n=0}^{\infty} a_n (z-p)^n$$

for some coefficients a_n and all z in $D_p(R)$. The integrated series

$$g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-p)^{n+1}$$

gives a primitive for f on $D_p(R)$. □

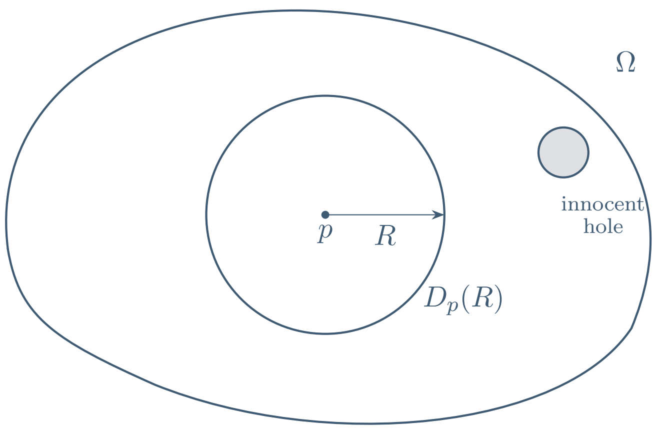

FIGURE 68

The configuration appears in Figure 68.

Corollary 23.4 can also be proven quickly by means of the baby version of Cauchy's theorem applied on the disk $D_p(R)$.

Cauchy's Estimates, Liouville's Theorem, and the Fundamental Theorem of Algebra

Some estimates can be made on the size of the Taylor coefficients $f^{(n)}(p)/n!$ of a holomorphic function.

Proposition 23.5 (Cauchy's estimates) If f is holomorphic on Ω, $D_p(R) \subseteq \Omega$ with $0 < R < \infty$, and M is an upper bound for $|f|$ on $D_p(R)$, then

$$\left| \frac{f^{(n)}(p)}{n!} \right| \leq \frac{M}{R^n}.$$

Proof. For each radius $r < R$ let $\gamma : [0, 2\pi] \to D_p(R)$ where $t \mapsto p + re^{it}$ is the usual circle. From Corollary 23.2 we get

$$\left|\frac{f^{(n)}(p)}{n!}\right| = \frac{1}{2\pi}\left|\int_\gamma \frac{f(\zeta)}{(\zeta - p)^{n+1}}\, \mathrm{d}\zeta\right| \leq \frac{1}{2\pi}\frac{\|f\|_{\gamma^*}}{r^{n+1}} 2\pi r = \frac{\|f\|_{\gamma^*}}{r^n} \leq \frac{M}{r^n}.$$

Since this holds for all $r < R$, we let $r \to R$ and get

$$\left|\frac{f^{(n)}(p)}{n!}\right| \leq \frac{M}{R^n}.$$

$\square$

Here comes something nice.

Theorem 23.6 (Liouville's theorem) If f is bounded and entire, then f is constant.

Proof. Let M be a bound for $|f|$ over $\mathbb{C}$. We have a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

valid on $\mathbb{C}$, and

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

By Cauchy's estimates,

$$\left|\frac{f^{(n)}(0)}{n!}\right| \leq \frac{M}{R^n}$$

where R is any radius such that $D_0(R) \subseteq \mathbb{C}$. In other words, R is arbitrary. Let $R \to \infty$ to obtain $a_n = 0$ when $n = 1, 2, 3, \ldots$. Consequently $f(z) = a_0$, a constant. $\square$

Next comes a lovely application of Theorem 23.6.

Theorem 23.7 (Fundamental theorem of algebra) Every complex polynomial of positive degree has a complex root.

Proof. It does no harm to suppose our polynomial is monic, and so we write

$$f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

where $a_j \in \mathbb{C}$ and $n \geq 1$. Suppose $f(z)$ has no root in $\mathbb{C}$. In that case, the reciprocal function $g(z) = 1/f(z)$ is entire. Let us verify that $g(z)$ is bounded on $\mathbb{C}$. For $z \neq 0$:

$$|f(z)| = |z|^n \left|1 + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}\right| \geq |z|^n \left(1 - \left(\frac{|a_{n-1}|}{|z|} + \frac{|a_{n-2}|}{|z|^2} + \cdots + \frac{|a_0|}{|z|^n}\right)\right).$$

By inspection,

$$|z|^n \left(1 - \left(\frac{|a_{n-1}|}{|z|} + \frac{|a_{n-2}|}{|z|^2} + \cdots + \frac{|a_0|}{|z|^n}\right)\right) \to \infty$$

as $|z| \to \infty$. Thus $|f(z)| \to \infty$ as $|z| \to \infty$. This tells us that there exists $R_0 > 0$ such that $|f(z)| \geq 1$ for all $|z| \geq R_0$. Therefore $|g(z)| \leq 1$ when $|z| \geq R_0$. But on the compact set $\overline{D_0(R_0)} = \{z : |z| \leq R_0\}$, $g(z)$ is bounded because g is continuous. Therefore $g(z)$ is bounded on all of $\mathbb{C}$. By Theorem 23.6, $g(z)$ is constant. Hence $f(z) = 1/g(z)$ is constant too. But this cannot be since f has degree at least 1. The only way out is to conclude that $f(z)$ has a root in $\mathbb{C}$. $\square$

Lecture 24 — Friday, March 07, 2014

The Identity Theorem, the Maximum Modulus Principle, and Morera's Theorem

The consequences of Theorem 21.2 keep on coming.

Definition 24.1 Let A be a set of complex numbers. A point $p \in \mathbb{C}$ is a **cluster point** of A if every disk $D_p(\varepsilon)$ contains points of A other than p.

This is the same as saying that there is a sequence z_n in A such that $z_n \to p$ but $z_n \neq p$.

Definition 24.2 If a point p in A is not a cluster point, we call it an **isolated point** of A.

Thus p in A is isolated when there is a disk $D_p(\varepsilon)$ containing only the element p from A; that is, $D_p(\varepsilon) \cap A = \{p\}$.

Example 24.3 The set $A = \{m + ni : m, n \in \mathbb{Z}\}$ consisting of Gaussian integers only has isolated points.

Example 24.4 $A = \{0, 1, 1/2, 1/3, \dots, 1/n, \dots\}$ has 0 as a cluster point, while the $1/n$ are all isolated.

Definition 24.5 For a function $f : \Omega \to \mathbb{C}$, define its **set of zeroes** to be

$$Z(f) = \{p \in \Omega : f(p) = 0\}.$$

Example 24.6 If $f(z) = e^z$, then $Z(f) = \emptyset$. If $f(z) = \sin z$, then we know that $Z(f) = \pi\mathbb{Z}$. For a more offbeat example, let

$$f(x) = \begin{cases} e^{-\left(\frac{1}{x-1}\right)^2}, & x > 1, \\ 0, & -1 \le x \le 1, \\ e^{-\left(\frac{1}{x+1}\right)^2}, & x < -1. \end{cases}$$

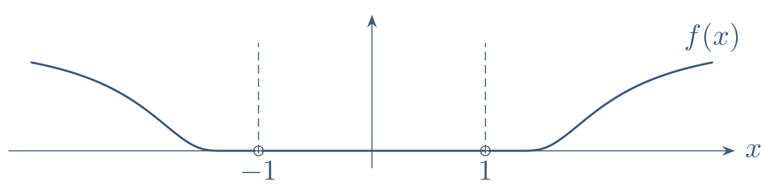

Figure 69

Figure 69 illustrates the setup. Then $Z(f) = [-1, 1]$.

Proposition 24.7 If f is nonzero and analytic on a region Ω, then every p in $Z(f)$ is isolated.

Proof. Let N be the set of $p \in Z(f)$ such that p is *not* isolated. The plan is to show that $N = \emptyset$. The first step is to show that $\Omega \setminus N$ is open. Clearly $\Omega \setminus N \neq \emptyset$ since $f \not\equiv 0$. So take $p \in \Omega \setminus N$. Either p is not a zero of f, or p is an isolated zero of f. If p isn't a zero, real analysis gives some $D_p(\varepsilon)$ such that f is never zero on $D_p(\varepsilon)$ because f is continuous. Hence $D_p(\varepsilon) \subseteq \Omega \setminus N$. On the other hand, if p is an isolated zero, then take $D_p(\varepsilon)$ such that $D_p^*(\varepsilon) \cap Z(f) = \emptyset$. So

$$D_p(\varepsilon) = \{p\} \cup D_p^*(\varepsilon) \subseteq \Omega \setminus N.$$

In either case, $\Omega \setminus N \neq \emptyset$ and $\Omega \setminus N$ is open. The second step is to show that N itself is open. Take $p \in N$. Since f is analytic, there is some $D_p(r) \subseteq \Omega$ such that

$$f(z) = a_0 + a_1(z-p) + a_2(z-p)^2 + \cdots .$$

We certainly have $a_0 = f(p) = 0$. If some of the other a_j are nonzero, write

$$f(z) = \sum_{k=1}^{\infty} a_k(z-p)^k = (z-p)^k \Big(a_k + a_{k+1}(z-p) + a_{k+2}(z-p)^2 + \cdots \Big)$$

on $D_p(r)$, where k is the first index with $a_k \neq 0$. The function

$$g(z) = a_k + a_{k+1}(z-p) + a_{k+2}(z-p)^2 + \cdots$$

is such that $g(p) = a_k \neq 0$. By the same real analysis fact, there is some $D_p(\varepsilon) \subset D_p(r)$ on which g is never zero. Thus f itself is nonzero on $D_p^*(\varepsilon)$, but this contradicts the fact that $p \in N$ (since p would then be an isolated zero). The only way out is for *all* of the a_j to be zero, so $f \equiv 0$ on $D_p(r)$. Hence $D_p(r)$ consists entirely of zeros of f. But then every $z \in D_p(r)$ is a nonisolated zero, so $D_p(r) \subseteq N$. Thus N is open. Finally, $\Omega = N \cup (\Omega \setminus N)$. Since Ω is connected, N is open, and $\Omega \setminus N$ is open and nonempty, it follows that $N = \emptyset$. □

Lecture 25 — Monday, March 10, 2014

Theorem 25.1 (The identity theorem) If f, g are holomorphic (that is, analytic) functions on a region Ω, A is a subset of Ω with a cluster point in Ω, and $f(z) = g(z)$ for all z in A, then $f = g$ on Ω.

Proof. The function $h(z) = f(z) - g(z)$ is analytic and vanishes on a set with a cluster point. Since h has at least one nonisolated zero, h must be zero on Ω by Proposition 24.7, and so $f = g$ on Ω. □

This again reveals the fully shaped nature of holomorphic functions. In order for two of them to be equal, all they have to do is agree on a set with a cluster point (that set could be a curve, a little disk, or even a sequence together with its distinct limit).

Just to pause from so much theory, let us relax by calculating an integral.

Example 25.2 If $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$ gives the unit circle, let us find $\int_\gamma (\sin z)/z^4 \, \mathrm{d}z$.

Apply Corollary 23.2 to the entire function $f(z) = \sin z$ at the center 0 to get

$$\int_\gamma \frac{\sin z}{z^4} \, \mathrm{d}z = 2\pi i \frac{f^{(3)}(0)}{3!} = \frac{\pi i}{3}(-\cos 0) = -\frac{\pi i}{3}.$$

Here comes yet another peculiarity of analytic (i.e., holomorphic) functions.

Theorem 25.3 (Maximum modulus principle) If f is holomorphic and nonconstant on a region Ω, then $|f|$ has no local maximum in Ω.

Proof. Suppose, on the contrary, that a local maximum for $|f|$ occurs at p in Ω. Hence $|f(z)| \leq |f(p)|$ for all z in some $D_p(R) \subseteq \Omega$. Since f is not constant on Ω, the identity theorem implies that f is not constant on $D_p(R)$. As a consequence of the Cauchy–Riemann equations, $|f|$ is not constant on $D_p(R)$. Thus $|f(w)| < |f(p)|$ for some $w \in D_p(R)$. Letting $r = |w - p|$, we see that $0 < r < R$. The points z on the circle of center p and radius r take the form $z = p + re^{it}$ where $t \in [0, 2\pi]$.

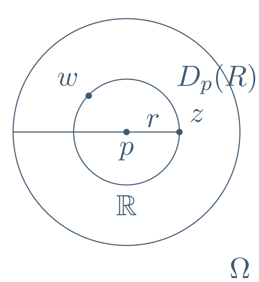

FIGURE 70

The construction is shown in Figure 70.

Thus $|f(p+re^{it})| \leq |f(p)|$ for all $t \in [0, 2\pi]$ and $|f(p+re^{is})| < |f(p)|$ for some $s \in [0, 2\pi]$. By standard properties of integrals of continuous functions we have

$$\int_0^{2\pi} |f(p+re^{it})|\,\mathrm{d}t < \int_0^{2\pi} |f(p)|\,\mathrm{d}t = 2\pi|f(p)|.$$

Proposition 19.2 applied to the circle of center p and radius r gives

$$f(p) = \frac{1}{2\pi}\int_0^{2\pi} f(p+re^{it})\,\mathrm{d}t.$$

Hence

$$|f(p)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(p+re^{it})\,\mathrm{d}t\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(p+re^{it})|\,\mathrm{d}t < |f(p)|,$$

a ridiculous contradiction. □

Quite evidently, since $|f|$ has no local maximum on Ω, then $|f|$ has no global maximum on Ω either.

Corollary 25.4 If A is a compact subset of Ω and f is nonconstant and analytic on Ω, then $|f|$ attains its maximum over A on the boundary of A.

Proof. $|f|$ does attain its maximum on A because $|f|$ is continuous and A is compact. By the maximum modulus principle, this maximum cannot occur in the interior of A. □

Here is a nice exam-style question that uses the maximum modulus principle:

Example 25.5 Let us find $\max_{|z|\leq 1} |z^2 + 3z - 1|$.

Since the function $|z^2+3z-1|$ is continuous and the closed disk $A = \{z : |z| \leq 1\}$ is compact, our function does have a maximum over this closed disk. Also, since $z^2 + 3z - 1$ is analytic (actually entire), this maximum must occur on the boundary $T = \{z : |z| = 1\}$ of our disk A.

For $|z| = 1$:

$$\begin{aligned}|z^2+3z-1|^2 &= (z^2+3z-1)(\bar{z}^2+3\bar{z}-1)\\ &= (z\bar{z})^2+3z^2\bar{z}-z^2+3z\bar{z}^2+9z\bar{z}-3z-\bar{z}^2-3\bar{z}+1\\ &= 1+3z-z^2+3\bar{z}+9-3z-\bar{z}^2-3\bar{z}+1\\ &= 11-(z^2+\bar{z}^2)\\ &= 11-2\left(\frac{z^2+\bar{z}^2}{2}\right).\end{aligned}$$

For $|z| = 1$ we can write $z = e^{i\theta}$ where $\theta \in \mathbb{R}$, so that $z^2 = e^{i2\theta}$ and $(z^2+\bar{z}^2)/2 = \cos 2\theta$. Then

$$|z^2+3z-1|^2 = 11-2\cos 2\theta.$$

The maximum of this occurs when $\cos 2\theta = -1$, giving a value of 13.

Thus $|z^2+3z-1|$ has maximum value $\sqrt{13}$ over the closed disk A.

Recall that Theorem 17.1 told us that if f is holomorphic in Ω and Δ is any (solid) triangle inside Ω, then

$$\int_{\partial\Delta} f = 0.$$

The converse of this statement is also true.

Theorem 25.6 (Morera's theorem) If f is continuous on a region Ω and if for every triangle $\Delta \subseteq \Omega$ we have $\int_{\partial\Delta} f = 0$, then f is analytic (that is, holomorphic) on Ω.

Lecture 26 — Wednesday, March 12, 2014

Proof. Take any $D_p(R) \subseteq \Omega$. According to the argument in Theorem 18.2 applied to $D_p(R)$, the function

$$g(z) = \int_{[p,z]} f$$

(where $[p, z]$ is the line from p to z) is a primitive for f on $D_p(R)$, so $g'(z) = f(z)$. Since g is holomorphic on $D_p(R)$, g has a power series representation

$$g(z) = \sum_{n=0}^{\infty} a_n (z-p)^n$$

for all z in $D_p(R)$. But then the derived series

$$\sum_{n=1}^{\infty} n a_n (z-p)^{n-1}$$

represents $f(z)$ on $D_p(R)$. Since f has a power series representation on every $D_p(R) \subseteq \Omega$, f is analytic on Ω. $\square$

Uniform Convergence on Compact Sets

We can exploit Theorem 25.6 to make more analytic functions.

Definition 26.1 A sequence of functions $f_n : \Omega \to \mathbb{C}$ is said to **converge uniformly on compact sets** to a function $f : \Omega \to \mathbb{C}$ provided that for every compact set $A \subseteq \Omega$, $\|f_n - f\|_A \to 0$ (that is, $f_n \to f$ uniformly on A).

Example 26.2 Let

$$s(z) = \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$$

on $D_0(1)$, and $s_n(z) = \sum_{k=0}^{n-1} z^k$. We saw

$$\|s - s_n\|_{D_0(1)} = \left\| \frac{z^n}{1-z} \right\|_{D_0(1)} = \infty \not\to 0,$$

so $s_n \not\to s$ uniformly on $D_0(1)$. But $D_0(1)$ is not compact! In contrast, $s_n \to s$ uniformly on compact subsets.

In fact, all power series converge uniformly on compact sets. We saw that for any power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

converging on some $D_0(R)$ that the sums

$$s_n(z) = \sum_{k=0}^{n} a_k z^k$$

converge uniformly to $f(z)$ on every closed disk $\overline{D_0(r)}$ where $r < R$. Since every compact subset A of $D_0(R)$ lies inside some $\overline{D_0(r)}$, the assertion holds true.

Proposition 26.3 Let

$$s = \sum_{k=0}^{\infty} a_k (z-p)^k \qquad \text{and} \qquad s_n = \sum_{k=0}^{n-1} a_k (z-p)^k, \quad \text{on } D_p(R).$$

Then $s_n \to s$ uniformly on compact sets.

Proof. Let $A \subseteq D_p(R)$ be compact. Notice that

$$A \subseteq D_p(R) = \bigcup_{0<r<R} D_p(r).$$

By compactness, A is contained in some finite union of these nested disks, which equals one $D_p(r_0)$ for some $0 < r_0 < R$. Choose r with $r_0 < r < R$. Then $A \subseteq \overline{D_p(r)}$. To see that $s_n \to s$ uniformly on A, it is enough to check it on $\overline{D_p(r)}$, as $\|s - s_n\|_A \leq \|s - s_n\|_{\overline{D_p(r)}}$. For $z \in \overline{D_p(r)}$, we have

$$|a_n (z-p)^n| \leq |a_n| r^n,$$

and

$$\sum_{n=0}^{\infty} |a_n| r^n$$

converges because $r < R$. By the Weierstrass M-test, $\|s - s_n\|_{\overline{D_p(r)}} \to 0$. □

Theorem 26.4 If $f_n : \Omega \to \mathbb{C}$ are holomorphic (that is, analytic) and $f_n \to f$ uniformly on compact subsets of Ω, then f is holomorphic (that is, analytic) too.

Proof. Note that f is continuous on Ω. Indeed, for each $p \in \Omega$, choose $s > 0$ with $\overline{D_p(s)} \subseteq \Omega$. Then $f_n \to f$ uniformly on the compact set $\overline{D_p(s)}$, so f is continuous at p, and thus on Ω. Let Δ

be any triangle in Ω. By Theorem 17.1,

$$\int_{\partial\Delta} f_n = 0.$$

However, Δ and $\partial\Delta$ are both compact, so

$$\int_{\partial\Delta} f_n \to \int_{\partial\Delta} f$$

because

$$\left|\int_{\partial\Delta} f_n - \int_{\partial\Delta} f\right| = \left|\int_{\partial\Delta} (f_n - f)\right| \le \|f_n - f\|_{\partial\Delta} \cdot \text{length}(\partial\Delta) \to 0,$$

and

$$\int_{\partial\Delta} f = 0$$

since we have a sequence of zeros. By Theorem 25.6, f is holomorphic too. $\square$

Lecture 27 — Friday, March 14, 2014

Let us get a little more practice verifying that power series converge uniformly on compact sets.

Example 27.1 For $z \in \mathbb{C}$, let $n^z = e^{z \log n}$. If $z = x + iy$, then

$$|n^z| = |e^{x \log n + iy \log n}| = e^{x \log n} = n^x.$$

If $x > 1$, we know from the study of p-series in calculus that

$$\sum_{n=1}^{\infty} \frac{1}{n^x}$$

converges. For instance, when $x = 2$, Euler famously showed

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Thus

$$\sum_{n=1}^{\infty} 1/n^z$$

converges absolutely on $H = \{x + iy : x > 1\}$.

Let us verify that this series converges uniformly on every compact subset A of H.

First, we check for such A that there is an $\varepsilon > 0$ such that $A \subseteq H_\varepsilon = \{x + iy : x \geq 1 + \varepsilon\}$. If there is no such ε, then we have a sequence $x_n + iy_n$ in A such that $1 < x_n < 1 + 1/n$. Since A is compact, we pick up a subsequence $x_{n_k} + iy_{n_k}$ such that

$$x_{n_k} + iy_{n_k} \to x + iy \in A.$$

But $1 < x_{n_k} < 1 + 1/n_k$ implies $x = 1$. However, $1 + iy \notin A$ since $A \subseteq H$, a contradiction.

Now take our $\varepsilon > 0$ for which every $z = x + iy$ in A has $x \geq 1 + \varepsilon$. We have $|n^{x+iy}| = n^x \geq n^{1+\varepsilon}$. Thus

$$\left|\frac{1}{n^z}\right| \leq \frac{1}{n^{1+\varepsilon}}$$

for all z in A.

