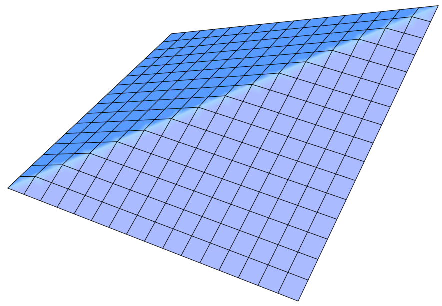

PMATH 451: Measure and Integration

Prof. Christopher Schafhauser • Winter 2016 • University of Waterloo

TRF 8:30–9:20AM in RCH 109

Transcribed, LaTeXed, and edited by Rob Zimmerman

Note: These are personal lecture notes, not official course materials. They may contain errors (which by default you should attribute to me, Rob Zimmerman, rather than the course instructor). The permanent home of these — and all of my other lecture notes — is `https://rob-zimmerman.github.io/coursenotes`.

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Lecture 1 — Tuesday, January 05, 2016

1. Measure Theory Foundations

Measure Theory Foundations

Let X be a compact metric space and let $f : X \to \mathbb{R}$ be continuous. We would like a notion of integrating a real-valued function on X,

$$\int f.$$

At a minimum, this notion should have the following basic properties:

1. We should have
$$\int (af + bg) = a \int f + b \int g,$$
for all $a, b \in \mathbb{R}$.
2. We should also have
$$\int f_n \to \int f$$
whenever $f_n : X \to \mathbb{R}$ are continuous and $f_n \to f$ uniformly.

If $f(X) \subseteq [a, b]$, choose a partition
$$a = t_0 < t_1 < \cdots < t_n = b,$$
and consider the sum
$$\sum_{i=0}^{n-1} t_i \cdot (\text{“size of } f^{-1}([t_i, t_{i+1}))\text{”}).$$
Figure 1 shows how the partition of the range slices the graph of f.

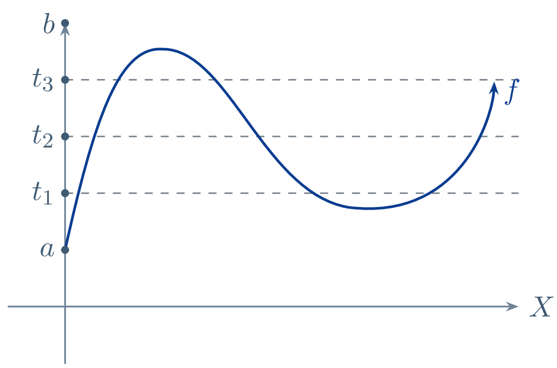

FIGURE 1

The integral
$$\int f$$
should be defined as the limit of these sums. The main issue is to determine exactly what the word “size” should mean. We have a ways to go.

σ-Algebras

Notation 1.1 Let X be a set and let $E \subseteq X$. We write E^c for $X \setminus E$.

Definition 1.2 Let X be a set. An **algebra** $\mathcal{A}$ on X is a collection of subsets of X such that the following hold:

1. $\emptyset \in \mathcal{A}$.
2. If $E_1, E_2 \in \mathcal{A}$, then $E_1 \cup E_2 \in \mathcal{A}$.
3. If $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$.

Fact 1.3 If $\mathcal{A}$ is an algebra on X, then $\mathcal{A}$ is closed under finite intersections and set differences, and $X \in \mathcal{A}$.

Proof. If $E, F \in \mathcal{A}$, then

$$E \cap F = (E^c \cup F^c)^c \in \mathcal{A}.$$

Also,

$$X = \emptyset^c \in \mathcal{A}.$$

Finally, if $E, F \in \mathcal{A}$, then

$$E \setminus F = E \cap F^c \in \mathcal{A}.$$

□

Definition 1.4 A ***σ*-algebra** on X is a collection $\mathcal{A}$ of subsets of X such that the following hold:

1. $\emptyset \in \mathcal{A}$.
2. If $E_1, E_2, \dots \in \mathcal{A}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$.
3. If $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$.

The second condition in the definition of a σ-algebra is stronger than the corresponding condition for an algebra. Countability is king.

Proposition 1.5 Suppose $\mathcal{A}$ is an algebra such that whenever $(E_n)_{n=1}^{\infty} \subseteq \mathcal{A}$ is pairwise disjoint, we have $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$. Then $\mathcal{A}$ is a σ-algebra.

Proof. Fix $(E_n)_{n=1}^{\infty} \subseteq \mathcal{A}$. Define

$$F_1 = E_1 \qquad \text{and} \qquad F_n = E_n \setminus \bigcup_{i=1}^{n-1} E_i, \quad \text{for } n \geq 2.$$

Then the sets F_n are pairwise disjoint, each $F_n \in \mathcal{A}$, and

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n.$$

By assumption, $\bigcup_{n=1}^{\infty} F_n \in \mathcal{A}$, so $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$. Therefore $\mathcal{A}$ is a σ-algebra. $\square$

Proposition 1.6 Suppose $\mathcal{A}$ is an algebra such that whenever $(E_n)_{n=1}^{\infty} \subseteq \mathcal{A}$ satisfies

$$E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots,$$

we have $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$. Then $\mathcal{A}$ is a σ-algebra.