The series

$$\sum_{n=1}^{\infty} 1/n^{1+\varepsilon}$$

converges, since $1 + \varepsilon > 1$. By the Weierstrass M-test,

$$\sum_{n=1}^{\infty} 1/n^z$$

converges uniformly on A. Each function

$$\frac{1}{n^z} = \frac{1}{e^{z \log n}} = e^{-z \log n}$$

is analytic (and, in fact, entire). By uniform convergence on compact subsets of H, our series

$$\sum_{n=1}^{\infty} 1/n^z$$

defines an analytic function on H. That function is the renowned **Riemann zeta function**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

(Number theorists like to use s for a complex number.)

9. Singularities

We will need again to work with punctured disks:

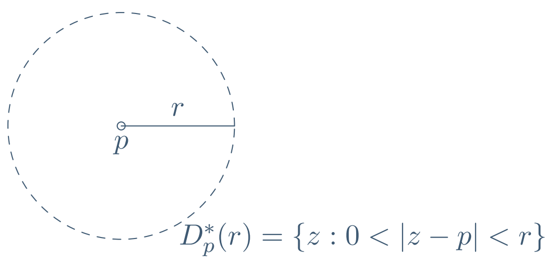

FIGURE 71

Figure 71 records the relevant geometry.

Definition 27.2 An analytic function f on a region Ω has a **singularity** at p when $p \notin \Omega$ but some $D_p^*(r) \subseteq \Omega$.

That is, $p \notin \Omega$ but some punctured disk around p is inside Ω.

Example 27.3 $f(z) = 1/z$ is analytic on $\mathbb{C} \setminus \{0\}$ with a singularity at 0. The principal branch $\log z$ is analytic on the cut plane $\Omega = \mathbb{C} \setminus \{x : x \le 0\}$, but the points $x \le 0$ on the cut are not called singularities of $\log z$. Also, $1/(z^2+1)$ has singularities at $\pm i$, and $e^{1/z}$ has a singularity at 0.

Recall that if f is holomorphic and not constant on Ω, then the set $Z(f)$ of zeros of f consists of isolated points by Proposition 24.7. That is, for every zero p of f there is a punctured disk $D_p^*(r)$ on which f never vanishes. Hence $1/f$ is analytic on $\Omega \setminus Z(f)$, and the points of $Z(f)$ are singularities of $1/f$.

Example 27.4 Let us find the singularities of $1/\sin z$. According to the previous discussion, these are the zeros of $\sin z$.

Now,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = 0 \iff e^{iz} = e^{-iz}.$$

Writing $z = x + iy$ we get

$$\begin{aligned} e^{iz} = e^{-iz} &\iff e^{i(x+iy)} = e^{-i(x+iy)} \\ &\iff e^{-y+ix} = e^{y-ix} \\ &\iff e^{-2y}e^{i2x} = 1 \\ &\iff y = 0 \text{ and } 2x = 2\pi k \text{ for some integer } k \\ &\iff z = \pi k + i0 \text{ for some integer } k. \end{aligned}$$

So the set of singularies is $Z(\sin z) = \pi\mathbb{Z}$.

FIGURE 72

The configuration appears in Figure 72.

We also say that p is a singularity of f, even though p is not in the domain of f.

Removable Singularities

Definition 27.5 If f is analytic on Ω, a singularity p of f is called **removable** when it is possible to define $f(p)$ in such a way that f becomes analytic on $\Omega \cup \{p\}$.

As soon as a removable singularity is detected, we will agree to remove as soon as possible and do not call it a singularity anymore. We will then be much happier.

Example 27.6 The function

$$f(z) = \frac{\sin z}{z}$$

has a removable singularity at 0. Indeed, for all $z \in \mathbb{C}$, we know

$$\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \frac{1}{7!}z^7 + \dots$$

Thus for all nonzero z we have

$$\frac{\sin z}{z} = 1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \frac{1}{7!}z^6 + \dots$$

If we put $f(0) = 1$, then

$$f(z) = 1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \dots \quad \text{for all } z \in \mathbb{C}.$$

The singularity that f had at 0 has thus been removed.

Proposition 27.7 Let f be analytic on Ω with a singularity at p. The singularity is removable if and only if f is bounded on some $D_p^*(r) \subseteq \Omega$.

Proof. *Boundedness implies removability.* Say $|f(z)| \leq M$ on $D_p^*(r)$. Put

$$g(z) = \begin{cases} (z-p)^2 f(z), & z \in \Omega, \\ 0, & z = p. \end{cases}$$

Then for $z \in D_p^*(r)$,

$$\left| \frac{g(z) - g(p)}{z - p} \right| = |z-p||f(z)| \leq |z-p|M \to 0 \text{ as } z \to p.$$

Thus $g'(p)$ exists and equals 0. Since g is analytic on $\Omega \cup \{p\}$, we have

$$g(z) = a_0 + a_1(z-p) + a_2(z-p)^2 + \dots$$

for some coefficients a_n and all z in $D_p(r) \subseteq \Omega \cup \{p\}$. Since $g(p) = g'(p) = 0$, we see that $a_0 = a_1 = 0$, and so

$$g(z) = (z-p)^2 \left(a_2 + a_3(z-p) + a_4(z-p)^2 + \dots \right)$$

for all z in $D_p(r)$. Cancel $(z-p)^2$ to get

$$f(z) = a_2 + a_3(z-p) + a_4(z-p)^2 + \dots \quad \text{for all } z \in D_p^*(r).$$

Define $f(p) = a_2$, so that f now has a power series representation on $D_p(r)$. Thereby f becomes analytic on $\Omega \cup \{p\}$. The singularity at p has been removed.

Removability implies boundedness. If f has a removable singularity at p, then remove it. The resulting function is continuous at p and therefore bounded on some disk $D_p(\varepsilon)$. After all,

$$|f(z) - f(p)| \le 1 \quad \text{when } z \in \text{some } D_p(\varepsilon).$$

□

Lecture 28 — Monday, March 17, 2014

The Casorati–Weierstrass Theorem

More interesting behavior comes from singularities that are not removable. For instance, $1/z$ is not bounded on any $D_0(\varepsilon)$. Thus the singularity that $1/z$ has at 0 is not removable.

The nonremovable singularities are of two types.

Theorem 28.1 (Casorati–Weierstrass) Let f be analytic on Ω with a nonremovable singularity at p. Then p has one of two mutually exclusive options:

1. For every punctured disk $D_p^*(\varepsilon)$ inside Ω, the image $f(D_p^*(\varepsilon))$ is dense in $\mathbb{C}$.
2. There is a positive integer m such that $(z-p)^m f(z)$ has a removable singularity at p.

The failure of (1) means that for some punctured disk $D_p^*(\varepsilon)$ there are a $w \in \mathbb{C}$ and a $\delta > 0$ such that $|f(z) - w| \ge \delta$ for all $z \in D_p^*(\varepsilon)$; in other words, $D_w(\delta) \cap f(D_p^*(\varepsilon)) = \emptyset$.

Proof. It is enough to show two implications: if (2) holds, then (1) fails; and if (1) fails, then (2) holds.

Suppose (2) holds. Let m be the least integer ≥ 1 such that $(z-p)^m f(z)$ has a removable singularity at p. Let

$$g(z) = (z-p)^m f(z),$$

with $g(p)$ defined so that g is analytic on $\Omega \cup \{p\}$. Thus

$$g(z) = a_0 + a_1(z-p) + a_2(z-p)^2 + \dots$$

for all z in some $D_p(r) \subseteq \Omega \cup \{p\}$. If $a_0 = g(p) = 0$, then

$$(z-p)^m f(z) = (z-p)\left(a_1 + a_2(z-p) + a_3(z-p)^2 + \dots\right)$$

for all $z \in D_p^*(r)$. Cancel $z-p$ to get

$$(z-p)^{m-1} f(z) = a_1 + a_2(z-p) + a_3(z-p)^2 + \dots$$

for all $z \in D_p^*(r)$. In this function we could remove the singularity at p, contradicting the minimality of m. Thus $a_0 = g(p) \neq 0$. Since $|g|$ is continuous and $|g(p)| > 0$, there is a disk $D_p(\varepsilon) \subseteq \Omega \cup \{p\}$ on which $|g|$ stays bounded away from 0. That is, $|g(z)| \geq B > 0$ for $z \in D_p(\varepsilon)$. Thus for $z \in D_p^*(\varepsilon)$ we have

$$|f(z)| = \left|\frac{g(z)}{(z-p)^m}\right| \geq \frac{B}{\varepsilon^m} > 0.$$

The image $f(D_p^*(\varepsilon))$ is not dense in $\mathbb{C}$, since its values are bounded away from 0. Hence (1) fails.

Conversely, suppose (1) fails. Thus $f(D_p^*(\varepsilon))$ is not dense in $\mathbb{C}$ for some punctured disk $D_p^*(\varepsilon)$. Then there is a point $w \in \mathbb{C}$ and a $\delta > 0$ such that

$$|f(z) - w| \geq \delta \quad \text{for all } z \in D_p^*(\varepsilon).$$

Examine

$$g(z) = \frac{1}{f(z) - w}$$

on $D_p^*(\varepsilon)$. Notice that $0 < |g(z)| \leq 1/\delta$ and $f(z) = w + 1/g(z)$ for $z \in D_p^*(\varepsilon)$. Thus we can define $g(p)$ so as to make g analytic on the full $D_p(\varepsilon)$. Look at the power series representation for $g(z)$ on $D_p(\varepsilon)$:

$$g(z) = a_0 + a_1(z-p) + a_2(z-p)^2 + \dots \quad \text{for all } z \in D_p(\varepsilon).$$

The function g cannot vanish identically, since it is nonzero on the punctured disk. Let a_m be its first nonzero coefficient. So $m \geq 0$ and $a_m \neq 0$. By factoring we have

$$g(z) = (z-p)^m \left(a_m + a_{m+1}(z-p) + a_{m+2}(z-p)^2 + \dots\right) \quad \text{for all } z \in D_p(\varepsilon).$$

Let

$$h(z) = a_m + a_{m+1}(z-p) + a_{m+2}(z-p)^2 + \dots,$$

an analytic function on $D_p(\varepsilon)$. Notice that $h(z) \neq 0$ for $z \in D_p^*(\varepsilon)$ since $g(z) \neq 0$ there. Also $h(p) = a_m \neq 0$. Thus h never vanishes on $D_p(\varepsilon)$. From the previous expression we get

$$f(z) = w + \frac{1}{(z-p)^m h(z)} \quad \text{for all } z \in D_p^*(\varepsilon).$$

Hence

$$(z-p)^m f(z) = w(z-p)^m + \frac{1}{h(z)} \quad \text{for all } z \in D_p^*(\varepsilon).$$

By looking at the right-hand side, we see that the singularity of $(z-p)^m f(z)$ at p is removable. It remains to check that $m \geq 1$. If $m = 0$, then $f(z) = (z-p)^0 f(z)$ would have a removable singularity at p, contrary to assumption. □

Essential Singularities and Picard's Theorem

Definition 28.2 Suppose that a function $f : \Omega \to \mathbb{C}$ has a singularity at p. When the singularity not removable, but for some $m \geq 1$ the singularity of $(z-p)^m f(z)$ at p it *is* removable, the singularity is called a **pole** of f. When this is not possible and all $f(D_p^*(\varepsilon))$ are dense in $\mathbb{C}$, the singularity is called an **essential singularity**.

Proposition 27.7 and Theorem 28.1 tell us that a singularity for f must be one of three types: a removable singularity, an essential singularity, or a pole.

Lecture 29 — Wednesday, March 19, 2014

Example 29.1 The singularity of $e^{1/z}$ at 0 is essential.

This is because for any $m \geq 1$, the function $z^m e^{1/z}$ remains unbounded on every $D_0^*(r)$. Indeed, $1/n \to 0$, and yet

$$\left(\frac{1}{n}\right)^m e^{1/(1/n)} = \frac{e^n}{n^m} \to \infty \quad \text{as } n \to \infty.$$

Regarding essential singularities, there is a major theorem due to Picard:

Theorem 29.2 (Picard's theorem) If f is analytic on Ω and has an essential singularity at p, then on every punctured disk $D_p^*(\varepsilon) \subseteq \Omega$, the function f takes every complex value, with at most one exception, infinitely often.

Needless to say, the proof is hard, so we will skip it. But as an easier exercise, at least show that $e^{1/z}$ maps every $D_0^*(\varepsilon)$ onto the punctured plane.

Poles and Residues

Now let us turn our attention to poles. If f is analytic on Ω with a pole at p, then $(z-p)^m f(z)$ has a removable singularity at p, for some $m \geq 1$. We take m to be the least positive integer for which the function

$$g(z) = (z-p)^m f(z)$$

can be made analytic on $\Omega \cup \{p\}$. We can write

$$g(z) = a_0 + a_1(z-p) + a_2(z-p)^2 + \dots$$

for z in some $D_p(r)$. If $g(p) = a_0 = 0$, then we would get

$$(z-p)^m f(z) = (z-p)\left(a_1 + a_2(z-p) + a_3(z-p)^2 + \dots\right)$$

for z in $D_p^*(r)$.

Cancel $z-p$ to get

$$(z-p)^{m-1} f(z) = a_1 + a_2(z-p) + a_3(z-p)^2 + \dots$$

when $z \in D_p^*(r)$. But this contradicts the minimality of m, since the right side above tells us how to remove the singularity of $(z-p)^{m-1} f(z)$ at p. Thus $g(p) = a_0 \neq 0$.

Then for z in $D_p^*(r)$, we get

$$\begin{aligned} f(z) &= \frac{1}{(z-p)^m} g(z) \\ &= \frac{a_0}{(z-p)^m} + \frac{a_1}{(z-p)^{m-1}} + \dots \quad + \frac{a_{m-1}}{z-p} \quad + \left(a_m + a_{m+1}(z-p) + a_{m+2}(z-p)^2 + \dots\right). \end{aligned}$$

with $m \geq 1$ and $a_0 \neq 0$.

The function
$$h(z) = f(z) - \left(\frac{a_{m-1}}{z-p} + \cdots + \frac{a_0}{(z-p)^m}\right)$$
is analytic on Ω. From the displayed formula we see that h is actually analytic on $\Omega \cup \{p\}$, that is, the singularity for h at p is removable.

Thus we have learned the following:

Proposition 29.3 If f is analytic on Ω and has a pole at p, then there is an analytic function h on $\Omega \cup \{p\}$, a positive integer m, and scalars $b_1, b_2, \ldots, b_m$ with $b_m \neq 0$ such that
$$f(z) = h(z) + \frac{b_1}{z-p} + \frac{b_2}{(z-p)^2} + \cdots + \frac{b_m}{(z-p)^m}.$$

Definition 29.4 If m is the smallest positive integer m that causes $(z-p)^m f(z)$ to have a removable singularity at p, we say that f has a **pole of order m** at p.

The function
$$\frac{b_1}{z-p} + \frac{b_2}{(z-p)^2} + \cdots + \frac{b_m}{(z-p)^m}$$
is called the **principal part of f at p**. The special coefficient b_1 is called the **residue of f at p**.

Residues are very important. Suppose f is analytic on Ω with a pole at p and represented as
$$f(z) = h(z) + \frac{b_1}{z-p} + \frac{b_2}{(z-p)^2} + \cdots + \frac{b_m}{(z-p)^m}, \quad m \geq 1,\ b_m \neq 0,$$
with h analytic on $\Omega \cup \{p\}$. Take any $D_p(R) \subseteq \Omega \cup \{p\}$ and let γ be any closed path in $D_p(R)$ that misses p.

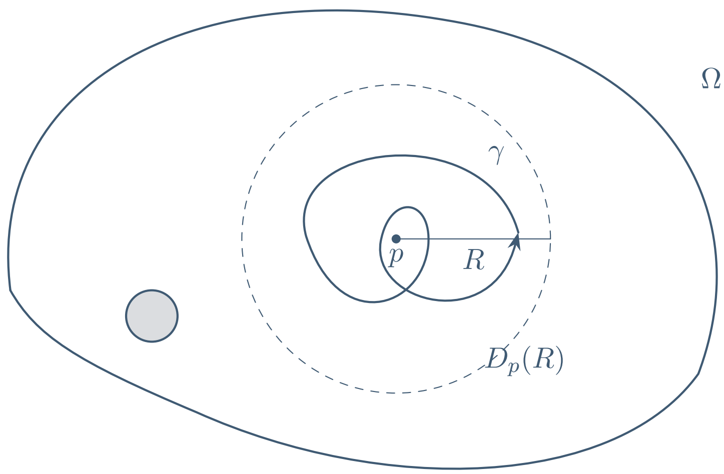

FIGURE 73

Figure 73 illustrates the setup.

Inside $D_p(R)$ the function h has a power series representation $\sum_{n=0}^{\infty} a_n(z-p)^n$, and its integrated series $\sum_{n=0}^{\infty} a_n/(n+1)(z-p)^{n+1}$ gives a primitive for h on $D_p(R)$.

For $j = 2, \ldots, m$, the functions $1/(z-p)^j$ have a primitive on $\Omega \setminus \{p\}$, namely

$$\frac{1}{-(j-1)(z-p)^{j-1}}.$$

Therefore,

$$\int_\gamma f = \int_\gamma h + b_1 \int_\gamma \frac{\mathrm{d}z}{z-p} + b_2 \int_\gamma \frac{\mathrm{d}z}{(z-p)^2} + \cdots + b_m \int_\gamma \frac{\mathrm{d}z}{(z-p)^m}.$$

This simplifies to

$$\int_\gamma f = b_1 \int_\gamma \frac{\mathrm{d}z}{z-p}$$

by Theorem 18.2, because the other functions have primitives on $D_p^*(R)$ where γ travels. Thus,

$$\int_\gamma f = b_1 2\pi i \operatorname{ind}_\gamma(p).$$

Since residues can help us find integrals, it is useful to look for ways of detecting residues. In the case of poles of order 1, this is easy.

Proposition 29.5 Suppose f is analytic on $\Omega \setminus \{p\}$. A singularity at p is a pole of order 1 if and only if

$$(z-p)f(z) \to b \neq 0 \quad \text{as } z \to p.$$

In that case, b is the residue of f at p.

Proof. If p is a pole of order 1, we have

$$f(z) = h(z) + \frac{b}{z-p}$$

with residue $b \neq 0$, and h analytic on $\Omega \cup \{p\}$. Clearly then

$$(z-p)f(z) = (z-p)h(z) + b \to 0 \cdot h(p) + b = b \neq 0.$$

If $(z-p)f(z) \to b \neq 0$ as $z \to p$, then $(z-p)f(z)$, being bounded on some $D_p^*(\varepsilon)$, has a removable singularity at p. Thus p is a pole of order 1, unless f itself had a removable singularity at p. If the latter happened, f would be bounded on some $D_p^*(\varepsilon)$. Say $|f| \leq B$ there. Then

$$|(z-p)f(z)| \leq B|z-p| \to 0 \text{ as } z \to p,$$

contrary to $|(z-p)f(z)| \to |b| > 0$. Thus f has a pole of order 1 at p. □

Example 29.6 Let us prove that

$$f(z) = \frac{z}{1-\cos z}$$

has a pole of order 1 at 0 and find the residue. We know that

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \quad \text{for } z \in \mathbb{C}.$$

Hence for $z \neq 0$,

$$zf(z) = \frac{z^2}{1-\cos z} = \frac{z^2}{\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \dots} = \frac{1}{\frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \dots} \to 2 \text{ as } z \to 0.$$

Therefore the singularity at 0 is a pole of order 1, with residue 2.

Here is a little specialization of Proposition 29.5. Suppose f, g are analytic on $\Omega \cup \{p\}$, and $g(p) = 0$ while $f(p) \neq 0$ and $g'(p) \neq 0$. Then f/g has a pole of order 1 at p with residue $f(p)/g'(p)$.

Indeed, for $z \neq p$, we have

$$(z-p)\frac{f(z)}{g(z)} = \frac{f(z)}{\frac{g(z)-g(p)}{z-p}} \to \frac{f(p)}{g'(p)} \neq 0$$

as $z \to p$. Thus $f(p)/g'(p)$ is our residue.

Example 29.7 We know that $1/\sin z$ has a singularity at each integer multiple $k\pi$ of π. Clearly $\sin(k\pi) = 0$ while its derivative $\cos(k\pi) \neq 0$. Thus $1/\sin z$ has a pole of order 1 at $k\pi$, and the residue is

$$\frac{1}{\cos(k\pi)} = \begin{cases} 1, & \text{when } k \text{ is even,} \\ -1, & \text{when } k \text{ is odd.} \end{cases}$$

Proposition 29.8 If $(z-p)^2 f(z) \to a \neq 0$ as $z \to p$, then f has a pole at p of order 2, and moreover

$$\operatorname{Res}(f, p) = \lim_{z \to p} \left((z-p)^2 f(z) \right)'.$$

Proof. Since $(z-p)^2 f(z) \to a \neq 0$, the function f has a pole of order 2 at p. Therefore there are constants $a, b \in \mathbb{C}$ and a function h analytic on a neighborhood of p such that, for $z \neq p$ near p,

$$f(z) = h(z) + \frac{b}{z-p} + \frac{a}{(z-p)^2}.$$

By definition, the residue is $\operatorname{Res}(f, p) = b$.