Proof. Let $(E_n)_{n=1}^{\infty} \subseteq \mathcal{A}$ be arbitrary, and define

$$F_n = \bigcup_{k=1}^{n} E_k.$$

Since $\mathcal{A}$ is an algebra, each $F_n \in \mathcal{A}$ and

$$F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots.$$

By assumption,

$$\bigcup_{n=1}^{\infty} F_n \in \mathcal{A}.$$

But

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n.$$

Thus $\mathcal{A}$ is closed under countable unions, so it is a σ-algebra. $\square$

Example 1.7

1. $\mathcal{P}(X) = \{\text{subsets of } X\}$ is a σ-algebra.
2. $\{\emptyset, X\}$ is trivially a σ-algebra.
3. $\mathcal{A} = \{E \subseteq X \mid E \text{ countable or } E^c \text{ countable}\}$ is a σ-algebra. (This is called the **countable-uncountable σ-algebra**.)
4. If $(\mathcal{A}_\lambda)_{\lambda\in\Lambda}$ is a collection of σ-algebras on a set X, then $\bigcap_{\lambda\in\Lambda} \mathcal{A}_\lambda$ is still a σ-algebra.

Proof. (1) and (2) are trivial.

For (3), we verify the conditions in the definition of a σ-algebra. Since $\emptyset$ is countable, $\emptyset \in \mathcal{A}$. If $E \in \mathcal{A}$, then either E or E^c is countable, so the same is true after taking the complement. Finally, suppose $(E_n)_{n=1}^\infty \subseteq \mathcal{A}$. If every E_n is countable, then so is $\bigcup_{n=1}^\infty E_n$. Otherwise, some E_j is uncountable. Since $E_j \in \mathcal{A}$, its complement is countable, and

$$\left(\bigcup_{n=1}^\infty E_n\right)^c \subseteq E_j^c.$$

Thus the complement of the union is countable. In either case, $\bigcup_{n=1}^\infty E_n \in \mathcal{A}$.

For (4), we again verify the conditions in the definition of a σ-algebra. Since $\emptyset \in \mathcal{A}_\lambda$ for every λ, we have $\emptyset \in \bigcap_{\lambda\in\Lambda} \mathcal{A}_\lambda$. If $E \in \bigcap_{\lambda\in\Lambda} \mathcal{A}_\lambda$, then $E^c \in \mathcal{A}_\lambda$ for every λ, so $E^c \in \bigcap_{\lambda\in\Lambda} \mathcal{A}_\lambda$. Finally, if $(E_n)_{n=1}^\infty \subseteq \bigcap_{\lambda\in\Lambda} \mathcal{A}_\lambda$, then $E_n \in \mathcal{A}_\lambda$ for every n and every λ, so $\bigcup_{n=1}^\infty E_n \in \mathcal{A}_\lambda$ for every λ, and hence $\bigcup_{n=1}^\infty E_n \in \bigcap_{\lambda\in\Lambda} \mathcal{A}_\lambda$. $\square$

Definition 1.8 Given a collection $\mathcal{E}$ of subsets of X, the **σ-algebra generated by $\mathcal{E}$**, denoted $\mathcal{M}(\mathcal{E})$, is the intersection of all σ-algebras containing $\mathcal{E}$.

Figure 2 emphasizes the minimality in this definition: $\mathcal{M}(\mathcal{E})$ contains the generators and every set forced by the two closure operations, but nothing else unless those operations require it.

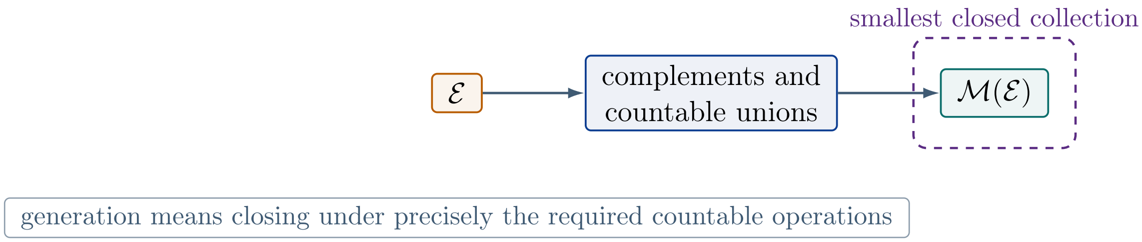

FIGURE 2

Definition 1.9 Let X be a topological space. The **Borel σ-algebra** on X is the σ-algebra generated by the open sets of X. We denote it by $\mathcal{B}_X$, and its elements are called the **Borel sets** of X.

Fact 1.10 Open sets are Borel, and closed sets are Borel.