Now multiply by $(z-p)^2$:

$$(z-p)^2 f(z) = (z-p)^2 h(z) + b(z-p) + a.$$

Differentiate to get

$$\left((z-p)^2 f(z)\right)' = 2(z-p)h(z) + (z-p)^2 h'(z) + b.$$

As $z \to p$, the first two terms tend to 0 because h and h' are bounded near p. Hence

$$\lim_{z\to p} \left((z-p)^2 f(z)\right)' = b = \operatorname{Res}(f,p).$$

□

Example 29.9 Let us find

$$\int_\gamma \frac{\mathrm{d}z}{(z^2+4)^2},$$

where $\gamma(t) = i + 2e^{it}$ and $t \in [0, 2\pi]$. Here $\Omega = \mathbb{C} \setminus \{\pm 2i\}$. Note

$$(z-2i)^2 \frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2} \to \frac{1}{(4i)^2} \neq 0,$$

so $f(z) = (z^2+4)^{-2}$ has a pole of order 2 at $p = 2i$. So

$$\int_\gamma f = 2\pi i \cdot \operatorname{ind}_\gamma(2i) \cdot \operatorname{Res}(f, 2i).$$

By inspection, $\operatorname{ind}_\gamma(2i) = 1$, and

$$\operatorname{Res}(f,2i) = \lim_{z\to 2i} \left((z-2i)^2 f(z)\right)' = \lim_{z\to 2i} \left(\frac{1}{(z+2i)^2}\right)' = \lim_{z\to 2i} \frac{-2}{(z+2i)^3} = \frac{-2}{(4i)^3} = \frac{1}{32i}.$$

So

$$\int_\gamma f = 2\pi i \cdot 1 \cdot \frac{1}{32i} = \frac{\pi}{16}.$$

10. Cauchy's Theorem (Adult Version)

Chains and Homology

The general context for a proper discussion of Cauchy's theorem is that of a chain. A **chain** γ is a finite list of closed paths $\gamma_1, \gamma_2, \ldots, \gamma_n$, denoted by

$$\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n.$$

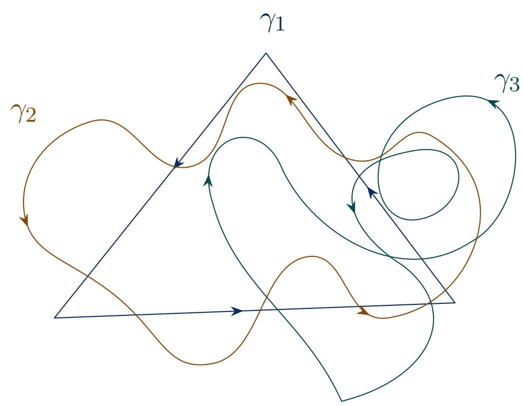

Figure 74

The construction is shown in Figure 74.

The image of γ is

$$\gamma^* = \bigcup_{j=1}^{n} \gamma_j^*.$$

For any continuous f on γ^* we define

$$\int_\gamma f = \sum_{j=1}^{n} \int_{\gamma_j} f.$$

The usual linearity property holds, as well as the estimate

$$\left| \int_\gamma f \right| \leq \|f\|_{\gamma^*} \sum_{j=1}^{n} \operatorname{length}(\gamma_j) = \|f\|_{\gamma^*} \operatorname{length}(\gamma),$$

where the length of the chain is defined by $\operatorname{length}(\gamma) = \sum_{j=1}^{n} \operatorname{length}(\gamma_j)$. This counts repeated traversals, as an integral estimate must.

Definition 29.10 If $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ is a chain and $p \notin \gamma^*$, the **index** (or **winding number**) of γ about p is

$$\operatorname{ind}_\gamma(p) = \sum_{j=1}^{n} \operatorname{ind}_{\gamma_j}(p) = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{\mathrm{d}z}{z-p} \in \mathbb{Z}.$$

The index is constant on the connected components of $\mathbb{C} \setminus \gamma^*$.

Example 29.11 Let

$$\gamma_1(t) = e^{2it} \qquad \text{and} \qquad \gamma_2(t) = 1 + e^{-it}, \quad t \in [0, 2\pi]$$

and let $\gamma = \gamma_1 + \gamma_2$. This is a chain made up of two circles.

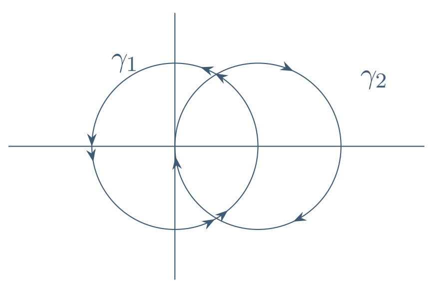

FIGURE 75

Figure 75 records the relevant geometry.

Let us compute the index on the four components of $\mathbb{C} \setminus \gamma^*$.

If p is outside both circles, then

$$\operatorname{ind}_\gamma(p) = \operatorname{ind}_{\gamma_1}(p) + \operatorname{ind}_{\gamma_2}(p) = 0 + 0 = 0.$$

If p is inside circle γ_1 and outside γ_2, then

$$\operatorname{ind}_\gamma(p) = \operatorname{ind}_{\gamma_1}(p) + \operatorname{ind}_{\gamma_2}(p) = 2 + 0 = 2.$$

If p is inside γ_2 and outside γ_1, then

$$\text{ind}_\gamma(p) = 0 - 1 = -1.$$

If p is inside both γ_1 and γ_2, then

$$\text{ind}_\gamma(p) = 2 - 1 = 1.$$

Definition 29.12 Two chains γ and β are said to be **homologous** in a region Ω when both γ^* and β^* are inside Ω, and $\text{ind}_\gamma(p) = \text{ind}_\beta(p)$ for all $p \notin \Omega$.

A single chain γ is **homologous to 0** in Ω when $\gamma^* \subset \Omega$, and $\text{ind}_\gamma(p) = 0$ for all $p \notin \Omega$.

We have to develop a good feeling for what homologous means. Let us discuss a few examples.

Proposition 29.13 If Ω is convex from a point, then every chain γ in Ω is homologous to 0.

Proof. It is enough to show this when γ is a closed path in Ω. For every $p \notin \Omega$, the function

$$f(z) = \frac{1}{z-p}$$

is analytic on Ω. Apply Theorem 18.2 to f and get

$$\text{ind}_\gamma(p) = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z-p} = 0.$$

□

Definition 29.14 A region Ω is called **simply connected** when all closed paths (and thus all chains) in Ω are homologous to 0.

Simply connected regions have no holes around which paths can wind.

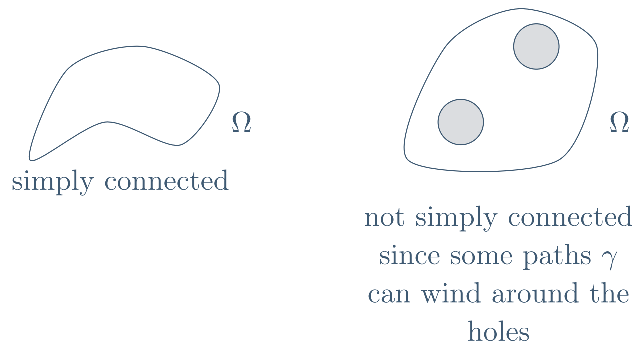

Figure 76

The configuration appears in Figure 76.

So all regions convex from a point are simply connected, but some simply connected regions are not convex from a point.

Example 29.15 Let $0 \le r < R \le \infty$ and take the annulus

$$\Omega = \{z : r < |z - p| < R\}$$

centered at some $p \in \mathbb{C}$. Then pick radii r_1, r_2 such that $r < r_1 < r_2 < R$, and define

$$\gamma_1(t) = p + r_1 e^{it} \qquad \text{and} \qquad \gamma_2(t) = p + r_2 e^{it}, \quad t \in [0, 2\pi].$$

Think of γ_1 and γ_2 as chains in Ω.

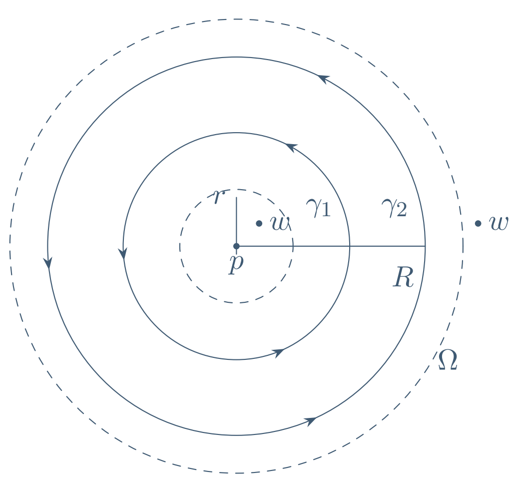

Figure 77

Figure 77 illustrates the setup.

The chain γ_1 is homologous to γ_2 in Ω. To see this, let $w \notin \Omega$. Thus $w \in \overline{D_p(r)}$ or $w \in \mathbb{C} \setminus D_p(R)$. If $w \in \overline{D_p(r)}$, then

$$\operatorname{ind}_{\gamma_1}(w) = 1 = \operatorname{ind}_{\gamma_2}(w),$$

as in Example 29.11. If $w \in \mathbb{C} \setminus D_p(R)$, then w is in the unbounded component of both $\mathbb{C} \setminus \gamma_1^*$ and $\mathbb{C} \setminus \gamma_2^*$, so

$$\operatorname{ind}_{\gamma_1}(w) = \operatorname{ind}_{\gamma_2}(w) = 0.$$

Since $\operatorname{ind}_{\gamma_1}(w) = \operatorname{ind}_{\gamma_2}(w)$ for all $w \notin \Omega$, our chains are homologous.

Note that for some $w \in \Omega$ we have $\operatorname{ind}_{\gamma_1}(w) \neq \operatorname{ind}_{\gamma_2}(w)$, but that has nothing to do with being homologous in Ω.

If

$$\tilde{\gamma}_2(t) = p + r_2 e^{-it}, \quad t \in [0, 2\pi],$$

we get the opposite path to γ_2. It does not take much to see that the chain $\gamma_1 + \tilde{\gamma}_2$ is homologous to 0.

Example 29.16 Let Ω be any region and let $p_1, \ldots, p_n$ be points not in Ω but such that some $D^*_{p_j}(r_j) \subseteq \Omega$. So the p_j are punctures in the region Ω. Now suppose γ is a chain in Ω that is homologous to 0 in $\Omega \cup \{p_1, \ldots, p_n\}$. So γ can do some winding around the p_j, but no winding around any w outside $\Omega \cup \{p_1, \ldots, p_n\}$.

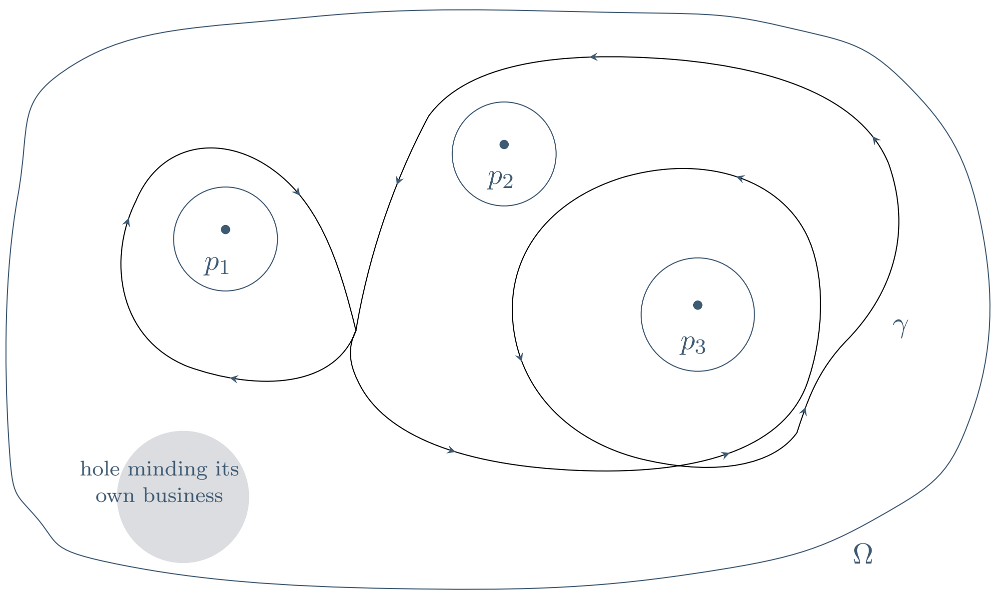

FIGURE 78

In Figure 78, $\operatorname{ind}_\gamma(p_1) = -1$, $\operatorname{ind}_\gamma(p_2) = 1$, and $\operatorname{ind}_\gamma(p_3) = 2$.

We want to replace our chain γ by a homologous chain β in $\Omega \cup \{p_1, \ldots, p_n\}$, but with a much more manageable form. For that, pick radii $r_j > 0$ so small that $\overline{D_{p_j}(r_j)} \subseteq \Omega \cup \{p_1, \ldots, p_n\}$ and the closed disks $\overline{D_{p_j}(r_j)}$ do not intersect each other.

Let

$$n_j = \operatorname{ind}_\gamma(p_j) \in \mathbb{Z}.$$

Define

$$\alpha_j : [0, 2\pi] \to \Omega, \qquad t \mapsto p_j + r_j e^{i n_j t}.$$

We calculate that

$$\operatorname{ind}_{\alpha_j}(p_j) = \frac{1}{2\pi i} \int_{\alpha_j} \frac{\mathrm{d}z}{z - p_j} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{i n_j r_j e^{i n_j t}}{p_j + r_j e^{i n_j t} - p_j} \, \mathrm{d}t = n_j.$$

Also, if $w = p_k \neq p_j$, then $\operatorname{ind}_{\alpha_j}(w) = 0$ since w is in the unbounded component of $\mathbb{C} \setminus \alpha_j^*$.

If $w \notin \Omega \cup \{p_1, \ldots, p_n\}$, then $\operatorname{ind}_{\alpha_j}(w) = 0$ for the same reason.

Let
$$\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

Now check that β is homologous to γ in Ω. If $w = p_j$ for some j, we have

$$\operatorname{ind}_\beta(w) = \sum_k \operatorname{ind}_{\alpha_k}(p_j) = n_j = \operatorname{ind}_\gamma(w).$$

If $w \in \mathbb{C} \setminus (\Omega \cup \{p_1, \ldots, p_n\})$, then

$$\operatorname{ind}_\beta(w) = \sum_k \operatorname{ind}_{\alpha_k}(w) = 0 + \cdots + 0 = 0 = \operatorname{ind}_\gamma(w).$$

Thus β is homologous to γ in Ω. But the chain β is made up of manageable little circles. The opposite path of each α_j is

$$\tilde{\alpha}_j : [0, 2\pi] \to \Omega, \qquad t \mapsto p_j + r_j e^{-in_j t}.$$

Clearly,
$$\operatorname{ind}_{\tilde{\alpha}_j}(p_j) = -n_j = -\operatorname{ind}_\gamma(p_j).$$

Now it does not take much to see that the chain

$$\gamma + \tilde{\alpha}_1 + \tilde{\alpha}_2 + \cdots + \tilde{\alpha}_n$$

is homologous to 0 in Ω.

Cauchy's Theorem (Adult Version)

Theorem 29.17 (Cauchy's theorem, adult version) Let γ be any chain in any region Ω. The chain γ is homologous to 0 in Ω if and only if

$$\int_\gamma f = 0 \quad \text{for every analytic } f \text{ on } \Omega.$$

In this case, the Cauchy integral formula from Theorem 21.2 generalizes to:

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = f(z)\,\mathrm{ind}_{\gamma}(z) \quad \text{for every } z \in \Omega \setminus \gamma^{*}.$$

The proof is *long* and takes some care, so we will skip it.

What does Theorem 29.17 tell us about chains γ, β that are homologous relative to Ω? Say

$$\beta = \beta_1 + \cdots + \beta_n,$$

where each β_j is a closed path in Ω.

Each β_j can be written as

$$\beta_j = \alpha_{j_1} + \cdots + \alpha_{j_m},$$

where $\alpha_{j_k} : [0,1] \to \Omega$ are smooth curves joined end to end.

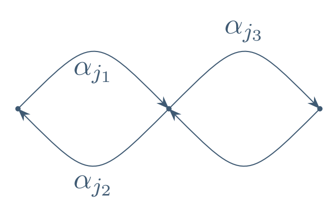

FIGURE 79

Figure 79 records the relevant geometry.

Each α_{j_k} has its opposite curve

$$\tilde{\alpha}_{j_k} : [0,1] \to \Omega, \qquad t \mapsto \alpha_{j_k}(1-t).$$

Together they give the opposite closed path

$$\tilde{\beta}_j = \tilde{\alpha}_{j_m} + \cdots + \tilde{\alpha}_{j_1}.$$

Then we get the opposite chain

$$\tilde{\beta} = \tilde{\beta}_n + \cdots + \tilde{\beta}_1.$$

It does not take much to see that for any continuous function f on $\beta^* = \tilde{\beta}^*$ we have

$$\int_{\tilde{\beta}} f = -\int_{\beta} f.$$

When

$$f(z) = \frac{1}{z-w}$$

and $w \notin \Omega$, deduce that

$$\operatorname{ind}_{\tilde{\beta}}(w) = \frac{1}{2\pi i}\int_{\tilde{\beta}} \frac{\mathrm{d}z}{z-w} = -\frac{1}{2\pi i}\int_{\beta} \frac{\mathrm{d}z}{z-w} = -\operatorname{ind}_{\beta}(w).$$

Thus the enlarged chain $\gamma + \tilde{\beta}$ is homologous to 0 in Ω. Indeed, for every $w \notin \Omega$ we have

$$\operatorname{ind}_{\gamma+\tilde{\beta}}(w) = \operatorname{ind}_{\gamma}(w) + \operatorname{ind}_{\tilde{\beta}}(w) = \operatorname{ind}_{\gamma}(w) - \operatorname{ind}_{\beta}(w) = 0.$$

Then, for every analytic function f on Ω, Theorem 29.17 tells us

$$\int_{\gamma+\tilde{\beta}} f = 0,$$

and therefore

$$0 = \int_{\gamma+\tilde{\beta}} f = \int_{\gamma} f + \int_{\tilde{\beta}} f = \int_{\gamma} f - \int_{\beta} f.$$

Hence

$$\int_{\gamma} f = \int_{\beta} f$$

for every holomorphic f on Ω. Thus:

Proposition 29.18 The integral of any analytic function f on Ω over any chain γ equals the integral of f over any other chain β homologous to γ in Ω.

Example 29.19 Let γ be the diamond-shaped path given by

$$\gamma = [2i, -2] + [-2, -2i] + [-2i, 2] + [2, 2i].$$

Let us find

$$\int_\gamma \frac{e^z}{z^2+1}\,\mathrm{d}z.$$

The function

$$\frac{e^z}{z^2+1}$$

is analytic on $\mathbb{C}\setminus\{\pm i\}$ with poles at $\pm i$. Let us replace γ by a chain β homologous to γ in $\mathbb{C}\setminus\{\pm i\}$, but easier to work with, and then use the fact that

$$\int_\gamma \frac{e^z}{z^2+1}\,\mathrm{d}z = \int_\beta \frac{e^z}{z^2+1}\,\mathrm{d}z.$$

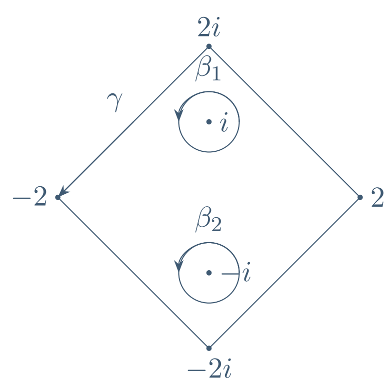

FIGURE 80

The configuration appears in Figure 80.

Let r_1, r_2 be two small radii such that the closed disks $\overline{D_i(r_1)}$ and $\overline{D_{-i}(r_2)}$ do not intersect and contain no pole other than their respective centers (for example, $r_1 = r_2 = 1/2$ are fine). Let

$$\beta_1(t) = i + r_1 e^{it} \qquad \text{and} \qquad \beta_2(t) = -i + r_2 e^{it}, \quad 0 \le t \le 2\pi.$$

By inspection, $\beta_1 + \beta_2$ is homologous to γ in $\mathbb{C}\setminus\{\pm i\}$.

Hence

$$\int_\gamma \frac{e^z}{z^2+1}\,\mathrm{d}z = \int_{\beta_1+\beta_2} \frac{e^z}{z^2+1}\,\mathrm{d}z$$

$$\begin{aligned}
&= \int_{\beta_1} \frac{e^z}{z^2+1}\,\mathrm{d}z + \int_{\beta_2} \frac{e^z}{z^2+1}\,\mathrm{d}z \\
&= \int_{\beta_1} \frac{e^z/(z+i)}{z-i}\,\mathrm{d}z + \int_{\beta_2} \frac{e^z/(z-i)}{z+i}\,\mathrm{d}z \\
&= 2\pi i \frac{e^i}{2i} + 2\pi i \frac{e^{-i}}{-2i} \quad \text{by Proposition 19.2} \\
&= 2\pi i \left(\frac{e^i - e^{-i}}{2i} \right) \\
&= 2\pi i \sin(1).
\end{aligned}$$

The method of Example 29.19 leads to a general result known as the residue theorem, which we will get to after a long and painful detour.

11. Homotopy Implies Homology

Warning This material requires patience.

Homotopy

Proposition 29.18 told us that if γ, β are closed paths in a region Ω and

$$\mathrm{ind}_\gamma(w) = \mathrm{ind}_\beta(w) \quad \text{for all } w \notin \Omega,$$

then for any analytic function f on Ω,

$$\int_\gamma f = \int_\beta f.$$

This also holds for chains. The ability to replace one chain γ by another chain β greatly facilitates the calculation of integrals. Useful as this is, we are still left with the nagging problem of verifying the sufficient condition

$$\mathrm{ind}_\gamma(w) = \mathrm{ind}_\beta(w).$$

Even for a single smooth closed curve γ,

$$\mathrm{ind}_\gamma(w) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z-w} = \frac{1}{2\pi i} \int_0^1 \frac{\gamma'(t)}{\gamma(t)-w}\,\mathrm{d}t$$

may not be easy to calculate. There is, however, a geometric condition on γ and β that ensures

$$\mathrm{ind}_\gamma(w) = \mathrm{ind}_\beta(w) \quad \text{when } w \notin \Omega.$$

This is the concept of **homotopy**.

For this discussion, we think of a path γ as a single piecewise smooth curve

$$\gamma : [0,1] \to \mathbb{C}.$$

In other words, γ is continuous, and there is a subdivision

$$0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$$

of $[0,1]$ such that

$$\gamma : [x_{j-1}, x_j] \to \mathbb{C}$$

is differentiable with continuous derivative.