Proof. Since $\mathcal{B}_X$ is generated by the open sets, every open set is a Borel set. Also, if $E \subseteq X$ is closed, then E^c is open and hence belongs to $\mathcal{B}_X$. Since $\mathcal{B}_X$ is a σ-algebra, $E = (E^c)^c \in \mathcal{B}_X$. Thus every closed set is a Borel set. $\square$

Definition 1.11 A countable intersection of open sets is called a ***G_δ*-set**, and a countable union of closed sets is called an ***F_σ*-set**. A countable union of G_δ-sets is called a ***$G_{\delta\sigma}$*-set**. Similarly, we define $F_{\sigma\delta}$, $G_{\delta\sigma\delta}$, $F_{\sigma\delta\sigma}$, and so on.

The names G_δ and F_σ are derived from the German words *Gebiet* (open set) and *Fläche* (closed set), respectively.

Proposition 1.12 $\mathcal{B}_{\mathbb{R}}$, the Borel σ-algebra on $\mathbb{R}$, is generated by any of the following collections:

$$\begin{aligned}
\mathcal{E}_1 &= \{(a,b) \mid a,b \in \mathbb{R}\} \\
\mathcal{E}_2 &= \{[a,b] \mid a,b \in \mathbb{R}\} \\
\mathcal{E}_3 &= \{(a,b] \mid a,b \in \mathbb{R}\} \\
\mathcal{E}_4 &= \{[a,b) \mid a,b \in \mathbb{R}\} \\
\mathcal{E}_5 &= \{(a,\infty) \mid a \in \mathbb{R}\} \\
\mathcal{E}_6 &= \{(-\infty,a) \mid a \in \mathbb{R}\}
\end{aligned}$$

$$\mathcal{E}_7 = \{[a,\infty) \mid a \in \mathbb{R}\}$$
$$\mathcal{E}_8 = \{(-\infty,a] \mid a \in \mathbb{R}\}$$

Proof. We prove, for example, that $\mathcal{M}(\mathcal{E}_1) = \mathcal{M}(\mathcal{E}_2)$. Fix a nonempty interval $(a,b) \in \mathcal{E}_1$. Then

$$(a,b) = \bigcup_{\substack{n\in\mathbb{N} \\ n>2/(b-a)}} [a+1/n, b-1/n].$$

Since each interval $[a+1/n, b-1/n]$ belongs to $\mathcal{E}_2$, we have $(a,b) \in \mathcal{M}(\mathcal{E}_2)$. Hence $\mathcal{E}_1 \subseteq \mathcal{M}(\mathcal{E}_2)$, and therefore

$$\mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_2).$$

Now fix $[a,b] \in \mathcal{E}_2$. Then

$$[a,b] = \bigcap_{n=1}^{\infty} (a-1/n, b+1/n).$$

Since each interval $(a-1/n, b+1/n)$ belongs to $\mathcal{E}_1$, we have $[a,b] \in \mathcal{M}(\mathcal{E}_1)$. Hence $\mathcal{E}_2 \subseteq \mathcal{M}(\mathcal{E}_1)$, and therefore

$$\mathcal{M}(\mathcal{E}_2) \subseteq \mathcal{M}(\mathcal{E}_1).$$

The other cases are proved similarly. □

Lecture 2 — Thursday, January 07, 2016

Definition 2.1 If $X \neq \emptyset$ and $\mathcal{M}$ is a σ-algebra on X, then the pair $(X,\mathcal{M})$ is called a **measurable space**. A subset $E \subseteq X$ is called **measurable** if $E \in \mathcal{M}$.

Definition 2.2 Let $(X,\mathcal{M})$ be a measurable space. A **measure** is a function $\mu : \mathcal{M} \to [0,\infty]$ such that the following hold:

1. $\mu(\emptyset) = 0$.

2. If $(E_k)_{k=1}^\infty \subseteq \mathcal{M}$ is pairwise disjoint, then
$$\mu\left(\bigcup_{k=1}^\infty E_k\right) = \sum_{k=1}^\infty \mu(E_k).$$

The triple $(X, \mathcal{M}, \mu)$ is then called a **measure space**.

Example 2.3

1. Let X be a set and consider $(X, \mathcal{P}(X))$. Define the **counting measure** $\mu : \mathcal{P}(X) \to [0, \infty]$ by
$$\mu(E) = \begin{cases} |E|, & |E| < \infty, \\ \infty, & |E| = \infty. \end{cases}$$
Plainly, $\mu(\emptyset) = 0$. If $(E_k)_{k=1}^\infty$ is pairwise disjoint and only finitely many points occur in the union, both sides of
$$\mu\left(\bigcup_{k=1}^\infty E_k\right) = \sum_{k=1}^\infty \mu(E_k)$$
count those points. If the union is infinite, then either some E_k is infinite or infinitely many E_k are nonempty; in either case, both sides are infinite. Thus μ is countably additive.