As far as integration is concerned, this concept of path is equivalent to the notion of path as a finite number of smooth curves

$$\alpha_j : [0,1] \to \mathbb{C}, \qquad \alpha_j(0) = \alpha_{j-1}(1).$$

It is just a matter of reparametrization.

Our path $\gamma : [0,1] \to \mathbb{C}$ is closed when $\gamma(0) = \gamma(1)$.

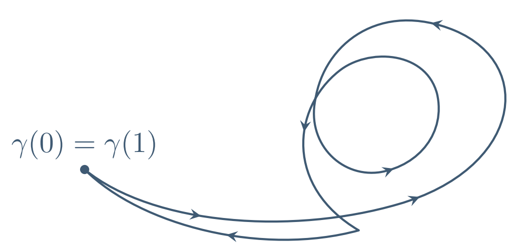

Figure 81

Figure 81 illustrates the setup.

For $w \notin \gamma^*$, we have the integer

$$\operatorname{ind}_\gamma(w) = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z-w} = \frac{1}{2\pi i}\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j} \frac{\gamma'(t)}{\gamma(t)-w}\,\mathrm{d}t.$$

Proposition 29.20 If $\alpha, \beta : [0,1] \to \mathbb{C}$ are closed paths and w is a complex number such that

$$|\alpha(t)-\beta(t)| < |w-\beta(t)| \quad \text{for all } t \in [0,1],$$

then $\operatorname{ind}_\alpha(w) = \operatorname{ind}_\beta(w)$.

Proof. One way to picture the situation is shown here.

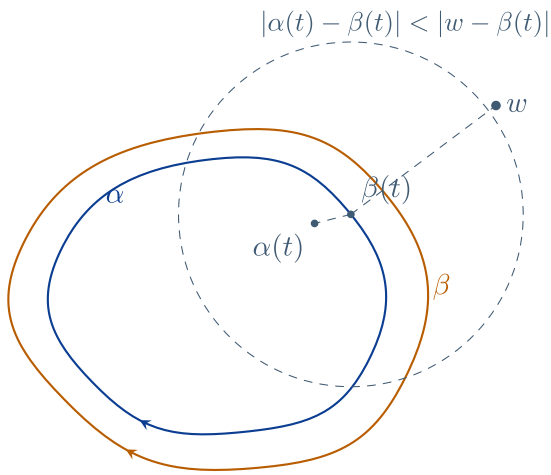

FIGURE 82

The construction is shown in Figure 82.

For simplicity of notation, we write

$$\operatorname{ind}_\alpha(w) = \frac{1}{2\pi i}\int_0^1 \frac{\alpha'(t)}{\alpha(t)-w}\,\mathrm{d}t,$$

keeping in mind that $\alpha'(t)$ has two possible interpretations at subdivision points where left and right derivatives may disagree. Integrals are not affected by values at finitely many points. Since $|w-\beta(t)| > 0$ for all $t \in [0,1]$, we see that $w \notin \beta^*$. Also, if $w \in \alpha^*$, then $w = \alpha(s)$ for some

$s \in [0,1]$, which would lead to

$$|w - \beta(s)| = |\alpha(s) - \beta(s)| < |w - \beta(s)|,$$

a contradiction. So $w \notin \alpha^*$ and $w \notin \beta^*$. Next define the closed path

$$\gamma : [0,1] \to \mathbb{C}, \qquad \gamma(t) = \frac{\alpha(t) - w}{\beta(t) - w}.$$

(One can check that γ is piecewise differentiable and closed.) Except at finitely many points of $[0,1]$, we have

$$\frac{\gamma'(t)}{\gamma(t)} = \frac{\beta(t) - w}{\alpha(t) - w}\left(\frac{(\beta(t) - w)\alpha'(t) - (\alpha(t) - w)\beta'(t)}{(\beta(t) - w)^2}\right) = \frac{\alpha'(t)}{\alpha(t) - w} - \frac{\beta'(t)}{\beta(t) - w}. \tag{$\star$}$$

From $|\alpha(t) - \beta(t)| < |\beta(t) - w|$ we get

$$\left|\frac{\alpha(t) - \beta(t)}{\beta(t) - w}\right| < 1, \qquad \left|\frac{\alpha(t) - w}{\beta(t) - w} - 1\right| < 1, \qquad \text{and} \qquad |\gamma(t) - 1| < 1 \quad \text{for all } t \in [0,1].$$

Thus γ is a closed path in $D_1(1)$. Since 0 is in the unbounded component of $\mathbb{C} \setminus \gamma^*$, we have

$$\operatorname{ind}_\gamma(0) = 0.$$

But using ($\star$), this tells us that

$$\begin{aligned} 0 = \operatorname{ind}_\gamma(0) = \frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t) - 0}\,\mathrm{d}t &= \frac{1}{2\pi i}\int_0^1 \frac{\alpha'(t)}{\alpha(t) - w}\,\mathrm{d}t - \frac{1}{2\pi i}\int_0^1 \frac{\beta'(t)}{\beta(t) - w}\,\mathrm{d}t \\ &= \operatorname{ind}_\alpha(w) - \operatorname{ind}_\beta(w). \end{aligned}$$

Hence $\operatorname{ind}_\alpha(w) = \operatorname{ind}_\beta(w)$. □

Definition 29.21 Two closed paths $\alpha, \beta : [0,1] \to \Omega$ are **homotopic** in Ω when there is a continuous function of two variables

$$H : [0,1] \times [0,1] \to \Omega$$

such that

(1) $H(s,0) = \alpha(s)$ and $H(s,1) = \beta(s)$ for all $s \in [0,1]$, and

(2) $H(0,t) = H(1,t)$ for all $t \in [0,1]$.

The function H is called a **homotopy**.

For every $t \in [0,1]$, we get a curve

$$\gamma_t : [0,1] \to \Omega, \qquad s \mapsto H(s,t).$$

Condition (2) says that $\gamma_t(0) = \gamma_t(1)$, meaning that each γ_t is closed. Condition (1) says that $\gamma_0 = \alpha$ and $\gamma_1 = \beta$. Thus H is a continuous deformation of the closed curve α into the closed curve β through closed curves.

Here is one way to picture this:

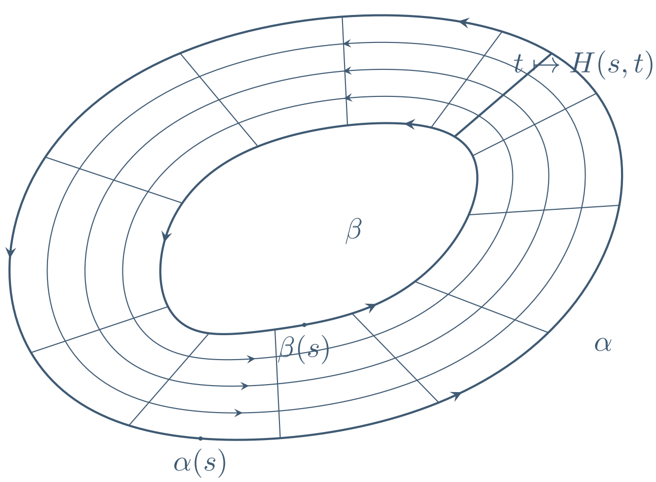

FIGURE 83

Figure 83 records the relevant geometry. The inner curves are the γ_t, and the ribs joining each $\alpha(s)$ to $\beta(s)$ are the functions $t \mapsto H(s,t)$.

Example 29.22 Let Ω be any convex set. Then any two closed paths $\alpha, \beta : [0,1] \to \Omega$ are homotopic in Ω. The homotopy is

$$H(s,t) = (1-t)\alpha(s) + t\beta(s).$$

The convexity of Ω ensures $H(s,t) \in \Omega$. Also $H(s,0) = \alpha(s)$, $H(s,1) = \beta(s)$, and

$$H(0,t) = (1-t)\alpha(0) + t\beta(0) = (1-t)\alpha(1) + t\beta(1) = H(1,t).$$

The continuity of H is immediate.

If Ω is convex from a point p, α is any closed path in Ω, and γ is the constant path $\gamma(s) = p$, then α is homotopic to γ via

$$H(s,t) = (1-t)\alpha(s) + tp.$$

Since homotopy is transitive, any two closed paths in Ω are homotopic.

As a side point, let us verify the transitivity of homotopy. Suppose

$$H : [0,1] \times [0,1] \to \Omega$$

is a homotopy from α to β, and

$$K : [0,1] \times [0,1] \to \Omega$$

is a homotopy from β to γ. Define

$$L : [0,1] \times [0,1] \to \Omega$$

by

$$L(s,t) = \begin{cases} H(s,2t), & 0 \le t \le \dfrac{1}{2}, \\ K(s,2t-1), & \dfrac{1}{2} \le t \le 1. \end{cases}$$

Then

$$L(s,0) = H(s,0) = \alpha(s), \qquad L(s,1) = K(s,1) = \gamma(s), \qquad \text{and} \qquad L(0,t) = L(1,t),$$

where the last equality follows from the corresponding property for H and K on each half of the interval.

The only point to check is the junction $t = 1/2$. But

$$H\left(s, 2 \cdot \frac{1}{2}\right) = H(s,1) = \beta(s) = K(s,0) = K\left(s, 2 \cdot \frac{1}{2} - 1\right),$$

so L is well defined and continuous. Hence L is a homotopy from α to γ.

Example 29.23 In $\Omega = \mathbb{C} \setminus \{0\}$, take the circles $\alpha, \beta : [0,1] \to \Omega$ defined by

$$\alpha(s) = e^{2\pi i s},$$
$$\beta(s) = 2e^{2\pi i s}.$$

Then α is homotopic to β using

$$H(s,t) = (1+t)e^{2\pi i s}, \quad s,t \in [0,1].$$

Indeed, $H(s,0) = e^{2\pi i s} = \alpha(s)$ and $H(s,1) = 2e^{2\pi i s} = \beta(s)$, while $H(s,t) \neq 0$ for all $s,t \in [0,1]$ because $1+t \in [1,2]$.

Homotopy Implies Homology

Now comes a neat, though not easy, connection between paths that are homotopic and paths that are homologous in Ω.

Theorem 29.24 (Homotopy implies homology) If α and β are closed paths in Ω that are homotopic, then

$$\operatorname{ind}_\alpha(w) = \operatorname{ind}_\beta(w) \quad \text{for all } w \notin \Omega.$$

Thus homotopic closed paths are homologous. Consequently, by Proposition 29.18, integrals of analytic functions over closed paths α can be replaced by integrals over paths β that are homotopic to α in Ω.

For instance, Figure 84 shows the rectangular path α being homotopic to the circular path β, all within the punctured plane.

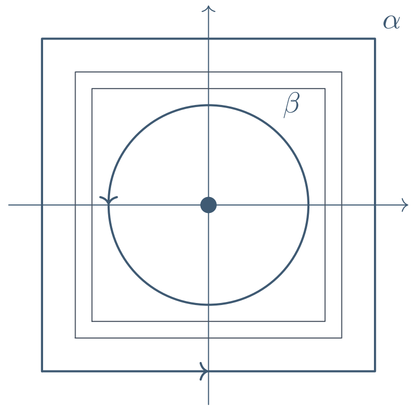

Figure 84

Proof. Let

$$H : [0,1] \times [0,1] \to \Omega$$

be a homotopy satisfying (1) and (2). To first see the main idea, assume temporarily that each closed curve

$$\gamma_t : [0,1] \to \Omega, \qquad s \mapsto H(s,t),$$

is piecewise differentiable. This is certainly not given, and later we will remove this assumption. Take $w \notin \Omega$. Since $w \neq H(s,t)$ for all $(s,t) \in [0,1] \times [0,1]$, and $|H(s,t) - w|$ is continuous and strictly positive on a compact set, there exists $\varepsilon > 0$ such that

$$|H(s,t) - w| > 2\varepsilon \quad \text{for all } (s,t) \in [0,1] \times [0,1].$$

(The factor 2 will be useful shortly.) Because H is uniformly continuous on $[0,1] \times [0,1]$, for this ε there exists a positive integer n such that

$$|H(s,t) - H(s',t')| < \varepsilon$$

whenever

$$|s - s'| \leq \frac{1}{n} \qquad \text{and} \qquad |t - t'| \leq \frac{1}{n}.$$

For $k = 0, \ldots, n$, define closed curves

$$\gamma_k : [0,1] \to \Omega, \qquad s \mapsto H\left(s, \frac{k}{n}\right).$$

In particular, $\gamma_0(s) = H(s,0) = \alpha(s)$ and $\gamma_n(s) = H(s,1) = \beta(s)$. Under our temporary differentiability assumption on the γ_k, we have for each $k = 1, \dots, n$ and all $s \in [0,1]$:

$$\begin{aligned} |w - \gamma_k(s)| &= \left| w - H\left(s, \frac{k}{n}\right) \right| > 2\varepsilon, \\ |\gamma_{k-1}(s) - \gamma_k(s)| &= \left| H\left(s, \frac{k-1}{n}\right) - H\left(s, \frac{k}{n}\right) \right| < \varepsilon, \end{aligned}$$

so

$$|\gamma_{k-1}(s) - \gamma_k(s)| < |w - \gamma_k(s)| \qquad (\dagger)$$

By repeated use of Proposition 29.20,

$$\operatorname{ind}_\alpha(w) = \operatorname{ind}_{\gamma_0}(w) = \operatorname{ind}_{\gamma_1}(w) = \cdots = \operatorname{ind}_{\gamma_n}(w) = \operatorname{ind}_\beta(w).$$

Figure 85 illustrates the idea for $n = 3$.

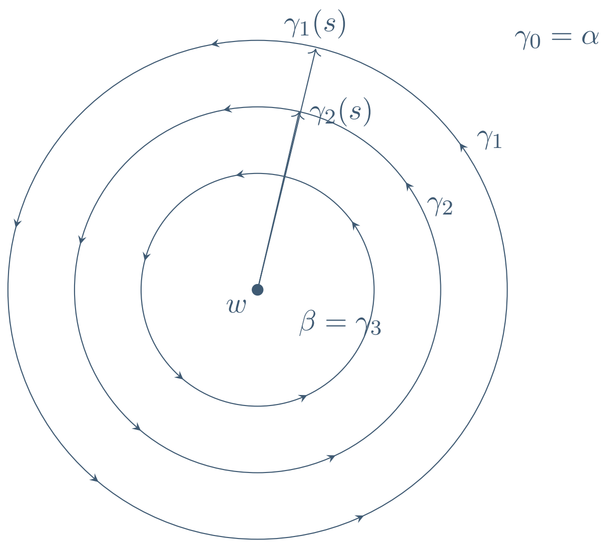

Figure 85

To remove the unjustified assumption that $\gamma_0, \dots, \gamma_n$ are piecewise differentiable, replace them by closed polygonal paths σ_k that are piecewise differentiable and close to the corresponding γ_k. For $k = 0, \dots, n$, let σ_k be the closed polygonal path through

$$\gamma_k(0),\ \gamma_k\left(\frac{1}{n}\right),\ \gamma_k\left(\frac{2}{n}\right),\ \dots,\ \gamma_k\left(\frac{n-1}{n}\right),\ \gamma_k(1) = \gamma_k(0).$$

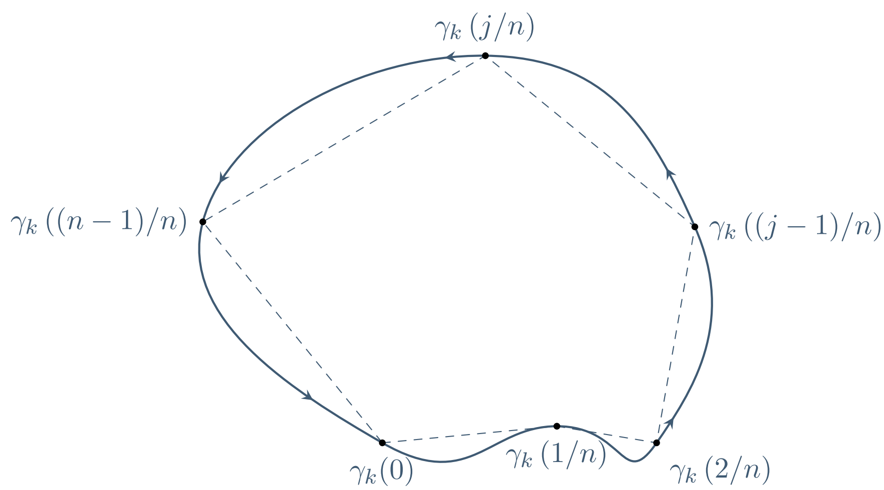

FIGURE 86

The construction is shown in Figure 86.

The straight-line segment from $\gamma_k\,((j-1)/n)$ to $\gamma_k\,(j/n)$ on $[(j-1)/n, j/n]$ is parametrized by

$$s \mapsto (ns+1-j)\gamma_k\left(\frac{j}{n}\right) + (j-ns)\gamma_k\left(\frac{j-1}{n}\right),$$

where $j = 1, \ldots, n$. Thus

$$\sigma_k = \left[\gamma_k(0), \gamma_k\left(\frac{1}{n}\right)\right] + \left[\gamma_k\left(\frac{1}{n}\right), \gamma_k\left(\frac{2}{n}\right)\right] + \cdots + \left[\gamma_k\left(\frac{n-1}{n}\right), \gamma_k(1)\right]$$

(using the notation for straight-line paths). Now it turns out that for all $s \in [0,1]$:

$$|\sigma_0(s) - \alpha(s)| < |w - \alpha(s)|, \tag{$\star$}$$
$$|\sigma_n(s) - \beta(s)| < |w - \beta(s)|, \tag{$\star\star$}$$

and

$$\left.\begin{aligned} |\sigma_0(s) - \sigma_1(s)| &< |w - \sigma_1(s)|, \\ |\sigma_1(s) - \sigma_2(s)| &< |w - \sigma_2(s)|, \\ &\vdots \\ |\sigma_{k-1}(s) - \sigma_k(s)| &< |w - \sigma_k(s)|, \\ &\vdots \\ |\sigma_{n-1}(s) - \sigma_n(s)| &< |w - \sigma_n(s)|, \end{aligned}\right\} \tag{$\star\star\star$}$$

for $k = 1, \ldots, n$. Once these inequalities are checked, we apply Proposition 29.20 $n+2$ times and conclude that

$$\operatorname{ind}_\alpha(w) = \operatorname{ind}_\beta(w).$$

Before checking $(\star)$, $(\star\star)$, and $(\star\star\star)$, note that $(\star\star\star)$ implies $w \notin \sigma_k^*$, so each index $\operatorname{ind}_{\sigma_k}(w)$ is defined. Also, it is not important whether $\sigma_k(s) \in \Omega$. The key fact is that

$$H(s,t) \in \Omega \qquad \text{and} \qquad w \notin \Omega,$$

which gave us

$$|w - H(s,t)| > 2\varepsilon$$

and the uniform continuity estimate

$$|H(s,t) - H(s',t')| < \varepsilon$$

when

$$|s - s'| \leq \frac{1}{n} \qquad \text{and} \qquad |t - t'| \leq \frac{1}{n}.$$

Now we check $(\star)$. Fix $s \in [0,1]$, and choose $j \in \{1, \ldots, n\}$ such that

$$s \in \left[\frac{j-1}{n}, \frac{j}{n}\right].$$

Then

$$0 \leq j - ns \leq 1 \qquad \text{and} \qquad 0 \leq ns + 1 - j \leq 1,$$

and, because $\alpha = \gamma_0$, the definition of σ_0 on this subinterval gives

$$\sigma_0(s) = (ns + 1 - j)\gamma_0\left(\frac{j}{n}\right) + (j - ns)\gamma_0\left(\frac{j-1}{n}\right).$$

Since

$$\left|s - \frac{j}{n}\right| \leq \frac{1}{n} \qquad \text{and} \qquad \left|s - \frac{j-1}{n}\right| \leq \frac{1}{n},$$

the uniform continuity estimate for H with $t = t' = 0$ yields

$$\left|\gamma_0\left(\frac{j}{n}\right) - \gamma_0(s)\right| < \varepsilon \qquad \text{and} \qquad \left|\gamma_0\left(\frac{j-1}{n}\right) - \gamma_0(s)\right| < \varepsilon.$$

Therefore

$$\begin{aligned}|\sigma_0(s) - \alpha(s)| &= \left|(ns+1-j)\left(\gamma_0\left(\frac{j}{n}\right) - \gamma_0(s)\right) + (j-ns)\left(\gamma_0\left(\frac{j-1}{n}\right) - \gamma_0(s)\right)\right| \\ &\le (ns+1-j)\left|\gamma_0\left(\frac{j}{n}\right) - \gamma_0(s)\right| + (j-ns)\left|\gamma_0\left(\frac{j-1}{n}\right) - \gamma_0(s)\right| \\ &< (ns+1-j)\varepsilon + (j-ns)\varepsilon = \varepsilon.\end{aligned}$$

Hence

$$|\sigma_0(s) - \alpha(s)| < \varepsilon < 2\varepsilon < |w - H(s,0)| = |w - \alpha(s)|,$$

which proves ($\star$).

We now prove ($\star\star$) with the same level of detail. Fix $s \in [0,1]$, choose the same j, and note that $\beta = \gamma_n$. On the interval $[(j-1)/n, j/n]$ we have

$$\sigma_n(s) = (ns+1-j)\gamma_n\left(\frac{j}{n}\right) + (j-ns)\gamma_n\left(\frac{j-1}{n}\right).$$

Again,

$$\left|s - \frac{j}{n}\right| \le \frac{1}{n} \qquad \text{and} \qquad \left|s - \frac{j-1}{n}\right| \le \frac{1}{n},$$

so uniform continuity with $t = t' = 1$ gives

$$\left|\gamma_n\left(\frac{j}{n}\right) - \gamma_n(s)\right| < \varepsilon \qquad \text{and} \qquad \left|\gamma_n\left(\frac{j-1}{n}\right) - \gamma_n(s)\right| < \varepsilon.$$

Therefore

$$\begin{aligned}|\sigma_n(s) - \beta(s)| &= \left|(ns+1-j)\left(\gamma_n\left(\frac{j}{n}\right) - \gamma_n(s)\right) + (j-ns)\left(\gamma_n\left(\frac{j-1}{n}\right) - \gamma_n(s)\right)\right| \\ &\le (ns+1-j)\left|\gamma_n\left(\frac{j}{n}\right) - \gamma_n(s)\right| + (j-ns)\left|\gamma_n\left(\frac{j-1}{n}\right) - \gamma_n(s)\right| \\ &< (ns+1-j)\varepsilon + (j-ns)\varepsilon = \varepsilon.\end{aligned}$$

Hence

$$|\sigma_n(s) - \beta(s)| < \varepsilon < 2\varepsilon < |w - H(s,1)| = |w - \beta(s)|,$$

which proves ($\star\star$).

Finally we check ($\star\star\star$). Fix $k \in \{1, \dots, n\}$ and $s \in [0,1]$, and keep the same j as above. We have

$$|\sigma_{k-1}(s) - \sigma_k(s)| \leq (ns+1-j)\left|\gamma_{k-1}\left(\frac{j}{n}\right) - \gamma_k\left(\frac{j}{n}\right)\right| + (j-ns)\left|\gamma_{k-1}\left(\frac{j-1}{n}\right) - \gamma_k\left(\frac{j-1}{n}\right)\right|.$$

Now

$$\left|\gamma_{k-1}\left(\frac{j}{n}\right) - \gamma_k\left(\frac{j}{n}\right)\right| = \left|H\left(\frac{j}{n}, \frac{k-1}{n}\right) - H\left(\frac{j}{n}, \frac{k}{n}\right)\right| < \varepsilon,$$

and likewise

$$\left|\gamma_{k-1}\left(\frac{j-1}{n}\right) - \gamma_k\left(\frac{j-1}{n}\right)\right| < \varepsilon,$$

because the t-coordinates differ by $1/n$ and the s-coordinates are equal. Hence

$$|\sigma_{k-1}(s) - \sigma_k(s)| < (ns+1-j)\varepsilon + (j-ns)\varepsilon = \varepsilon.$$

For the lower bound on $|w - \sigma_k(s)|$, write

$$\begin{aligned} |w - \sigma_k(s)| &= \left|\left(w - \gamma_k\left(\frac{j}{n}\right)\right) + (j-ns)\left(\gamma_k\left(\frac{j}{n}\right) - \gamma_k\left(\frac{j-1}{n}\right)\right)\right| \\ &\geq \left|w - \gamma_k\left(\frac{j}{n}\right)\right| - (j-ns)\left|\gamma_k\left(\frac{j}{n}\right) - \gamma_k\left(\frac{j-1}{n}\right)\right|. \end{aligned}$$

The first term satisfies

$$\left|w - \gamma_k\left(\frac{j}{n}\right)\right| = \left|w - H\left(\frac{j}{n}, \frac{k}{n}\right)\right| > 2\varepsilon,$$

and the second difference satisfies

$$\left|\gamma_k\left(\frac{j}{n}\right) - \gamma_k\left(\frac{j-1}{n}\right)\right| = \left|H\left(\frac{j}{n}, \frac{k}{n}\right) - H\left(\frac{j-1}{n}, \frac{k}{n}\right)\right| < \varepsilon,$$

because the s-coordinates differ by $1/n$ and the t-coordinates are equal. Therefore

$$|w - \sigma_k(s)| > 2\varepsilon - (j-ns)\varepsilon \geq \varepsilon.$$

Combining both bounds gives

$$|\sigma_{k-1}(s) - \sigma_k(s)| < \varepsilon < |w - \sigma_k(s)|,$$

which is exactly ($\star\star\star$). Therefore ($\star$), ($\star\star$), and ($\star\star\star$) all hold, and the theorem follows. $\square$

Another intriguing question is this. If α, β are closed paths in Ω and

$$\operatorname{ind}_\alpha(w) = \operatorname{ind}_\beta(w) \quad \text{for all } w \notin \Omega,$$

does there exist a homotopy H from α to β? In general, no: homology remembers net winding numbers but forgets the order in which loops are traversed. For example, in $\mathbb{C} \setminus \{-1, 1\}$, the commutator obtained by looping around -1, then around 1, and then undoing those two loops in reverse order has winding number 0 about both punctures, so it is homologous to a constant path. It is nevertheless not null-homotopic. Thus homotopy is strictly stronger than homology in a multiply punctured region.

Simple Connectedness for Geometers

Most geometers say that a region Ω is simply connected when every closed path α in Ω is homotopic to a constant path

$$\beta(s) = p \quad \text{for all } s \in [0,1].$$

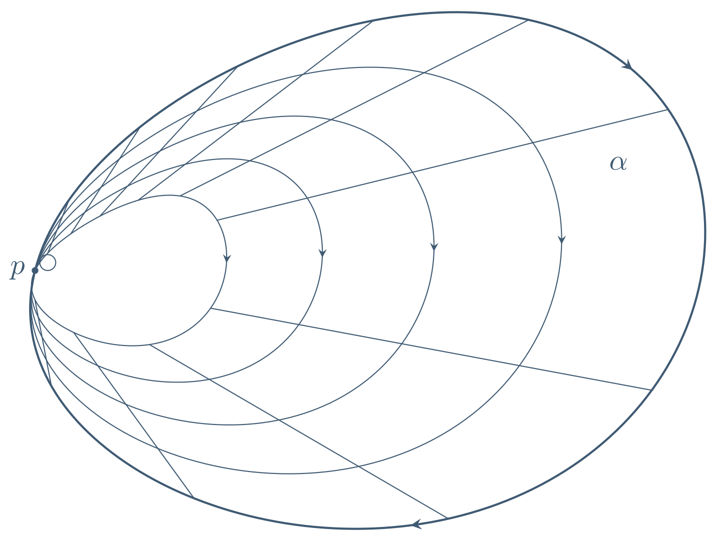

FIGURE 87

Figure 87 records the relevant geometry.

By transitivity of homotopy, all closed paths in a simply connected region are homotopic to each other.

By Theorem 29.24, all closed paths in a simply connected Ω are homologous to each other. That is,

$$\operatorname{ind}_\alpha(w) = \operatorname{ind}_\beta(w)$$

for all closed paths α, β in Ω and all $w \notin \Omega$. So the geometers' concept of simple connectedness implies our concept based on homologous curves. (The converse is also true, but we will skip it.)

When a region has a hole, it cannot be simply connected, since a hole would tangle up a closed path around it.

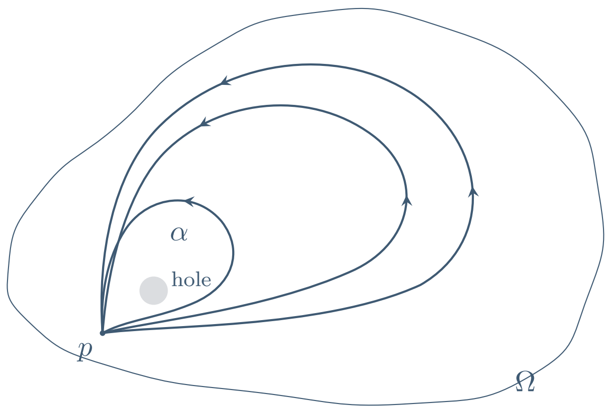

FIGURE 88

The configuration appears in Figure 88.

As α gets pulled toward p by a homotopy H, the homotopy gets blocked by the hole that α wraps around.

It must be confessed that homotopies between closed curves can be challenging to write explicitly.

Example 29.25 Let us try writing a homotopy from α to β as shown in $\Omega = \mathbb{C} \setminus \{i\}$, and deducing the value of $\operatorname{ind}_\alpha(i)$.

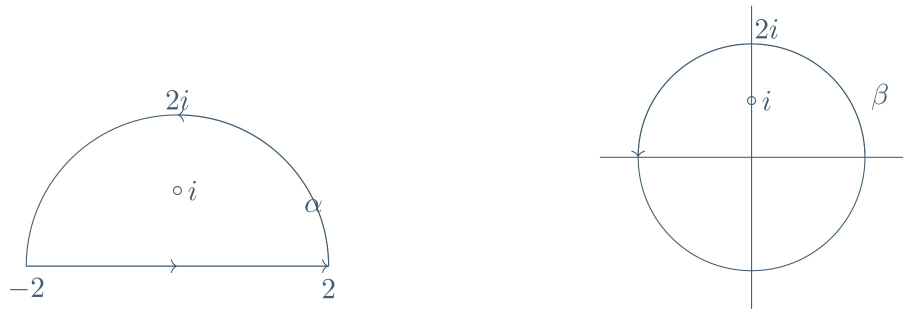

Figure 89

Figure 89 illustrates the setup.

First, parametrize α by

$$\alpha(s) = \begin{cases} -2+8s, & 0 \le s \le \dfrac{1}{2}, \\ 2e^{i\pi(2s-1)}, & \dfrac{1}{2} \le s \le 1. \end{cases}$$

Thus α goes from -2 to 2 along the real axis and then returns from 2 to -2 along the upper semicircle of radius 2.

Now define a homotopy $H : [0,1] \times [0,1] \to \mathbb{C}$ by

$$H(s,t) = \begin{cases} \big(-2+8s\big) - it\sqrt{4-\big(-2+8s\big)^2}, & 0 \le s \le \dfrac{1}{2}, \\ 2e^{i\pi(2s-1)}, & \dfrac{1}{2} \le s \le 1. \end{cases}$$

For $s \le 1/2$, this bends the diameter downward from the segment of the real axis (at $t = 0$) to the lower semicircle (at $t = 1$). For $s \ge 1/2$, the upper semicircle is fixed.

Hence

$$H(s,0) = \alpha(s),$$

and

$$H(s,1) = \beta(s),$$

where β is the full circle of radius 2 centered at 0, traversed once in the positive direction.

We also check that $H(s,t) \ne i$ for every (s,t). On the first piece, the imaginary part is nonpositive, so the value cannot be i. On the second piece, points have modulus 2, while

$|i| = 1$, so again the value cannot be i. Therefore $H([0,1] \times [0,1]) \subseteq \Omega$, so α and β are homotopic in Ω.

By Theorem 29.24,

$$\operatorname{ind}_\alpha(i) = \operatorname{ind}_\beta(i).$$

Since β winds once counterclockwise around i, we have

$$\operatorname{ind}_\beta(i) = 1.$$

Consequently,

$$\operatorname{ind}_\alpha(i) = 1.$$

Lecture 31 — Monday, March 24, 2014

12. The Residue Theorem

The Theorem Itself

If f is analytic on Ω and has a pole $p \notin \Omega$ of order m, this means that

$$f(z) = h(z) + \frac{b_1}{z-p} + \cdots + \frac{b_m}{(z-p)^m},$$

where $m \geq 1$, $b_m \neq 0$, and h is analytic on $\Omega \cup \{p\}$. The function

$$\frac{b_1}{z-p} + \cdots + \frac{b_m}{(z-p)^m}$$

is the principal part of f at p. The residue of f at p is b_1, denoted $b_1 = \operatorname{res}(f, p)$. Now suppose $\overline{D_p(r)}$ is a small closed disk inside $\Omega \cup \{p\}$, and let n be any integer. Define the circle path

$$\beta : [0, 2\pi] \to \Omega, \qquad t \mapsto p + re^{int},$$

which winds n times around p.

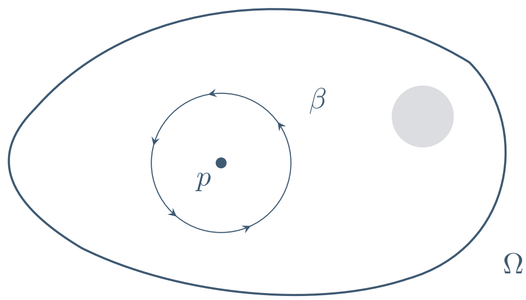

Figure 90

The construction is shown in Figure 90.

Theorem 31.1 (Residue theorem) Suppose f is analytic on Ω, and $p_1, \ldots, p_n$ are poles of f. Let γ be a chain in Ω that is homologous to 0 in $\Omega \cup \{p_1, \ldots, p_n\}$; that is, γ does not wind around any w outside $\Omega \cup \{p_1, \ldots, p_n\}$, but it may wind around the p_j. Then

$$\int_\gamma f = 2\pi i \sum_{j=1}^{n} \operatorname{res}(f, p_j)\ \operatorname{ind}_\gamma(p_j).$$

Proof. Let

$$n_j = \operatorname{ind}_\gamma(p_j).$$

Choose small closed disks $\overline{D_{p_j}(r_j)}$ that are pairwise disjoint and contained in $\Omega \cup \{p_1, \ldots, p_n\}$. Define

$$\beta_j : [0, 2\pi] \to \Omega, \qquad t \mapsto p_j + r_j e^{i n_j t}.$$

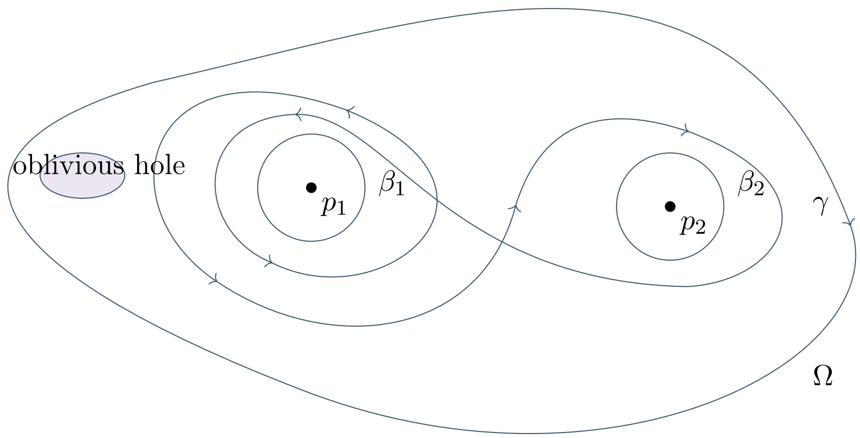

FIGURE 91

Figure 91 records the relevant geometry.

Now fix one pole p with residue $b_1 = \operatorname{res}(f, p)$ and corresponding winding number n. Near p,

$$f(z) = h(z) + \frac{b_1}{z-p} + \frac{b_2}{(z-p)^2} + \cdots + \frac{b_m}{(z-p)^m},$$

where h is analytic across p. Therefore, on a small winding circle β around p,

$$\int_\beta f = \int_\beta h + b_1 \int_\beta \frac{\mathrm{d}z}{z-p} + b_2 \int_\beta \frac{\mathrm{d}z}{(z-p)^2} + \cdots + b_m \int_\beta \frac{\mathrm{d}z}{(z-p)^m}.$$

The h-term is zero by Theorem 18.2, and for $j \geq 2$, each

$$\frac{1}{(z-p)^j}$$

has a primitive on the punctured disk by Cauchy's theorem (Theorem 29.17), so those integrals are zero. Hence

$$\int_\beta f = b_1 \int_\beta \frac{\mathrm{d}z}{z-p} = b_1\, 2\pi i\, n = 2\pi i \operatorname{res}(f, p)\, n.$$

Applying this to each β_j gives

$$\int_{\beta_j} f = 2\pi i \operatorname{res}(f, p_j)\, n_j.$$

As discussed earlier, the chain

$$\beta_1 + \cdots + \beta_n$$

is homologous to γ in Ω. Therefore, by Proposition 29.18,

$$\int_\gamma f = \int_{\beta_1+\cdots+\beta_n} f = \sum_{j=1}^{n} \int_{\beta_j} f = \sum_{j=1}^{n} 2\pi i \operatorname{res}(f, p_j)\, n_j = 2\pi i \sum_{j=1}^{n} \operatorname{res}(f, p_j) \operatorname{ind}_\gamma(p_j).$$

$\square$

Residue Calculus

Let us do some examples with Theorem 31.1.

Example 31.2 Let us find

$$\int_\gamma \frac{\mathrm{d}z}{1+z^4}$$

along the indicated semicircular shaped path:

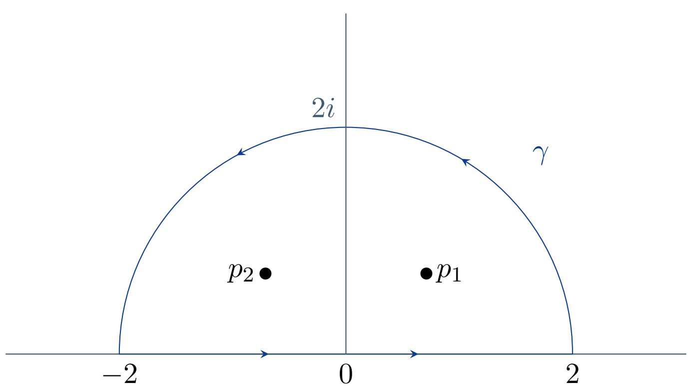

Figure 92

The configuration appears in Figure 92.

The poles of $1/(1+z^4)$ are the fourth roots of -1:

$$p_1 = e^{\pi i/4}, \qquad p_2 = e^{3\pi i/4}, \qquad p_3 = e^{5\pi i/4}, \qquad \text{and} \qquad p_4 = e^{7\pi i/4}.$$

The chain γ is homologous to 0 in $\mathbb{C}$ (as every closed chain is). By Theorem 31.1,

$$\int_\gamma \frac{\mathrm{d}z}{1+z^4} = 2\pi i \sum_{j=1}^{4} \operatorname{res}\left(\frac{1}{1+z^4}, p_j\right) \operatorname{ind}_\gamma(p_j) = 2\pi i \left(\operatorname{res}\left(\frac{1}{1+z^4}, p_1\right) + \operatorname{res}\left(\frac{1}{1+z^4}, p_2\right)\right),$$

since $\operatorname{ind}_\gamma(p_3) = \operatorname{ind}_\gamma(p_4) = 0$ and $\operatorname{ind}_\gamma(p_1) = \operatorname{ind}_\gamma(p_2) = 1$.

These are simple poles. Let $F(z) = 1 + z^4$. Then

$$\operatorname{res}\left(\frac{1}{1+z^4}, p_1\right) = \lim_{z\to p_1} (z - p_1)\frac{1}{F(z)} = \frac{1}{F'(p_1)} = \frac{1}{4p_1^3} = \frac{1}{4e^{3\pi i/4}} = -\frac{1}{4}e^{\pi i/4}.$$

Similarly,

$$\operatorname{res}\left(\frac{1}{1+z^4}, p_2\right) = \frac{1}{4p_2^3} = \frac{1}{4e^{9\pi i/4}} = \frac{1}{4}e^{-\pi i/4}.$$

Hence

$$\int_\gamma \frac{\mathrm{d}z}{1+z^4} = 2\pi i\left(-\frac{1}{4}e^{\pi i/4} + \frac{1}{4}e^{-\pi i/4}\right) = \pi\left(\frac{e^{\pi i/4} - e^{-\pi i/4}}{2i}\right) = \pi \sin\left(\frac{\pi}{4}\right) = \frac{\pi}{\sqrt{2}}.$$

Example 31.3 A useful spinoff from the computation in Example 31.2 is the real improper integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^4} := \lim_{R\to\infty} \int_{-R}^{R} \frac{\mathrm{d}x}{1+x^4}.$$

Replace 2 by R in the same semicircular path. The residue computation in the upper half-plane is unchanged.

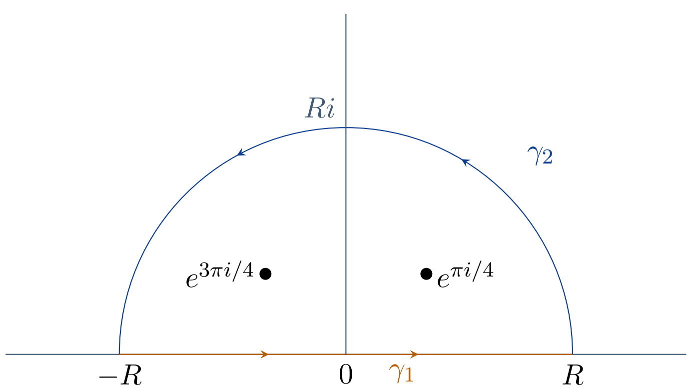

FIGURE 93

Figure 93 illustrates the setup.

For $R > 1$, let the semicircular path be parametrized by

$$\begin{aligned}\gamma_1 : [-R, R] &\to \mathbb{C} \setminus \{e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4}\},\\ x &\mapsto x,\end{aligned}$$

and

$$\begin{aligned}\gamma_2 : [0, \pi] &\to \mathbb{C} \setminus \{e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4}\},\\ t &\mapsto Re^{it}.\end{aligned}$$

Then

$$\frac{\pi}{\sqrt{2}} = \int_{\gamma_1} \frac{1}{1+z^4}\,\mathrm{d}z + \int_{\gamma_2} \frac{1}{1+z^4}\,\mathrm{d}z = \int_{-R}^{R} \frac{\mathrm{d}x}{1+x^4} + \int_{\gamma_2} \frac{1}{1+z^4}\,\mathrm{d}z.$$

Also,

$$\left|\int_{\gamma_2} \frac{1}{1+z^4}\,\mathrm{d}z\right| \leq \left\|\frac{1}{1+z^4}\right\|_{\gamma_2^*} \mathrm{length}(\gamma_2^*) \leq \frac{1}{R^4-1}\,\pi R,$$

because $|1+z^4| \geq |z|^4 - 1 = R^4 - 1$ on γ_2^*. Since

$$\frac{\pi R}{R^4 - 1} \to 0 \quad (R \to \infty),$$

we conclude that

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^4} = \frac{\pi}{\sqrt{2}}.$$

Example 31.4 Let us find

$$\int_{\gamma} \frac{1+z}{1-\cos z}\,\mathrm{d}z$$

around the indicated diamond path γ.