2. Let X be a set and consider $(X, \mathcal{P}(X))$. Fix $x_0 \in X$. Let $\delta_{x_0} : \mathcal{P}(X) \to [0, \infty]$ be given by
$$\delta_{x_0}(E) = \begin{cases} 1, & x_0 \in E, \\ 0, & x_0 \notin E. \end{cases}$$
This is a measure because $\delta_{x_0}(\emptyset) = 0$, and if $E_1, E_2, \ldots \in \mathcal{P}(X)$ are disjoint, then $\bigcup_{k=1}^\infty E_k$ contains x_0 if and only if some E_k contains x_0. Hence δ_{x_0} is countably additive.

3. Let X be infinite and consider $(X, \mathcal{P}(X))$. Define $\mu : \mathcal{P}(X) \to [0, \infty]$ by
$$\mu(E) = \begin{cases} 0 & |E| \leq \aleph_0 \text{ (countable)} \\ \infty & |E| > \aleph_0 \text{ (uncountable)} \end{cases}.$$

Note that $|\emptyset| = 0$, so $\mu(\emptyset) = 0$. If $(E_k) \subseteq \mathcal{P}(X)$ is disjoint, then $\bigcup_{k=1}^{\infty} E_k$ is countable if and only if every E_k is countable. It follows that μ is countably additive.

4. Let $\mathcal{C} = \{E \subseteq X : \text{either } E \text{ is countable or } E^c \text{ is countable}\}$. This is the countable-uncountable σ-algebra from Example 1.7(3). In the previous example, we could restrict our attention to $\mathcal{C}$ and get a measure space $(X, \mathcal{C}, \mu)$.

5. Define $\mu : \mathcal{P}(X) \to [0, 1]$ by

$$\mu(E) = \begin{cases} 0, & |E| < \infty, \\ 1, & |E| = \infty. \end{cases}$$

We have $\mu(\emptyset) = 0$, but μ is not countably additive. If X is infinite, choose distinct points $x_1, x_2, \dots \in X$. Then

$$\mu\left(\bigcup_{k=1}^{\infty} \{x_k\}\right) = 1 \neq \sum_{k=1}^{\infty} \mu(\{x_k\}) = 0.$$

Theorem 2.4 Let $(X, \mathcal{M}, \mu)$ be a measure space. Then the following hold:

1. (**Monotonicity**) If $E \subseteq F$, $E, F \in \mathcal{M}$, then $\mu(E) \leq \mu(F)$.
2. (**$\boldsymbol{\sigma}$-subadditivity**) If $(E_k)_{k=1}^{\infty} \subseteq \mathcal{M}$, then

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu(E_k).$$

3. (**Continuity from below**) If $E_1 \subseteq E_2 \subseteq \dots$ and each $E_k \in \mathcal{M}$, then

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{k \to \infty} \mu(E_k).$$

4. (**Continuity from above**) If $E_1 \supseteq E_2 \supseteq \dots$, each $E_k \in \mathcal{M}$, and if each $\mu(E_k) < \infty$, then $\mu(\bigcap_{k=1}^{\infty} E_k) = \lim_{k \to \infty} \mu(E_k)$.

Proof. For (1), write

$$F = E \cup (F \setminus E).$$

Then

$$\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E).$$

For (2), let $F_1 = E_1$, and for $k \geq 2$, let

$$F_k = E_k \setminus \bigcup_{j=1}^{k-1} E_j.$$

Then $F_k \subseteq E_k$, the sets F_k are pairwise disjoint, and

$$\bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k.$$

Thus

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \sum_{k=1}^{\infty} \mu(F_k) \leq \sum_{k=1}^{\infty} \mu(E_k).$$

For (3), if any $\mu(E_k) = \infty$, then

$$\lim_{k\to\infty} \mu(E_k) = \infty = \mu\left(\bigcup_{k=1}^{\infty} E_k\right).$$

Now suppose that $\mu(E_k) < \infty$ for all k. Since

$$\mu(E_{k+1}) = \mu(E_k) + \mu(E_{k+1} \setminus E_k),$$

we get

$$\mu(E_{k+1} \setminus E_k) = \mu(E_{k+1}) - \mu(E_k).$$

Also, since $E_1 \subseteq E_2 \subseteq \dots$, we have

$$\bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} (E_k \setminus E_{k-1})$$

where $E_0 = \emptyset$. Therefore

$$\begin{aligned}\mu\left(\bigcup_{k=1}^{\infty} E_k\right) &= \mu\left(\bigcup_{k=1}^{\infty} (E_k \setminus E_{k-1})\right) \\ &= \sum_{k=1}^{\infty} \mu(E_k \setminus E_{k-1})\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty}(\mu(E_k) - \mu(E_{k-1})) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}(\mu(E_k) - \mu(E_{k-1})) \\
&= \lim_{n\to\infty}(\mu(E_n) - \mu(E_0)) \\
&= \lim_{n\to\infty}\mu(E_n).
\end{aligned}
$$

For (4), set $F_k = E_1 \setminus E_k$. Then

$$F_1 = \emptyset \subseteq F_2 \subseteq F_3 \subseteq \dots.$$

By (3),

$$\mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \lim_{n\to\infty}\mu(F_n).$$