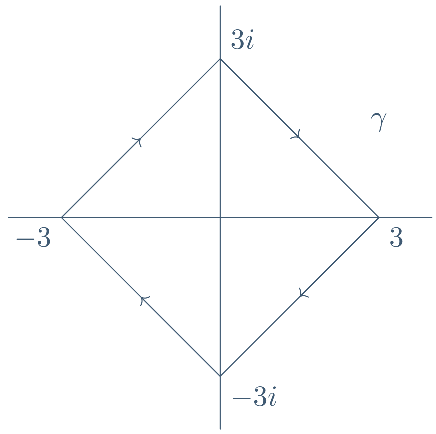

FIGURE 94

The construction is shown in Figure 94.

To use Theorem 31.1, we need the singularities of

$$f(z) = \frac{1+z}{1-\cos z}$$

that γ winds around. Singularities occur when $\cos z = 1$. Solving:

$$\begin{aligned} e^{iz} + e^{-iz} &= 2, \\ e^{iz} - 2 + e^{-iz} &= 0, \\ (e^{iz})^2 - 2e^{iz} + 1 &= 0, \\ (e^{iz} - 1)^2 &= 0, \\ e^{iz} &= 1, \\ z &= 2\pi k, \quad k \in \mathbb{Z}. \end{aligned}$$

The only singularity that γ winds around is 0, and $\mathrm{ind}_\gamma(0) = 1$. Let us determine the residue at 0. Using the power series for $\cos z$, for $z \neq 2\pi k$ we have

$$f(z) = \frac{1+z}{1-\cos z} = \frac{1}{z^2} \frac{1+z}{\frac{1}{2} - \frac{z^2}{4!} + \frac{z^4}{6!} - \dots}.$$

Thus $z^2 f(z)$ has a removable singularity at 0, while $zf(z)$ still has a singularity there, so 0 is a pole of order 2.

Write

$$z^2 f(z) = \frac{z^2(1+z)}{1-\cos z} = a_0 + a_1 z + a_2 z^2 + \dots$$

near 0. Then

$$f(z) = \frac{a_0}{z^2} + \frac{a_1}{z} + a_2 + a_3 z + \dots,$$

so

$$\operatorname{res}(f, 0) = a_1.$$

To find a_1, use

$$z^2 + z^3 = (1-\cos z)(a_0 + a_1 z + a_2 z^2 + \dots) = \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \dots\right)(a_0 + a_1 z + a_2 z^2 + \dots).$$

Hence

$$z^2 + z^3 = \frac{a_0}{2} z^2 + \frac{a_1}{2} z^3 + \left(\frac{a_2}{2} - \frac{a_0}{24}\right) z^4 + \dots$$

Comparing coefficients gives

$$\frac{a_0}{2} = 1 \qquad \text{and} \qquad \frac{a_1}{2} = 1 \implies a_1 = 2.$$

So

$$\operatorname{res}\left(\frac{1+z}{1-\cos z}, 0\right) = 2,$$

and therefore

$$\int_\gamma \frac{1+z}{1-\cos z}\,\mathrm{d}z = 2\pi i \operatorname{res}\left(\frac{1+z}{1-\cos z}, 0\right) = 4\pi i.$$

Example 31.5 For $-1 < a < 1$ and $a \neq 0$, let us find

$$\int_0^{2\pi} \frac{\mathrm{d}t}{1 + a\cos t}.$$

We compute

$$\begin{aligned}\int_0^{2\pi} \frac{\mathrm{d}t}{1 + a\left(\dfrac{e^{it}+e^{-it}}{2}\right)} &= \int_0^{2\pi} \frac{2\,\mathrm{d}t}{2 + ae^{it} + ae^{-it}} \\ &= \int_0^{2\pi} \frac{2e^{it}\,\mathrm{d}t}{a(e^{it})^2 + 2e^{it} + a} \\ &= -2i \int_0^{2\pi} \frac{ie^{it}\,\mathrm{d}t}{a(e^{it})^2 + 2e^{it} + a} \\ &= -2i \int_\gamma \frac{\mathrm{d}z}{az^2 + 2z + a},\end{aligned}$$

where $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$.

So we need the poles of $1/(az^2 + 2z + a)$, the residues at those poles, and the indices of γ around those poles.

The roots of $az^2 + 2z + a$ are

$$\frac{-2 \pm \sqrt{4 - 4a^2}}{2a} = \frac{-1 \pm \sqrt{1 - a^2}}{a}.$$

Let

$$p_1 = \frac{-1 - \sqrt{1-a^2}}{a} \qquad \text{and} \qquad p_2 = \frac{-1 + \sqrt{1-a^2}}{a}.$$

By inspection,

$$|p_1| = \left| \frac{-1 - \sqrt{1-a^2}}{a} \right| > \frac{1}{|a|} > 1.$$

For p_2 we expect $|p_2| < 1$. Indeed,

$$\begin{aligned}\left| \frac{-1 + \sqrt{1-a^2}}{a} \right| < 1 &\iff 1 - \sqrt{1-a^2} < |a| \\ &\iff 1 < |a| + \sqrt{1-a^2} \\ &\iff 1 < a^2 + 2|a|\sqrt{1-a^2} + (1 - a^2) \\ &\iff 0 < 2|a|\sqrt{1-a^2},\end{aligned}$$

which is true for $-1 < a < 1$ and $a \neq 0$.

Thus $|p_2| < 1$, so $\operatorname{ind}_\gamma(p_2) = 1$. This is a simple pole, and

$$\begin{aligned}\operatorname{res}\left(\frac{1}{az^2+2z+a}, p_2\right) &= \lim_{z\to p_2} \frac{z-p_2}{az^2+2z+a} = \lim_{z\to p_2} \frac{z-p_2}{a(z-p_1)(z-p_2)} \\ &= \frac{1}{a(p_2-p_1)} = \frac{1}{a\left(\dfrac{2\sqrt{1-a^2}}{a}\right)} = \frac{1}{2\sqrt{1-a^2}}.\end{aligned}$$

Therefore

$$\int_0^{2\pi} \frac{\mathrm{d}t}{1+a\cos t} = -2i(2\pi i)\frac{1}{2\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}}.$$

Here is a refinement of a result we saw long ago in Proposition 29.8.

Proposition 31.6 An analytic function f on Ω with a singularity at p has a pole of order 2 at p if and only if

$$(z-p)^2 f(z) \to b \neq 0 \quad \text{as } z \to p.$$

In that case,

$$\operatorname{res}(f,p) = \lim_{z\to p} \frac{\mathrm{d}}{\mathrm{d}z}\left((z-p)^2 f(z)\right).$$

Proof. If p is a pole of order 2, then

$$f(z) = \frac{b_1}{z-p} + \frac{b_2}{(z-p)^2} + h(z),$$

where $b_2 \neq 0$, h is analytic on $\Omega \cup \{p\}$, and $b_1 = \operatorname{res}(f,p)$. Then

$$(z-p)^2 f(z) = (z-p)b_1 + b_2 + (z-p)^2 h(z) \to b_2 \quad (z \to p),$$

so the stated limit is nonzero. Also,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left((z-p)^2 f(z)\right) = b_1 + 2(z-p)h(z) + (z-p)^2 h'(z) \to b_1 \quad (z \to p),$$

which gives the residue formula. Conversely, suppose

$$(z-p)^2 f(z) \to b \neq 0.$$

Then $(z-p)^2 f(z)$ is bounded on some punctured disk, so it has a removable singularity at p.

Hence f has a pole of order at most 2. If the order were less than 2, then $(z-p)f(z) \to c$ as $z \to p$ for some finite c, and therefore

$$(z-p)^2 f(z) \to 0 \cdot c = 0,$$

contradicting $b \neq 0$. So the order is exactly 2. $\square$

It is worth remembering this formula.

Example 31.7 For example, for

$$f(z) = \frac{\cos z}{z^2(z-\pi)},$$

we detect singularities at 0 and π. Since

$$z^2 f(z) = \frac{\cos z}{z-\pi} \to -\frac{1}{\pi} \neq 0 \quad (z \to 0),$$

the pole at 0 has order 2.

Then

$$\operatorname{res}(f,0) = \lim_{z\to 0}\left(\frac{\cos z}{z-\pi}\right)' = \lim_{z\to 0}\frac{(z-\pi)(-\sin z) - \cos z}{(z-\pi)^2} = -\frac{1}{\pi}.$$

Lecture 32 — Friday, March 28, 2014

Fourier Transforms

Let us exploit Theorem 31.1 further in computing some Fourier transforms.

Definition 32.1 If $y = f(x)$ is a continuous real-valued function on $\mathbb{R}$ and $\omega \in \mathbb{R}$, the **Fourier transform** of f is

$$\begin{aligned}\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(x)e^{i\omega x}\,\mathrm{d}x \\ &= \lim_{r\to\infty}\int_{-r}^{r} f(x)e^{i\omega x}\,\mathrm{d}x\end{aligned}$$

$$= \lim_{r\to\infty} \left(\int_{-r}^{r} f(x)\cos(\omega x)\,\mathrm{d}x + i\int_{-r}^{r} f(x)\sin(\omega x)\,\mathrm{d}x \right),$$

provided the limit exists.

The next result gives sufficient conditions for $\hat{f}(\omega)$ to exist when $\omega > 0$, and it tells us how to compute it.

Proposition 32.2 If f is analytic on

$$\Omega = \mathbb{C} \setminus \{p_1, \ldots, p_n\},$$

where the p_j are poles of f and $p_j \notin \mathbb{R}$, and if

$$|zf(z)| \leq M$$

for some bound M when $|z| \geq R$, then $\hat{f}(\omega)$ exists for all $\omega > 0$ and

$$\hat{f}(\omega) = 2\pi i \sum_{\substack{j=1 \\ \operatorname{Im} p_j > 0}}^{n} \operatorname{res}(f(z)e^{i\omega z}, p_j).$$

Proof. Take the closed rectangular path γ shown in Figure 95, with r large enough that the rectangle contains all poles of f in the upper half-plane. The dots are the poles of f.

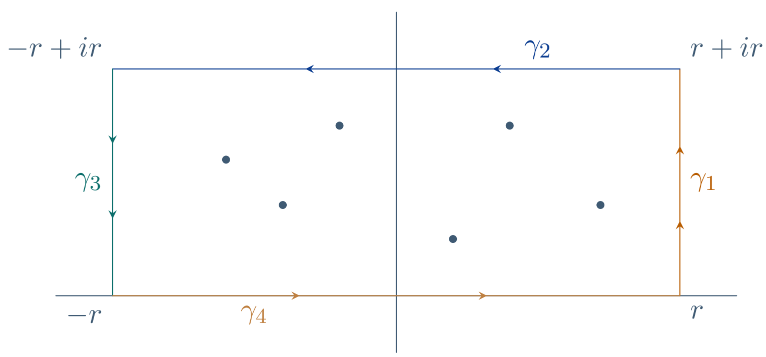

Figure 95

Write $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$. By Theorem 31.1,

$$\int_\gamma f(z)e^{i\omega z}\,\mathrm{d}z = 2\pi i \sum_{\substack{j=1 \\ \operatorname{Im} p_j > 0}}^{n} \operatorname{res}(f(z)e^{i\omega z}, p_j).$$

Let the right-hand side be A. Then

$$A = \int_{\gamma_1} f(z)e^{i\omega z}\,\mathrm{d}z + \int_{\gamma_2} f(z)e^{i\omega z}\,\mathrm{d}z + \int_{\gamma_3} f(z)e^{i\omega z}\,\mathrm{d}z + \int_{\gamma_4} f(z)e^{i\omega z}\,\mathrm{d}z.$$

Using the parametrization $t \mapsto t$ on $[-r, r]$,

$$\int_{\gamma_4} f(z)e^{i\omega z}\,\mathrm{d}z = \int_{-r}^{r} f(x)e^{i\omega x}\,\mathrm{d}x.$$

So

$$\int_{-r}^{r} f(x)e^{i\omega x}\,\mathrm{d}x = A - \int_{\gamma_1} f(z)e^{i\omega z}\,\mathrm{d}z - \int_{\gamma_2} f(z)e^{i\omega z}\,\mathrm{d}z - \int_{\gamma_3} f(z)e^{i\omega z}\,\mathrm{d}z.$$

It remains to show that the three contour integrals on the right tend to 0 as $r \to \infty$. For γ_1, use $t \mapsto r + it$ on $[0, r]$:

$$\begin{aligned}
\left|\int_{\gamma_1} f(z)e^{i\omega z}\,\mathrm{d}z\right| &= \left|\int_0^r e^{i\omega(r+it)} f(r+it) i\,\mathrm{d}t\right| \\
&\le \int_0^r e^{-\omega t}|f(r+it)|\,\mathrm{d}t \\
&\le \int_0^r e^{-\omega t}\frac{M}{|r+it|}\,\mathrm{d}t \quad (r \text{ large}) \\
&\le \frac{M}{r}\int_0^r e^{-\omega t}\,\mathrm{d}t \\
&= \frac{M}{r\omega}\left(1 - e^{-\omega r}\right) \to 0
\end{aligned}$$

as $r \to \infty$, since $\omega > 0$. A similar argument gives

$$\left|\int_{\gamma_3} f(z)e^{i\omega z}\,\mathrm{d}z\right| \to 0.$$

For γ_2, parametrize by $t \mapsto -t + ir$ on $[-r, r]$:

$$\left|\int_{\gamma_2} f(z)e^{i\omega z}\,\mathrm{d}z\right| = \left|\int_{-r}^{r} f(-t+ir)e^{i\omega(-t+ir)}(-1)\,\mathrm{d}t\right|$$

$$\begin{aligned}
&\leq \int_{-r}^{r} |f(-t+ir)|e^{-\omega r}\,\mathrm{d}t \\
&\leq \int_{-r}^{r} \frac{M}{|-t+ir|}e^{-\omega r}\,\mathrm{d}t \\
&\leq \frac{M}{r}\int_{-r}^{r} e^{-\omega r}\,\mathrm{d}t \\
&= 2Me^{-\omega r} \to 0.
\end{aligned}$$

Therefore

$$\lim_{r\to\infty}\int_{-r}^{r} f(x)e^{i\omega x}\,\mathrm{d}x = A,$$

which is exactly the stated formula for $\hat{f}(\omega)$. □

Example 32.3 Let us compute the Fourier transform of

$$f(x) = \frac{x}{x^2+1},$$

The function

$$f(z) = \frac{z}{z^2+1}$$

fits the conditions of Proposition 32.2. For instance,

$$|zf(z)| = \left|\frac{z^2}{z^2+1}\right| \to 1 \quad (|z|\to\infty),$$

so $|zf(z)|$ is bounded outside some disk. The only pole of

$$\frac{z}{z^2+1}e^{i\omega z}$$

in the upper half-plane is i. Hence

$$\hat{f}(\omega) = 2\pi i \operatorname{res}\left(\frac{z}{z^2+1}e^{i\omega z}, i\right) = 2\pi i \lim_{z\to i}(z-i)\frac{z}{z^2+1}e^{i\omega z} = 2\pi i\,\frac{i}{2i}e^{-\omega} = \pi i e^{-\omega}.$$

Lecture 33 — Wednesday, March 26, 2014

Computation of Series

Knowing that a series $\sum_{n=1}^{\infty} a_n$ converges is good, but knowing what it converges to is better. For instance, what numbers are represented by

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{1+n^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^4}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^6}, \qquad \text{or} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}?$$

Here is a setup that can help. Let f be analytic on

$$\Omega = \mathbb{C} \setminus \{p_1, \ldots, p_k\},$$

where the p_j are poles, and consider

$$g(z) = f(z)\,\pi\cot(\pi z).$$

The poles of g are of two types: poles of f, and zeros of $\sin(\pi z)$ that are not already poles of f (that is, integers that are not poles of f). (Note that if an integer n is also a zero of f, then n may fail to be a pole of g. One may think of this as a pole of order 0 with residue 0.)

If $n \in \mathbb{Z}$ is not a pole of f, then

$$\lim_{z\to n}(z-n)g(z) = \lim_{z\to n} f(z)(z-n)\pi\frac{\cos(\pi z)}{\sin(\pi z)} = \lim_{z\to n} f(z)\,\frac{\cos(\pi z)}{\dfrac{\sin(\pi z)-\sin(\pi n)}{\pi z-\pi n}} = f(n)\,\frac{\cos(\pi n)}{\cos(\pi n)} = f(n).$$

So if $f(n) \neq 0$, the integer n is a pole of order 1 for g with residue $f(n)$, and near n,

$$g(z) = \frac{f(n)}{z-n} + h(z),$$

where h has a removable singularity at n. Now let N be an integer so large that the rectangular path γ_N shown in Figure 96 encloses all poles p_j of f.

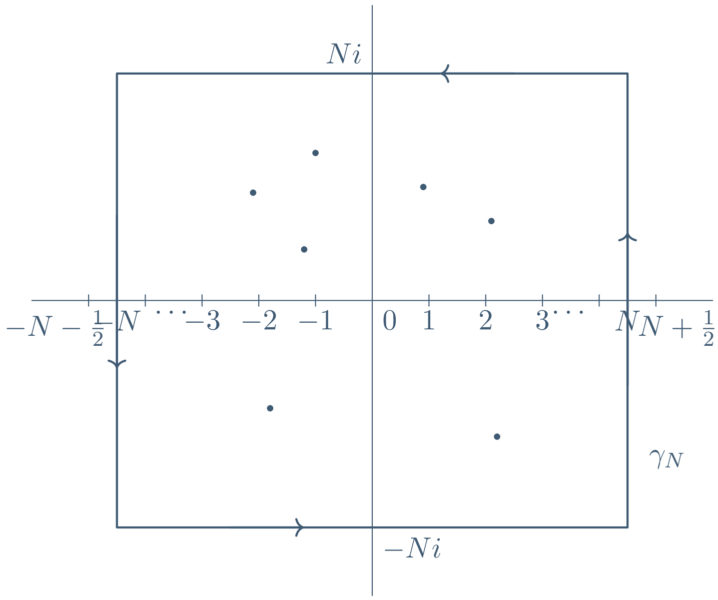

Figure 96

By Theorem 31.1 for g,

$$\int_{\gamma_N} g = 2\pi i \left(\sum_{j=1}^{k} \operatorname{res}(g, p_j) \underbrace{\operatorname{ind}_{\gamma_N}(p_j)}_{=1} + \sum_{\substack{n=-N \\ n \neq p_j}}^{N} f(n) \underbrace{\operatorname{ind}_{\gamma_N}(n)}_{=1} \right).$$

If we are lucky with f, we may get

$$\int_{\gamma_N} g \to 0 \quad \text{as } N \to \infty.$$

In that case,

$$\sum_{\substack{n=-\infty \\ n \neq p_j}}^{\infty} f(n) = -\sum_{j=1}^{k} \operatorname{res}(g, p_j). \tag{$\star$}$$

Here is one way to be lucky with f.

Proposition 33.1 If for some $\lambda > 1$ we have

$$|z|^\lambda |f(z)| \le M$$

for some bound M when $|z|$ is large enough, then

$$\int_{\gamma_N} f(z)\,\pi \frac{\cos(\pi z)}{\sin(\pi z)}\,\mathrm{d}z \to 0 \quad \text{as } N \to \infty.$$

Proof. We first verify that

$$\left\| \frac{\cos(\pi z)}{\sin(\pi z)} \right\|_{\gamma_N^*} \le 2 \quad \text{for all large } N. \tag{$\star\star$}$$

Once this is done,

$$\begin{aligned}
\left| \int_{\gamma_N} f(z)\,\pi \frac{\cos(\pi z)}{\sin(\pi z)}\,\mathrm{d}z \right| &\le \left\| f(z)\,\pi \frac{\cos(\pi z)}{\sin(\pi z)} \right\|_{\gamma_N^*} \text{length}(\gamma_N^*) \\
&\le \|f(z)\|_{\gamma_N^*} \cdot \pi \cdot 2 \cdot (8N+1) \\
&\le \left\| \frac{M}{|z|^\lambda} \right\|_{\gamma_N^*} (16\pi N + 2\pi) \quad (N \text{ large}) \\
&\le \frac{M}{N^\lambda}(16\pi N + 2\pi) \to 0
\end{aligned}$$

as $N \to \infty$, because $\lambda > 1$.