Also, by De Morgan's laws,

$$\bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty}(E_1 \setminus E_k) = E_1 \setminus \bigcap_{k=1}^{\infty} E_k$$

and since $\mu(E_1) < \infty$, we have

$$\mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \mu(E_1) - \mu\left(\bigcap_{k=1}^{\infty} E_k\right).$$

Hence

$$\mu(E_1) - \mu\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{n\to\infty}\mu(F_n) = \lim_{n\to\infty}(\mu(E_1) - \mu(E_n)).$$

Subtracting $\mu(E_1)$ from both sides gives

$$\mu\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{n\to\infty}\mu(E_n).$$

$\square$

Lecture 3 — Friday, January 08, 2016

We begin by recalling Theorem 2.4. In particular, the finiteness hypothesis in (4) cannot be removed without additional assumptions.

Example 3.1 Let $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu)$ be the counting measure space, and set

$$E_n = \{n, n+1, n+2, \dots\}.$$

Then

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots,$$

and each $\nu(E_n) = \infty$. Hence

$$\lim_{n\to\infty} \nu(E_n) = \infty.$$

However,

$$\bigcap_{n=1}^{\infty} E_n = \emptyset,$$

so

$$\nu\left(\bigcap_{n=1}^{\infty} E_n\right) = 0.$$

Thus continuity from above can fail when no finite measure hypothesis is available.

More generally, the proof of (4) still works if $\mu(E_k) < \infty$ for some k, since we may then begin the argument at that stage.

Definition 3.2 Let $(X, \mathcal{M}, \mu)$ be a measure space.

1. We say that μ is **finite** if $\mu(X) < \infty$.
2. We say that μ is **σ-finite** if there exists a sequence $(X_n)_{n=1}^{\infty} \subseteq \mathcal{M}$ such that

$$X = \bigcup_{n=1}^{\infty} X_n$$

and $\mu(X_n) < \infty$ for every n.

3. We say that μ is **semifinite** if whenever $E \in \mathcal{M}$ satisfies $\mu(E) = \infty$, there exists $F \subseteq E$ with $F \in \mathcal{M}$ and
$$0 < \mu(F) < \infty.$$

Fact 3.3 Every finite measure is σ-finite, and every σ-finite measure is semifinite.

Proof. The first implication is immediate. For the second, suppose μ is σ-finite and let $E \in \mathcal{M}$ satisfy $\mu(E) = \infty$. Choose sets $X_n \in \mathcal{M}$ such that

$$X = \bigcup_{n=1}^{\infty} X_n$$

and $\mu(X_n) < \infty$ for every n. Replacing X_n by $\bigcup_{k=1}^{n} X_k$ if necessary, we may assume that

$$X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots.$$

Set $E_n = E \cap X_n$. Then
$$E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots,$$
and each $\mu(E_n) \leq \mu(X_n) < \infty$. By continuity from below,

$$\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu(E) = \infty.$$

Therefore, at least one E_n satisfies
$$0 < \mu(E_n) < \infty.$$
This shows that μ is semifinite. □

Example 3.4 Let $X \neq \emptyset$ and let ν be counting measure on $(X, \mathcal{P}(X))$. Then ν is finite if and only if $|X| < \infty$, it is σ-finite if and only if $|X| \leq \aleph_0$, and it is always semifinite.

Proof. If $|X| < \infty$, then $\nu(X) = |X| < \infty$, so ν is finite. Conversely, if ν is finite, then $\nu(X) < \infty$, so $|X| < \infty$.

If X is at most countable, we can cover it by a sequence of singletons (and, if X is finite, empty sets), so ν is σ-finite. Conversely, if ν is σ-finite, then there exist sets $X_n \in \mathcal{P}(X)$ such that

$$X = \bigcup_{n=1}^{\infty} X_n$$

and $\nu(X_n) < \infty$ for every n. Since $\nu(X_n) = |X_n|$, we have $|X_n| < \infty$ for every n. Therefore

$$|X| = \left| \bigcup_{n=1}^{\infty} X_n \right| \le \aleph_0.$$

Finally, ν is semifinite since if $E \in \mathcal{P}(X)$ satisfies $\nu(E) = \infty$, then E is infinite, so we can find a finite subset $F \subseteq E$ such that $\nu(F) = |F| > 0$. $\square$

Example 3.5 Let $X \neq \emptyset$ and let $f : X \to [0, \infty]$. For each $E \subseteq X$, define

$$\sum_{x \in E} f(x) = \sup \left\{ \sum_{x \in F} f(x) : F \subseteq E \text{ finite} \right\}.$$

If $E = \{x_1, x_2, \ldots\}$ is countable, then

$$\sum_{x \in E} f(x) = \lim_{N \to \infty} \sum_{n=1}^{N} f(x_n).$$

This is the series version of Tonelli's theorem in (1).

Proof. If E is finite, then the definition of $\sum_{x \in E} f(x)$ is just the usual sum. If E is countably infinite, then for each N, we have

$$\sum_{n=1}^{N} f(x_n) \le \sup \left\{ \sum_{x \in F} f(x) : F \subseteq E \text{ finite} \right\} = \sum_{x \in E} f(x).$$

Thus

$$\lim_{N \to \infty} \sum_{n=1}^{N} f(x_n) \le \sum_{x \in E} f(x).$$

On the other hand, if $F \subseteq E$ is finite, then $F \subseteq \{x_1, \dots, x_N\}$ for some N, so

$$\sum_{x \in F} f(x) \leq \sum_{n=1}^{N} f(x_n).$$

Taking the supremum over all finite $F \subseteq E$ gives

$$\sum_{x \in E} f(x) \leq \lim_{N \to \infty} \sum_{n=1}^{N} f(x_n).$$

$\square$

Fact 3.6 Define $\mu : \mathcal{P}(X) \to [0, \infty]$ by $\mu(E) = \sum_{x \in E} f(x)$. Then μ is a measure. Moreover, μ is finite if and only if $\sum_{x \in X} f(x) < \infty$, it is σ-finite if and only if $\{x \in X : f(x) > 0\}$ is countable and $\{x \in X : f(x) = \infty\} = \emptyset$, and it is semifinite if and only if $\{x \in X : f(x) = \infty\} = \emptyset$.

Proof. We first show that μ is a measure. Plainly, $\mu(\emptyset) = 0$. If $(E_k)_{k=1}^{\infty} \subseteq \mathcal{P}(X)$ is pairwise disjoint, then

$$\begin{aligned}
\mu\left(\bigcup_{k=1}^{\infty} E_k\right) &= \sum_{x \in \bigcup_{k=1}^{\infty} E_k} f(x) \\
&= \sup\left\{\sum_{x \in F} f(x) : F \subseteq \bigcup_{k=1}^{\infty} E_k \text{ finite}\right\} \\
&= \sup\left\{\sum_{k=1}^{N} \sum_{x \in F_k} f(x) : N \geq 1, F_k \subseteq E_k \text{ finite}\right\} \\
&= \sup\left\{\sum_{k=1}^{N} \mu(F_k) : N \geq 1, F_k \subseteq E_k \text{ finite}\right\} \\
&= \sum_{k=1}^{\infty} \mu(E_k).
\end{aligned}$$

Thus μ is a measure.

Now, μ is finite if and only if $\sum_{x \in X} f(x) < \infty$ by definition.

If μ is σ-finite, then there exist sets $X_n \in \mathcal{P}(X)$ such that

$$X = \bigcup_{n=1}^{\infty} X_n$$

and $\mu(X_n) < \infty$ for every n. No point can have infinite weight, because any X_n containing such a point would have infinite measure. Moreover, for every $m \in \mathbb{N}$, the set

$$S_{n,m} = \{x \in X_n : f(x) \geq 1/m\}$$

is finite: otherwise $\mu(X_n) \geq \sum_{x \in S_{n,m}} 1/m = \infty$. Since

$$\{x \in X : f(x) > 0\} = \bigcup_{n,m=1}^{\infty} S_{n,m},$$

the positive support of f is countable. Conversely, suppose the positive support $S = \{x \in X : f(x) > 0\}$ is at most countable and f is finite everywhere. Cover S by its singletons and add the zero set $X \setminus S$. Every set in this countable cover has finite measure, so μ is σ-finite.

Finally, suppose that f is finite everywhere and $\mu(E) = \infty$. Then E contains some x with $f(x) > 0$, and

$$0 < \mu(\{x\}) = f(x) < \infty.$$

Thus μ is semifinite. Conversely, if $f(x) = \infty$ for some x, then $\{x\}$ has infinite measure but contains no subset of finite positive measure. Hence μ is semifinite if and only if f is finite everywhere. $\square$

Definition 3.7 A measure space $(X, \mathcal{M}, \mu)$ is called **complete** if whenever $N \in \mathcal{M}$ satisfies $\mu(N) = 0$ (i.e., N is **μ-null**) and $E \subseteq N$, we also have $E \in \mathcal{M}$.

Outer Measures and Constructions of Measures

Definition 3.8 An **outer measure** on a nonempty set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that the following hold:

1. $\mu^*(\emptyset) = 0$.

2. (**Monotonicity**) If $E \subseteq F$, then $\mu^*(E) \leq \mu^*(F)$.
3. (**$\boldsymbol{\sigma}$-subadditivity**) If $(E_n)_{n\geq 1} \subseteq \mathcal{P}(X)$, then
$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n).$$

For ordinary measures, σ-subadditivity is a theorem. For outer measures, σ-subadditivity is part of the definition.

Proposition 3.9 Let $\mathcal{E} \subseteq \mathcal{P}(X)$ and let $\rho : \mathcal{E} \to [0, \infty]$ be such that $\emptyset \in \mathcal{E}$ and $\rho(\emptyset) = 0$. Then for $A \subseteq X$, the quantity
$$\mu^*(A) = \inf\left\{\sum_{n=1}^{\infty} \rho(E_n) : A \subseteq \bigcup_{n=1}^{\infty} E_n, E_n \in \mathcal{E}\right\}$$
defines an outer measure (with the convention that $\inf \emptyset = \infty$).