It only remains to verify ($\star\star$). Across the top edge of γ_N, with $z = t + Ni$,

$$\left| \frac{\cos(\pi z)}{\sin(\pi z)} \right| = \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| \le \frac{|e^{i\pi z}| + |e^{-i\pi z}|}{\big||e^{i\pi z}| - |e^{-i\pi z}|\big|} = \frac{e^{-\pi N} + e^{\pi N}}{e^{\pi N} - e^{-\pi N}} = \frac{1 + e^{-2\pi N}}{1 - e^{-2\pi N}} \le 2.$$

for large N. Across the bottom edge, the same estimate holds. On the right edge, let $z = N + 1/2 + it$ with $-N \le t \le N$. Then

$$\begin{aligned}
\left| \frac{\cos(\pi z)}{\sin(\pi z)} \right| &= \left| \frac{e^{i\pi(N+1/2+it)} + e^{-i\pi(N+1/2+it)}}{e^{i\pi(N+1/2+it)} - e^{-i\pi(N+1/2+it)}} \right| \\
&= \left| \frac{e^{i\pi N} e^{i\pi/2} e^{-\pi t} + e^{-i\pi N} e^{-i\pi/2} e^{\pi t}}{e^{i\pi N} e^{i\pi/2} e^{-\pi t} - e^{-i\pi N} e^{-i\pi/2} e^{\pi t}} \right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \frac{e^{i\pi N} i(e^{-\pi t} - e^{\pi t})}{e^{i\pi N} i(e^{-\pi t} + e^{\pi t})} \right| \\
&= \left| \frac{e^{-\pi t} - e^{\pi t}}{e^{-\pi t} + e^{\pi t}} \right| \\
&\leq \frac{e^{-\pi t}}{e^{-\pi t} + e^{\pi t}} + \frac{e^{\pi t}}{e^{-\pi t} + e^{\pi t}} \\
&\leq 2.
\end{aligned}
$$

The left edge is analogous. □

Example 33.2 Let

$$f(z) = \frac{1}{z^2}.$$

Clearly f satisfies the hypotheses of Proposition 33.1, so formula ($\star$) applies. The only integer that is a pole of f is 0. Hence

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n^2} = -\operatorname{res}\left(\frac{1}{z^2}\, \pi \frac{\cos(\pi z)}{\sin(\pi z)}, 0 \right).$$

Here 0 is a pole of order 3, since

$$z^3 \frac{1}{z^2}\, \pi \frac{\cos(\pi z)}{\sin(\pi z)} = \pi z \frac{\cos(\pi z)}{\sin(\pi z)} \to 1 \neq 0 \quad (z \to 0).$$

Write

$$\frac{\pi z \cos(\pi z)}{\sin(\pi z)} = a_0 + a_1 z + a_2 z^2 + \ldots$$

on some $D_0(r)$. Then

$$g(z) = \frac{\pi \cos(\pi z)}{z^2 \sin(\pi z)} = \frac{a_0}{z^3} + \frac{a_1}{z^2} + \frac{a_2}{z} + a_3 + \ldots$$

on $D_0^*(r)$, so

$$\operatorname{res}\left(\frac{1}{z^2}\, \pi \frac{\cos(\pi z)}{\sin(\pi z)}, 0 \right) = a_2.$$

Now use

$$\pi z \cos(\pi z) = \sin(\pi z)(a_0 + a_1 z + a_2 z^2 + \dots).$$

That is,

$$\begin{aligned} \pi z - \frac{(\pi z)^3}{2!} + \frac{(\pi z)^5}{4!} - \dots &= \left(\pi z - \frac{(\pi z)^3}{3!} + \frac{(\pi z)^5}{5!} - \dots\right)(a_0 + a_1 z + a_2 z^2 + \dots) \\ &= \pi a_0 z + \pi a_1 z^2 + \left(\pi a_2 - \frac{\pi^3}{6} a_0\right) z^3 + \dots \end{aligned}$$

Comparing coefficients,

$$\pi = \pi a_0 \implies a_0 = 1 \qquad \text{and} \qquad -\frac{\pi^3}{2} = \pi a_2 - \frac{\pi^3}{6} a_0 = \pi a_2 - \frac{\pi^3}{6}.$$

so

$$\pi a_2 = -\frac{\pi^3}{3} \qquad \text{and} \qquad a_2 = -\frac{\pi^2}{3}.$$

Hence

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n^2} = -\left(-\frac{\pi^2}{3}\right) = \frac{\pi^2}{3}.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

which is Euler's famous formula.

Lecture 34 — Monday, March 31, 2014

13. Rouché's Theorem

The Argument Principle and Rouché's Theorem

Proposition 34.1 If f is analytic and nonconstant on Ω and p is a zero of f (i.e., $f(p) = 0$) with multiplicity $m \geq 1$, then p is a pole of f'/f having order 1, and $\text{Res}(f'/f, p) = m$.

Proof. If p is a zero of multiplicity m, then on some $D_p(r) \subseteq \Omega$ we can write

$$f(z) = a_m(z-p)^m + a_{m+1}(z-p)^{m+1} + \ldots,$$

where $a_m \neq 0$. Thus

$$f(z) = (z-p)^m h(z) \qquad \text{and} \qquad h(p) \neq 0,$$

and for z where $f(z) \neq 0$ we get

$$\frac{f'(z)}{f(z)} = \frac{m(z-p)^{m-1}h(z) + (z-p)^m h'(z)}{(z-p)^m h(z)} = \frac{m}{z-p} + \frac{h'(z)}{h(z)}.$$

Since $h(p) \neq 0$, the function

$$\frac{h'(z)}{h(z)}$$

is analytic on some disk about p. Therefore, p is a pole of f'/f of order 1, and

$$m = \text{res}\left(\frac{f'}{f}, p\right).$$

$\square$

Now let γ be a closed path in Ω. We can view it as a single piecewise continuously differentiable path

$$\gamma : [0,1] \to \mathbb{C}.$$

For every $w \notin \gamma^*$,

$$\text{ind}_\gamma(w) = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z-w} = \frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t)-w}\,\mathrm{d}t.$$

Note that the left and right derivatives of γ may fail to agree at finitely many parameter values. This does not affect the index computation, and it gives the same answer as treating γ as a concatenation of smooth paths.

We say that γ has an **interior** if for all $w \notin \gamma^*$,

$$\operatorname{ind}_\gamma(w) \in \{0, 1\}.$$

For instance, the unit circle $t \mapsto e^{it}$ has an interior. We define the interior of γ to be the set of points w with

$$\operatorname{ind}_\gamma(w) = 1.$$

This ad hoc definition avoids an explicit appeal to the Jordan curve theorem. We will also say that p is inside γ when $\operatorname{ind}_\gamma(p) = 1$.

Returning to our analytic function f on Ω, suppose γ has an interior, does not pass through zeros of f, and is homologous to 0 in Ω. Then only finitely many zeros of f can lie in the interior of γ. Indeed, the homology assumption places the interior inside Ω, while its boundary lies in the compact set $\gamma^* \subseteq \Omega$. Hence the closure of the interior is a compact subset of Ω. An infinite collection of zeros there would have a cluster point in Ω, contradicting the isolation of zeros of a nonconstant analytic function.

Proposition 34.2 (Argument principle) If f is analytic on Ω, γ is a closed path in Ω with interior and does not pass through zeros of f, and γ is homologous to 0 in Ω (that is, $\operatorname{ind}_\gamma = 0$ on $\mathbb{C} \setminus \Omega$), then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \text{the number of zeros of } f \text{ inside } \gamma\text{, counted with multiplicity.}$$

Proof. Apply Theorem 31.1 to f'/f. The homology assumption is exactly what allows this application. Hence

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \sum_{\substack{p \text{ is a zero of } f \\ \operatorname{ind}_\gamma(p)=1}} \operatorname{res}\left(\frac{f'}{f}, p\right).$$

If m_p is the multiplicity of the zero at p, then

$$m_p = \operatorname{res}\left(\frac{f'}{f}, p\right).$$

Therefore,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \sum_{\substack{p \text{ is a zero of } f \\ \operatorname{ind}_\gamma(p)=1}} m_p,$$

as required. $\square$

Lecture 35 — Wednesday, April 02, 2014

There is also a geometric interpretation of

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}.$$

Using $\gamma : [0,1] \to \mathbb{C}$,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \frac{1}{2\pi i}\int_0^1 \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))}\,\mathrm{d}t.$$

Now

$$(f\circ\gamma)'(t) = f'(\gamma(t))\gamma'(t),$$

so

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \frac{1}{2\pi i}\int_0^1 \frac{(f\circ\gamma)'(t)}{(f\circ\gamma)(t)}\,\mathrm{d}t = \frac{1}{2\pi i}\int_{f\circ\gamma}\frac{\mathrm{d}z}{z} = \mathrm{ind}_{f\circ\gamma}(0).$$

Thus this integral is the winding number of $f\circ\gamma$ about 0. It is often described as the change in argument of f along γ.

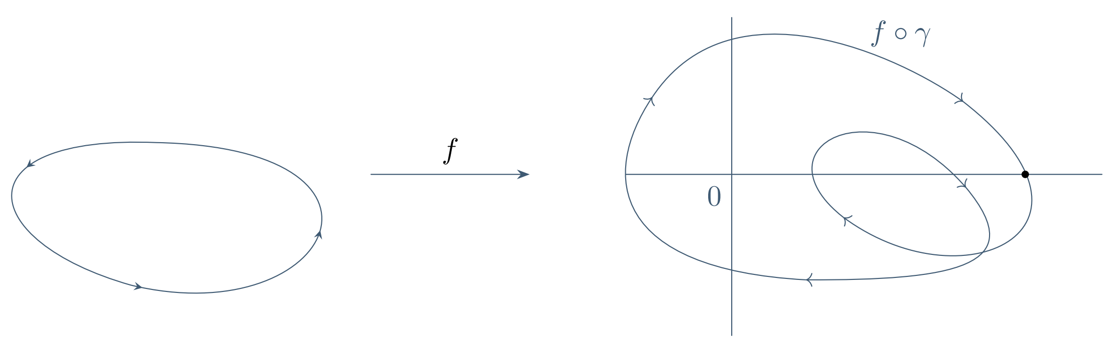

Figure 97

In Figure 97, if $f\circ\gamma(t)$ starts with argument 0 at $t=0$, it ends with argument $2\cdot 2\pi$ at $t=1$. So the total change in argument equals the number of zeros of f inside γ, counted with multiplicity.

This gives a useful result about winding numbers around 0.

Proposition 35.1 Suppose $\alpha, \beta : [0,1] \to \mathbb{C}$ are closed paths such that

$$|\alpha(t) - \beta(t)| < |\beta(t)| \quad \text{for all } t \in [0,1].$$

Then

$$\text{ind}_\alpha(0) = \text{ind}_\beta(0).$$

Before the proof, here is an allegory. You are β, walking around a fire hydrant (the point 0). Your dog is α. The leash length at time t is

$$|\alpha(t) - \beta(t)|,$$

and you keep it strictly less than your own distance to the hydrant, namely $|\beta(t)|$. Then the dog never has enough freedom to add extra loops around the hydrant. Intuitively, both of you must wind around 0 the same number of times.

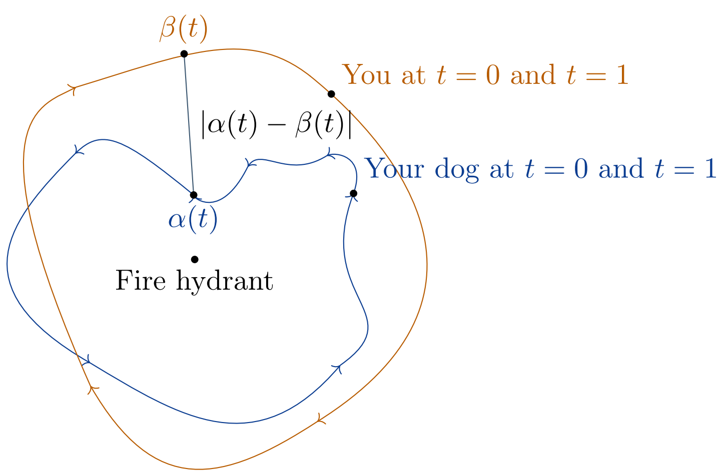

FIGURE 98

The construction is shown in Figure 98.

Proof. First, $|\beta(t)| > 0$ for all t, so $0 \notin \beta^*$. Also, if $\alpha(s) = 0$ for some $s \in [0,1]$, then

$$|\beta(s)| = |0 - \beta(s)| = |\alpha(s) - \beta(s)| < |\beta(s)|,$$

a contradiction. Hence $0 \notin \alpha^*$ as well. Define

$$\gamma(t) = \frac{\alpha(t)}{\beta(t)}.$$

This is a well-defined closed path, and still $0 \notin \gamma^*$. For $t \in [0, 1]$,

$$\frac{\gamma'(t)}{\gamma(t)} = \frac{\beta(t)}{\alpha(t)} \left(\frac{\beta(t)\alpha'(t) - \alpha(t)\beta'(t)}{\beta(t)^2} \right) = \frac{\alpha'(t)}{\alpha(t)} - \frac{\beta'(t)}{\beta(t)}. \tag{1}$$

Moreover,

$$\left| \frac{\alpha(t)}{\beta(t)} - 1 \right| < 1,$$

so $\gamma(t) \in D_1(1)$ for all t. Therefore $\mathrm{ind}_\gamma(0) = 0$.

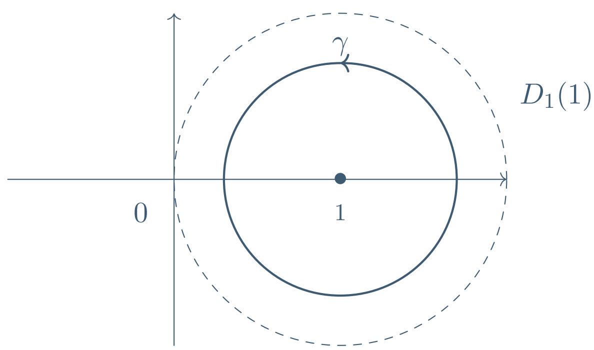

Figure 99

Figure 99 records the relevant geometry.

Now

$$0 = \int_\gamma \frac{\mathrm{d}z}{z} = \int_0^1 \frac{\gamma'(t)}{\gamma(t)}\,\mathrm{d}t = \int_0^1 \frac{\alpha'(t)}{\alpha(t)}\,\mathrm{d}t - \int_0^1 \frac{\beta'(t)}{\beta(t)}\,\mathrm{d}t = \int_\alpha \frac{\mathrm{d}z}{z} - \int_\beta \frac{\mathrm{d}z}{z},$$

by (1). Hence

$$\mathrm{ind}_\alpha(0) = \mathrm{ind}_\beta(0).$$

$\square$

Putting these pieces together gives Theorem 35.2.

Theorem 35.2 (Rouché's theorem) If f, g are analytic on Ω, $\gamma : [0,1] \to \Omega$ is a closed path with interior (that is, ind_γ only takes the values 0 and 1), γ is homologous to 0 in Ω (that is, $\mathrm{ind}_\gamma = 0$ on $\mathbb{C} \setminus \Omega$), and $|g(z) - f(z)| < |f(z)|$ for all $z \in \gamma^*$, then f and g have the same number of zeros in the interior of γ, counting multiplicities.

Proof. By Proposition 34.2, the number of zeros of f inside γ, counting multiplicities, is

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} = \mathrm{ind}_{f\circ\gamma}(0).$$

Similarly, the number of zeros of g inside γ equals

$$\mathrm{ind}_{g\circ\gamma}(0).$$

For all $t \in [0,1]$,

$$|g(\gamma(t)) - f(\gamma(t))| < |f(\gamma(t))|.$$

Applying Proposition 35.1 to the closed paths $f \circ \gamma$ and $g \circ \gamma$ gives

$$\mathrm{ind}_{f\circ\gamma}(0) = \mathrm{ind}_{g\circ\gamma}(0),$$

so f and g have the same number of zeros inside γ, counted with multiplicity. □

Coomputations Using Rouché's Theorem

Let us work through a few examples. The standard strategy is to write

$$g = f + h,$$

where h is a small perturbation of f on the boundary.

First verify

$$|h(z)| < |f(z)| \quad \text{on } \gamma^*,$$

and then conclude that f and g have the same number of zeros inside γ.

Example 35.3 How many roots does

$$g(z) = z^8 - 5z^3 + z - 2$$

have inside the unit circle?

Take

$$f(z) = -5z^3 \qquad \text{and} \qquad h(z) = z^8 + z - 2.$$

For $|z| = 1$,

$$|h(z)| = |z^8 + z - 2| \le |z|^8 + |z| + 2 = 4 < 5 = |-5z^3| = |f(z)|.$$

Since $-5z^3$ has exactly 3 zeros inside the unit circle (the zero at 0 with multiplicity 3), the polynomial $z^8 - 5z^3 + z - 2$ also has 3 zeros inside the unit circle. All 8 zeros are shown in Figure 100:

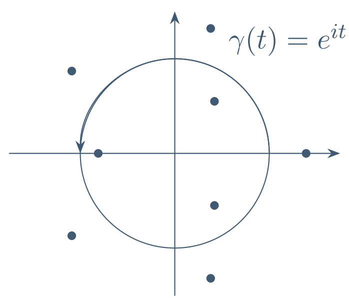

Figure 100

Example 35.4 Show that

$$z + 3 + 2e^z$$

has exactly one zero in the left half-plane.

Since $z + 3$ has one zero (-3) in the left half-plane, write

$$g(z) = z + 3 + 2e^z = f(z) + h(z),$$

where

$$f(z) = z + 3 \qquad \text{and} \qquad h(z) = 2e^z.$$

Take the usual large semicircular contour γ_r in the left half-plane.

On the diameter, $z = iy$ and

$$|2e^{iy}| = 2 < 3 \leq |3 + iy|.$$

On the semicircular arc, $|z| = r$ and $\operatorname{Re} z \leq 0$, so

$$|2e^z| = 2e^{\operatorname{Re} z} \leq 2 \qquad \text{and} \qquad |z + 3| \geq r - 3 > 2$$

when $r > 5$. Thus $|2e^z| < |z + 3|$ everywhere on γ_r^*. By Theorem 35.2, $z + 3 + 2e^z$ has exactly one zero inside γ_r, because $z + 3$ does. Since this holds for every $r > 5$, the function has exactly one zero in the left half-plane.

Example 35.5 Show all roots of $g(z) = z^6 - 5z^2 + 10$ lie in the annulus $1 < |z| < 2$.

First we show that all six roots lie in $D_0(2)$. Write

$$f(z) = z^6 \qquad \text{and} \qquad h(z) = -5z^2 + 10.$$

For $|z| = 2$,

$$|h(z)| = | - 5z^2 + 10| \leq 5|z|^2 + 10 = 30,$$

while

$$|f(z)| = |z^6| = 2^6 = 64 > 30.$$

By Theorem 35.2, all six roots of $z^6 - 5z^2 + 10$ lie in $D_0(2)$.

Now we show there are no roots in $D_0(1)$. Write

$$f(z) = 10 \qquad \text{and} \qquad h(z) = z^6 - 5z^2.$$

For $|z| = 1$,

$$|f(z)| = 10 \qquad \text{and} \qquad |h(z)| = |z^6 - 5z^2| \leq |z|^6 + 5|z|^2 \leq 6 < 10.$$

Therefore $z^6 - 5z^2 + 10$ has as many zeros in $D_0(1)$ as 10 does, namely none.

So all roots lie in

$$D_0(2) \setminus D_0(1).$$

It remains to rule out roots on $|z| = 1$. For $|z| = 1$,

$$|z^6 - 5z^2 + 10| \geq 10 - |z|^6 - 5|z|^2 = 4 > 0,$$

so no root lies on the unit circle. In fact, this same estimate shows there are no roots in the closed unit disk.

For example, this also reproves Theorem 23.7.

Let

$$g(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$$

be a polynomial of degree n. Think of g as a perturbation of

$$f(z) = z^n$$

by

$$h(z) = a_{n-1}z^{n-1} + \cdots + a_0.$$

For large R, on the circle $\gamma_R(t) = Re^{it}$,

$$|f(z)| = |z^n| = R^n,$$

and

$$|h(z)| \leq |a_{n-1}|R^{n-1} + \cdots + |a_1|R + |a_0|.$$

If

$$R > |a_{n-1}| + \cdots + |a_1| + |a_0| \qquad \text{and} \qquad R > 1,$$

then

$$R^n > |a_{n-1}|R^{n-1} + \cdots + |a_1|R^{n-1} + |a_0|R^{n-1} \geq |a_{n-1}|R^{n-1} + \cdots + |a_1|R + |a_0| \geq |h(z)|.$$

Hence $|h(z)| < |f(z)|$ on γ_R^*, so by Theorem 35.2, g has as many zeros inside γ_R as f does, namely n.

Since this holds for every $R > \max\{1, |a_{n-1}| + \cdots + |a_0|\}$, all roots of g lie in the closed disk centered at 0 of radius

$$\max\{1, |a_{n-1}| + \cdots + |a_1| + |a_0|\}.$$

Lecture 36 — Friday, April 04, 2014

Example 36.1 If $g(z) = 3 + z + 2e^z$, let us show that g has exactly one zero in the half-plane given by $\operatorname{Re}(z) \leq 0$.

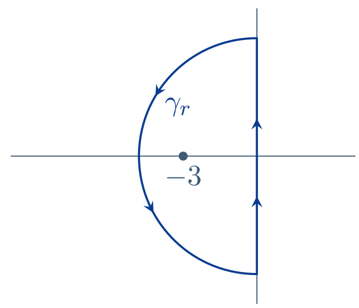

Figure 101

Figure 101 illustrates the setup. Consider $\gamma = \gamma_1 + \gamma_2$ for large $r > 0$. Our first instinct might be to use $2e^z$, but that won't work. Instead, try $f(z) = 3 + z$, which has exactly one zero (i.e., -3) inside γ when $r > 3$. We must check that

$$|g(z) - f(z)| = |2e^z| < |3 + z| = |f(z)|$$

on γ^*, and for this we just need to verify the middle inequality. If $z \in \gamma_1^*$, then $z = iy$ and so

$$\left|2e^{iy}\right| = 2 < 3 \leq \sqrt{3^2 + y^2} = |3 + iy|.$$

If $z \in \gamma_2^*$, then $z = x + iy$ where $x \leq 0$, and

$$|2e^z| = \left|2e^{x+iy}\right| = 2e^x \leq 2$$

while $|3 + z| \geq |z| - 3 = r - 3 > 2$ when $r > 5$. So for $r > 5$, Theorem 35.2 applies. Take $r \to \infty$ to conclude.

The Open Mapping Theorem

Theorem 36.2 (Open mapping theorem) If $f\colon \Omega \to \mathbb{C}$ is analytic and nonconstant, then $f(\Omega)$ is open.

Warning This is emphatically not true in $\mathbb{R}$!

Proof. We need to show that for all $p \in \Omega$, there exists $\varepsilon > 0$ with $D_\varepsilon(f(p)) \subseteq f(\Omega)$. First, suppose that $f(p) = 0$. Then p is an isolated zero of f (i.e., there exists $\overline{D_r(p)} \subseteq \Omega$ such that $f = 0$ only at p in that disk). Thus the minimum $\min\{|f(z)| : |z-p| = r\}$ can be called ε, which is strictly positive. Obviously then, $\varepsilon \le |f(z)|$ when $|z-p| = r$. For $w \in D_\varepsilon(0)$, put $g(z) = f(z) - w$. If $|z-p| = r$, then

$$|g(z) - f(z)| = |-w| < \varepsilon \le |f(z)|.$$

By Theorem 35.2, g has exactly as many zeros as f inside $|z-p| = r$, counted with multiplicity. This number is positive because p is a zero of f. Therefore $f(z) - w = 0$ for some $z \in D_r(p)$ (i.e., $w = f(z)$ for some $z \in \Omega$). Thus $D_\varepsilon(0) \subseteq f(\Omega)$. Now suppose that $f(p) \neq 0$. Then we can apply what we just did to $g(z) = f(z) - f(p)$, which is a fresh g with $g(p) = 0$. So $D_\varepsilon(0) \subseteq g(\Omega)$ for some ε. Then $D_\varepsilon(0) \subseteq f(\Omega) - f(p)$ (that's a translation), and hence

$$D_\varepsilon(f(p)) = D_\varepsilon(0) + f(p) \subseteq (f(\Omega) - f(p)) + f(p) = f(\Omega).$$

□

Proposition 36.3 If f is analytic on Ω and $f'(p) = 0$ for some $p \in \Omega$, then f is *not* one-to-one on Ω.