Proof. Since $\emptyset \in \mathcal{E}$, $\mu^*(\emptyset) = 0$. If $A \subseteq B$, then any cover of B by a countable family from $\mathcal{E}$ is also a cover of A. Thus $\mu^*(A) \leq \mu^*(B)$. Now, suppose $(A_n)_{n=1}^{\infty} \subseteq \mathcal{P}(X)$. If
$$\sum_{n=1}^{\infty} \mu^*(A_n) = \infty,$$
then the desired inequality follows from
$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) = \infty.$$
Let us suppose that
$$\sum_{n=1}^{\infty} \mu^*(A_n)$$
is finite, so that $\mu^*(A_n) < \infty$. Given $\varepsilon > 0$, let $(E_{n,k})_{k=1}^{\infty} \subseteq \mathcal{E}$ be such that $A_n \subseteq \bigcup_{k=1}^{\infty} E_{n,k}$ and
$$\sum_{k=1}^{\infty} \rho(E_{n,k}) < \mu^*(A_n) + \frac{\varepsilon}{2^n}.$$

Plainly,

$$\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty} E_{n,k}$$

with each $E_{n,k} \in \mathcal{E}$, and

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \rho(E_{n,k}) < \sum_{n=1}^{\infty}\left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon.$$

So

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \rho(E_{n,k}) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon.$$

Since $\varepsilon > 0$ may be chosen arbitrarily,

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

□

This is effectively the only way of constructing outer measures in practice. We will apply this construction to subsets of $\mathbb{R}$ in the next lecture.

Lecture 4 — Tuesday, January 12, 2016

Let us now use outer measures to produce honest measures.

Definition 4.1 Let μ^* be an outer measure on X. A set $A \subseteq X$ is called **μ^*-measurable** if for every set $E \subseteq X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Remark 4.2 Since

$$E = (E \cap A) \cup (E \cap A^c)$$

and μ^* is subadditive, we automatically have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Thus, to prove μ^*-measurability, it suffices to prove the reverse inequality.

Carathéodory's Theorem

Theorem 4.3 (Carathéodory's extension theorem) Let μ^* be an outer measure on a set X. Let $\mathcal{M}$ denote the collection of μ^*-measurable sets in X. Then $\mathcal{M}$ is a σ-algebra and $\mu = \mu^*|_{\mathcal{M}}$ is a complete measure on $(X, \mathcal{M})$. That is, whenever $E \in \mathcal{M}$, $\mu(E) = 0$, and $F \subseteq E$, we have $F \in \mathcal{M}$.

The construction in Figure 3 starts with a set function defined everywhere, identifies the sets on which it splits additively, and then restricts to obtain a complete measure.

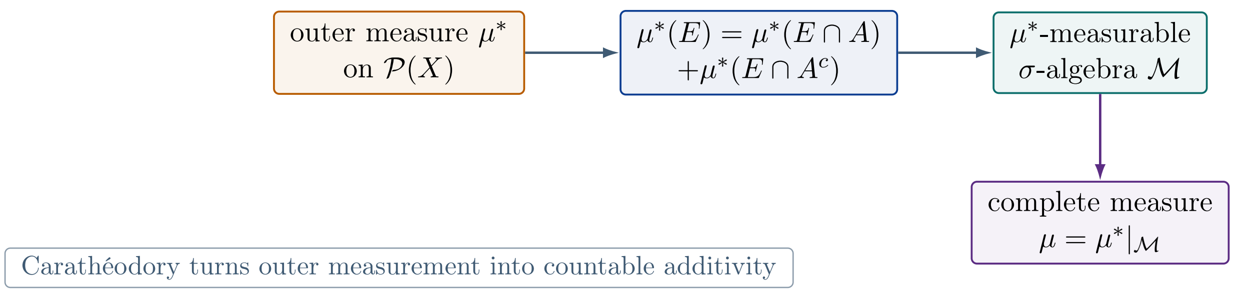

FIGURE 3

Proof. We break the proof into a series of claims.

Claim: $\emptyset \in \mathcal{M}$.

Verification: If $E \subseteq X$, then $\mu^*(E \cap \emptyset) + \mu^*(E \cap \emptyset^c) = \mu^*(\emptyset) + \mu^*(E) = \mu^*(E)$. So $\emptyset$ is μ^*-measurable.

Claim: If $A \in \mathcal{M}$ then $A^c \in \mathcal{M}$.

Verification: Fix $E \subseteq X$. Since $A \in \mathcal{M}$, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$. So $\mu^*(E) = \mu^*(E \cap A^c) + \mu^*(E \cap (A^c)^c)$, so A^c is μ^*-measurable and $A^c \in \mathcal{M}$.

Claim: If $A, B \in \mathcal{M}$, then $A \cup B \in \mathcal{M}$.