Again, this stands in stark contrast with functions of a real variable, where we can have one-to-one functions like $f(x) = x^3$ satisfying $f'(0) = 0$.

Proof. If f is constant, it is already not one-to-one on the region Ω. Suppose therefore that f is nonconstant, and replace f by $F = f - f(p)$. Then $F(p) = F'(p) = 0$, and injectivity is unchanged.

Neither F nor $F' = f'$ is identically zero, so their zeros are isolated. Choose $r > 0$ such that $\overline{D_p(r)} \subseteq \Omega$ and neither F nor F' vanishes on $0 < |z-p| \le r$. On the boundary circle $C = \{z :$

$|z - p| = r\}$, continuity gives

$$\varepsilon := \min_{z \in C} |F(z)| > 0.$$

Choose w with $0 < |w| < \varepsilon$ and put $G = F - w$. Since $|G - F| = |w| < |F|$ on C, Theorem 35.2 says that F and G have the same number of zeros in $D_p(r)$, counted with multiplicity. The zero of F at p has multiplicity at least 2 because $F'(p) = 0$, so G has at least two zeros counted with multiplicity.

These zeros cannot all coincide at one point q. If they did, then q would be a multiple zero of G, so $G'(q) = F'(q) = 0$. Since $w \neq 0$, we have $q \neq p$, contradicting the choice of r. Thus there are distinct $z_1, z_2 \in D_p(r)$ with

$$F(z_1) = F(z_2) = w.$$

Consequently $f(z_1) = f(z_2)$, so f is not one-to-one. □

Univalent Functions

Theorem 36.2 and Proposition 36.3 are very interesting.

Definition 36.4 An analytic function

$$f : \Omega \to \mathbb{C}$$

that is one-to-one is usually called **univalent** among complex analysis sophisticates.

Proposition 36.3 says that derivatives of univalent functions never vanish, and from Theorem 36.2 we see that univalent functions map open sets to open sets.

Let $\Delta = f(\Omega)$. This is an open set, and we have the bijection $f : \Omega \to \Delta$ with inverse mapping $g : \Delta \to \Omega$. This leads to a special theorem about the inverse g.

Theorem 36.5 Let $f : \Omega \to \mathbb{C}$ be univalent and let $\Delta = f(\Omega)$. Then Δ is an open region too, and the inverse mapping $g : \Delta \to \Omega$ is analytic. Furthermore if $p \in \Omega$ and $q = f(p)$, then

$$g'(q) = \frac{1}{f'(p)}$$

This is the usual rule for the derivative of the inverse.

Proof. We saw that f is an open mapping and that f' never vanishes on Ω. For every open set $V \subseteq \Omega$ the image $f(V)$ is open inside Δ. But $f(V) = g^{-1}(V)$, the inverse image of V under g. This tells us that the inverse map g is a continuous function on Δ. Now it's easy to see that $g'(q)$ exists for every q in Δ. Let $q = f(p)$ for its unique p in Ω, i.e. $p = g(q)$. For every $w \neq q$ in Δ we have $g(w) \neq g(q)$ and then

$$\frac{w-q}{g(w)-g(q)} = \frac{f(g(w)) - f(g(q))}{g(w)-g(q)}$$

Now as $w \to q$ we get $g(w) \to g(q)$ since g is continuous on Δ. (That's what f being an open map gave.) As $g(w) \to g(q)$ we see that the above tends to $f'(g(q)) = f'(p)$, and $f'(p) \neq 0$, so

$$\begin{aligned}\lim_{w\to q} \frac{g(w)-g(q)}{w-q} &= \lim_{w \to q} \frac{1}{\frac{w-q}{g(w)-g(q)}} \\ &= \frac{1}{f'(p)},\end{aligned}$$

and we have our derivative for g. □

Example 36.6 Let $\Omega = \{x + iy : -\pi < y < \pi\}$, and take the exponential function

$$\begin{aligned}\exp : \Omega &\to \mathbb{C} \\ z &\mapsto e^z\end{aligned}$$

On Ω, exp is univalent. As expected, $(e^z)' \neq 0$. The image $\exp(\Omega)$ is the cut plane

$$\Delta = \{x + iy : y \neq 0 \text{ or } x > 0\}.$$

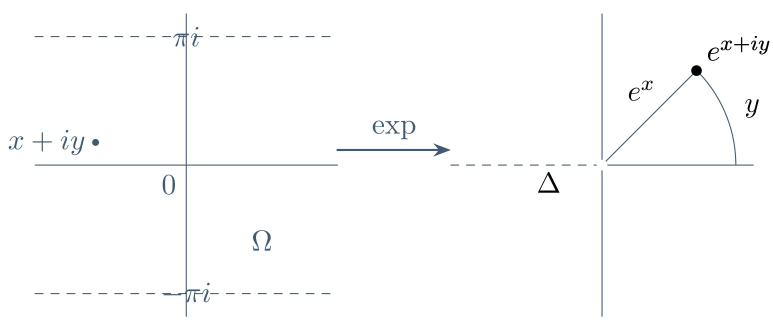

FIGURE 102

The construction is shown in Figure 102. To see that $\exp(\Omega) = \Delta$, notice that for $x + iy$ with $-\pi < y < \pi$ we have

$$e^{x+iy} = e^x e^{iy} = re^{iy}$$

where $r > 0$ covers every positive real and y gives every argument from $-\pi$ to π, omitting $\pm\pi$. That's the cut plane.

Naturally, the inverse of this $\exp : \Omega \to \Delta$ is the principal branch of the logarithm

$$\log : \Delta \to \Omega,$$

which of course is analytic.

Lecture 37 — Monday, April 07, 2014

14. Laurent Series

Laurent Expansions and Series

Given a power series

$$g(w) = \sum_{n=1}^{\infty} b_n w^n$$

with radius of convergence s, where $0 < s \leq \infty$, let

$$r = \frac{1}{s} \quad (\text{with } \frac{1}{\infty} = 0).$$

For $p \in \mathbb{C}$ define the unbounded annulus

$$\Omega = \{z : |z - p| > r\}.$$

Then we obtain an analytic function

$$\begin{aligned} f : \Omega &\to D_0(s) \to \mathbb{C} \\ z &\mapsto \frac{1}{z-p} \mapsto g\left(\frac{1}{z-p}\right). \end{aligned}$$

Indeed,

$$z \in \Omega \iff |z-p| > \frac{1}{s} \iff \left|\frac{1}{z-p}\right| < s \iff \frac{1}{z-p} \in D_0(s).$$

So f is represented by a series in negative powers of $z-p$:

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-p)^n}, \quad |z-p| > r.$$

By Proposition 4.8, for $|z-p| > r$,

$$\begin{aligned} f'(z) &= g'\left(\frac{1}{z-p}\right)\left(\frac{1}{z-p}\right)' \\ &= \left(\sum_{n=1}^{\infty} n b_n \left(\frac{1}{z-p}\right)^{n-1}\right)\left(-\frac{1}{(z-p)^2}\right) \\ &= \sum_{n=1}^{\infty} \frac{-n b_n}{(z-p)^{n+1}} \\ &= \sum_{n=1}^{\infty} b_n \left(\frac{1}{(z-p)^n}\right)'. \end{aligned}$$

This is the expected term-by-term differentiation.

If A is a compact subset of the unbounded annulus Ω, then

$$B = \left\{\frac{1}{z-p} : z \in A\right\}$$

is compact in $D_0(s)$, where g is defined. Since

$$\sum_{n=1}^{\infty} b_n w^n$$

converges uniformly on B to $g(w)$, it follows that

$$\sum_{n=1}^{\infty} \frac{b_n}{(z-p)^n}$$

converges uniformly on A. Equivalently, for every $\varepsilon > 0$,

$$\left|\sum_{k=1}^{n} b_k w^k - g(w)\right| < \varepsilon \quad \text{for all } w \in B$$

for all large n, which implies

$$\left|\sum_{k=1}^{n} \frac{b_k}{(z-p)^k} - f(z)\right| < \varepsilon \quad \text{for all } z \in A$$

for all large n.

Now suppose we also have $0 \le r < R \le \infty$ and an analytic function

$$h(z) = \sum_{n=0}^{\infty} a_n (z-p)^n \quad \text{on } D_p(R).$$

Then on the annulus

$$A_p(r, R) = \{z : r < |z-p| < R\}$$

we get

$$\ell(z) = f(z) + h(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-p)^n} + \sum_{n=0}^{\infty} a_n (z-p)^n.$$

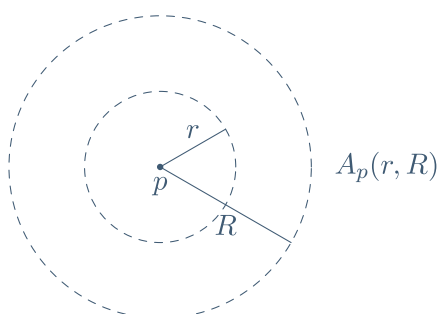

Figure 103

Figure 103 records the relevant geometry.

Definition 37.1 An analytic function on an annulus represented in this way is said to have a **Laurent expansion** on that annulus. The combined series in integer powers of $z-p$ is called a **Laurent series**.

Example 37.2

$$f(z) = \frac{1}{z(z-1)}$$

is analytic on $A_0(1, \infty)$.

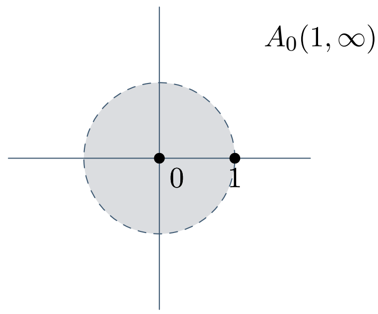

FIGURE 104

The configuration appears in Figure 104.

A Laurent expansion on $A_0(1, \infty)$ is

$$f(z) = \frac{1}{z}\left(\frac{1}{z\left(1-\frac{1}{z}\right)}\right) = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{n=2}^{\infty}\frac{1}{z^n}, \quad |z| > 1.$$

Example 37.3 For

$$f(z) = \frac{1}{z(z-1)}$$

on $A_0(0,1) = D_0^*(1)$,

$$f(z) = \frac{1}{z}\left(\frac{-1}{1-z}\right) = -\frac{1}{z}\sum_{n=0}^{\infty} z^n = -\frac{1}{z} - \sum_{n=0}^{\infty} z^n, \quad 0 < |z| < 1.$$

Example 37.4 For

$$f(z) = \frac{1}{z(z-1)}$$

on $A_1(0,1)$, the punctured disk centered at 1 with radius 1,

$$\begin{aligned} f(z) &= \frac{1}{z-1} \cdot \frac{1}{1+(z-1)} \\ &= \frac{1}{z-1} \sum_{n=0}^{\infty} (-1)^n (z-1)^n \\ &= \frac{1}{z-1} + \sum_{n=0}^{\infty} (-1)^{n+1} (z-1)^n, \quad 0 < |z-1| < 1. \end{aligned}$$

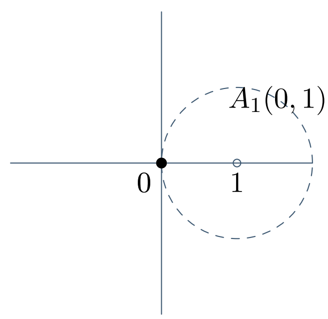

Figure 105

Figure 105 illustrates the setup.

For another example, if f is analytic on Ω and p is a pole of f, then

$$f(z) = \frac{b_m}{(z-p)^m} + \cdots + \frac{b_1}{z-p} + h(z),$$

where h is analytic on $\Omega \cup \{p\}$.

If $D_p(R)$ is the largest disk contained in $\Omega \cup \{p\}$, then

$$h(z) = \sum_{n=0}^{\infty} a_n (z-p)^n \quad \text{on } D_p(R).$$

Hence on $D_p^*(R) = A_p(0,R)$,

$$f(z) = \frac{b_m}{(z-p)^m} + \cdots + \frac{b_1}{z-p} + a_0 + a_1(z-p) + a_2(z-p)^2 + \ldots,$$

so the negative part terminates.

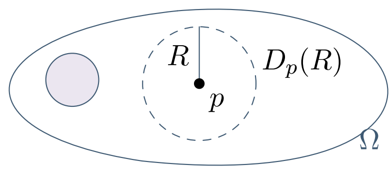

Figure 106

The construction is shown in Figure 106.

Uniqueness of Laurent Series and Laurent's Theorem

Before proving existence for general annuli, let us show uniqueness of Laurent expansions.

Proposition 37.5 The Laurent expansion of f on $A_p(r, R)$ is unique.

Proof. Suppose on $A_p(r, R)$,

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-p)^n} + \sum_{n=0}^{\infty} a_n (z-p)^n.$$

Choose s with $r < s < R$, and let

$$\gamma : [0, 2\pi] \to A_p(r, R), \qquad t \mapsto p + se^{it}.$$

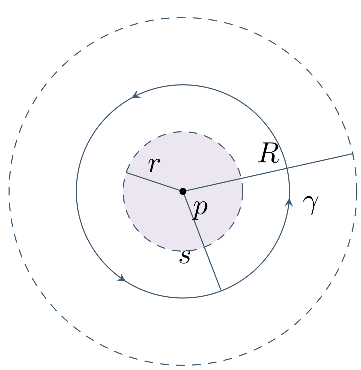

Figure 107

Figure 107 records the relevant geometry.

Because both series converge uniformly on γ^*, we may integrate term by term:

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{n=1}^{\infty} b_n \int_\gamma \frac{\mathrm{d}z}{(z-p)^n} + \sum_{n=0}^{\infty} a_n \int_\gamma (z-p)^n\,\mathrm{d}z = 2\pi i\, b_1.$$

Hence

$$b_1 = \frac{1}{2\pi i}\int_\gamma f(z)\,\mathrm{d}z.$$

Also,

$$(z-p)f(z) = b_1 + \frac{b_2}{z-p} + \frac{b_3}{(z-p)^2} + \cdots + a_0(z-p) + a_1(z-p)^2 + \dots,$$

so

$$b_2 = \frac{1}{2\pi i}\int_\gamma (z-p)f(z)\,\mathrm{d}z.$$

Similarly,

$$b_3 = \frac{1}{2\pi i}\int_\gamma (z-p)^2 f(z)\,\mathrm{d}z,$$

and in general,

$$b_n = \frac{1}{2\pi i}\int_\gamma (z-p)^{n-1} f(z)\,\mathrm{d}z. \tag{37.1}$$

On the other hand,

$$\frac{f(z)}{z-p} = \frac{b_1}{(z-p)^2} + \frac{b_2}{(z-p)^3} + \cdots + \frac{a_0}{z-p} + a_1 + a_2(z-p) + \dots,$$

so

$$a_0 = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-p}\,\mathrm{d}z,$$

and in general,

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-p)^{n+1}}\,\mathrm{d}z. \tag{37.2}$$

Thus f determines all Laurent coefficients by (37.1) and (37.2), so the Laurent expansion on $A_p(r,R)$ is unique. □

Now suppose f has a singularity at p, and on some punctured disk $D_p^*(R) = A_p(0,R)$ it has

Laurent expansion

$$f(z) = \frac{b_1}{z-p} + \frac{b_2}{(z-p)^2} + \cdots + \frac{b_n}{(z-p)^n} + \cdots + a_0 + a_1(z-p) + a_2(z-p)^2 + \ldots.$$

Then: p is removable if and only if all $b_n = 0$; p is a pole if and only if finitely many (but at least one) of the b_n are nonzero; and p is essential if and only if infinitely many b_n are nonzero. So Laurent expansions give a new viewpoint on singularities.

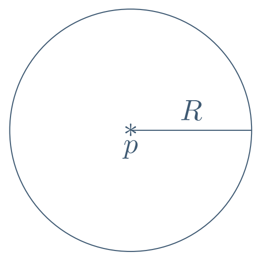

FIGURE 108

The configuration appears in Figure 108.

Let us now prove existence.

Theorem 37.6 (Laurent's theorem) If f is analytic on an annulus $A_p(r,R)$, then f has a unique Laurent expansion on $A_p(r,R)$.

Proof. Choose radii

$$r < r_1 < r_2 < R,$$

and define

$$\begin{aligned} \gamma_1 &: [0,2\pi] \to A_p(r,R), & t &\mapsto p + r_1 e^{it}, \\ \gamma_2 &: [0,2\pi] \to A_p(r,R), & t &\mapsto p + r_2 e^{it}. \end{aligned}$$

Set

$$\gamma = \tilde{\gamma}_1 + \gamma_2.$$

Then γ is homologous to 0 in $A_p(r,R)$. By the adult version of Cauchy's integral formula,

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = f(z)\,\mathrm{ind}_\gamma(z),$$

for all $z \in A_p(r, R) \setminus \gamma^*$.

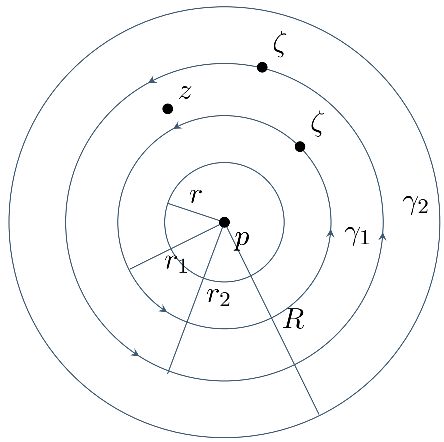

FIGURE 109

Figure 109 illustrates the setup.

If $z \in A_p(r_1, r_2)$, then $\operatorname{ind}_\gamma(z) = 1$, so

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta. \tag{37.3}$$

For $\zeta \in \gamma_1^*$,

$$\begin{aligned}
-\frac{f(\zeta)}{\zeta - z} &= -\frac{f(\zeta)}{(\zeta - p) + (p - z)} \\
&= \frac{f(\zeta)}{(z - p) - (\zeta - p)} \\
&= \frac{f(\zeta)}{z - p} \cdot \frac{1}{1 - \dfrac{\zeta - p}{z - p}} \\
&= \frac{f(\zeta)}{z - p} \sum_{n=0}^{\infty} \left(\frac{\zeta - p}{z - p}\right)^n \qquad \text{since } \left|\frac{\zeta - p}{z - p}\right| < 1 \\
&= \sum_{n=1}^{\infty} f(\zeta)(\zeta - p)^{n-1} \left(\frac{1}{z - p}\right)^n.
\end{aligned}$$

This converges uniformly in $\zeta \in \gamma_1^*$ by the M-test, because

$$\left\| f(\zeta)(\zeta - p)^{n-1} \left(\frac{1}{z-p} \right)^n \right\|_{\zeta \in \gamma_1^*} \leq \|f\|_{\gamma_1^*} \frac{r_1^{n-1}}{|z-p|^n} = \frac{\|f\|_{\gamma_1^*}}{r_1} \left| \frac{r_1}{z-p} \right|^n,$$

and $|r_1/(z-p)| < 1$. So we may integrate term by term:

$$-\frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta = \sum_{n=1}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_1} f(\zeta)(\zeta - p)^{n-1} \, \mathrm{d}\zeta \right) \left(\frac{1}{z-p} \right)^n. \tag{37.4}$$

This is the negative part of the Laurent expansion. For $\zeta \in \gamma_2^*$,

$$\begin{aligned}
\frac{f(\zeta)}{\zeta - z} &= \frac{f(\zeta)}{(\zeta - p) + (p - z)} \\
&= \frac{f(\zeta)}{\zeta - p} \cdot \frac{1}{1 - \dfrac{z-p}{\zeta - p}} \\
&= \frac{f(\zeta)}{\zeta - p} \sum_{n=0}^{\infty} \left(\frac{z-p}{\zeta - p} \right)^n \quad \text{since } \left| \frac{z-p}{\zeta-p} \right| < 1 \\
&= \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - p)^{n+1}} (z-p)^n.
\end{aligned}$$

This also converges uniformly on γ_2^* by the M-test, since

$$\left\| \frac{f(\zeta)}{(\zeta - p)^{n+1}} (z-p)^n \right\|_{\zeta \in \gamma_2^*} \leq \|f\|_{\gamma_2^*} \frac{|z-p|^n}{r_2^{n+1}} = \frac{\|f\|_{\gamma_2^*}}{r_2} \left| \frac{z-p}{r_2} \right|^n,$$

and $|(z-p)/r_2| < 1$. Hence term-by-term integration gives

$$\frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{(\zeta - p)^{n+1}} \, \mathrm{d}\zeta \right) (z-p)^n. \tag{37.5}$$

This is the positive part. Combining (37.3), (37.4), and (37.5), we obtain coefficients a_n, b_n such that

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-p)^n} + \sum_{n=0}^{\infty} a_n (z-p)^n$$

for all $z \in A_p(r_1, r_2)$. To extend from $A_p(r_1, r_2)$ to all of $A_p(r, R)$, observe that any two such intermediate annuli can be joined by a finite chain of intermediate annuli with consecutive overlaps. On each nonempty overlap, Proposition 37.5 forces the coefficients to agree, so the same coefficients propagate along the chain. Hence the coefficients are independent of the intermediate radii. Now every $z \in A_p(r, R)$ belongs to some $A_p(r_1, r_2)$ with

$$r < r_1 < |z - p| < r_2 < R,$$

so the same Laurent series represents f at every point of $A_p(r, R)$. $\square$

Appendix: Lecture and Tutorial Index