Verification: Fix $E \subseteq X$. Then

$$\mu^*(E) = \mu^*(E\cap A)+\mu^*(E\cap A^c) = \mu^*(E\cap A\cap B)+\mu^*(E\cap A\cap B^c)+\mu^*(E\cap A^c\cap B)+\mu^*(E\cap A^c\cap B^c).$$

Note that $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$. By the subadditivity of μ^*,

$$\mu^*(E \cap (A \cup B)) \leq \mu^*(E \cap (A \cap B)) + \mu^*(E \cap (A \cap B^c)) + \mu^*(E \cap (A^c \cap B)).$$

So

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A^c \cap B^c)) = \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$$

which is the inequality we want. So $A \cup B$ is μ^*-measurable and $A \cup B \in \mathcal{M}$.

Claim: If $A, B \in \mathcal{M}$ with $A \cap B = \emptyset$, then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

Verification: Since A is μ^*-measurable,

$$\mu^*(A \cup B) = \mu^*((A \cup B) \cap A) + \mu^*((A \cup B) \cap A^c) = \mu^*(A) + \mu^*(B).$$

Claim: If $(A_n)_{n=1}^\infty \subseteq \mathcal{M}$, then $\bigcup_{n=1}^\infty A_n \in \mathcal{M}$.

Verification: Since $\mathcal{M}$ is an algebra, we may assume that the A_n are disjoint. Set $B_n = A_1 \cup A_2 \cup \cdots \cup A_n$ and $B = \bigcup_{k=1}^\infty A_k$. Fix $E \subseteq X$. Since A_n are μ^*-measurable,

$$\mu^*(E \cap B_n) = \mu^*((E \cap B_n) \cap A_n) + \mu^*((E \cap B_n) \cap A_n^c) = \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}).$$

By induction, we have

$$\mu^*(E \cap B_n) = \sum_{k=1}^n \mu^*(E \cap A_k).$$

Since $B_n \in \mathcal{M}$ ($\mathcal{M}$ is closed under finite unions), we know that

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \geq \sum_{k=1}^n \mu^*(E \cap A_k) + \mu^*(E \cap B^c).$$

Taking the limit as $n \to \infty$,

$$\mu^*(E) \geq \sum_{k=1}^\infty \mu^*(E \cap A_k) + \mu^*(E \cap B^c)$$

$$\begin{aligned}
&\geq \mu^*\left(\bigcup_{k=1}^{\infty}(E \cap A_k)\right) + \mu^*(E \cap B^c) \\
&= \mu^*(E \cap B) + \mu^*(E \cap B^c) \\
&\geq \mu^*(E),
\end{aligned}$$

so all the inequalities are in fact equalities! In particular, $B = \bigcup_{k=1}^{\infty} A_k \in \mathcal{M}$.

Claim: If $(A_k) \subseteq \mathcal{M}$ is a disjoint sequence, then

$$\mu^*\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu^*(A_k).$$

Verification: By our previous work, for every $E \subseteq X$,

$$\mu^*(E) = \sum_{k=1}^{\infty} \mu^*(E \cap A_k) + \mu^*\left(E \cap \left(\bigcup_{k=1}^{\infty} A_k\right)^c\right).$$

When $E = \bigcup_{k=1}^{\infty} A_k$,

$$\mu^*\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu^*(A_k).$$

It only remains to prove completeness of $\mu^*|_{\mathcal{M}}$. Assume $A \in \mathcal{M}$ with $\mu^*(A) = 0$, and let $B \subseteq A$. Since $B \subseteq A$ and $\mu^*(A) = 0$, we also have $\mu^*(B) = 0$. For every $E \subseteq X$,

$$\mu^*(E \cap B) + \mu^*(E \cap B^c) \leq \mu^*(B) + \mu^*(E) = \mu^*(E).$$

The reverse inequality follows from subadditivity because $E = (E \cap B) \cup (E \cap B^c)$. Thus equality holds, so $B \in \mathcal{M}$. Therefore $\mu^*|_{\mathcal{M}}$ is complete. $\square$

Lecture 5 — Thursday, January 14, 2016

Definition 5.1 Let $\mathcal{A}$ be an algebra on a set X. A **pre-measure** on $\mathcal{A}$ is a function $\mu_0 : \mathcal{A} \to [0, \infty]$ such that the following hold:

1. $\mu_0(\emptyset) = 0$.
2. If $(A_n) \subseteq \mathcal{A}$ is a disjoint sequence and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$, then
$$\mu_0\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu_0(A_n).$$

Fact 5.2 If μ_0 is a pre-measure as above, there is an induced outer measure μ^* where
$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu_0(A_n) \mid A_n \in \mathcal{A}, E \subseteq \bigcup_{n=1}^{\infty} A_n\right\}.$$

Proof. This is a special case of Proposition 3.9, where we take $\mathcal{E} = \mathcal{A}$ and $\rho = \mu_0$. □

Lemma 5.3 With the notation above, the following two statements hold:

1. $\mu^*(A) = \mu_0(A)$ for every $A \in \mathcal{A}$.
2. If $A \in \mathcal{A}$, then A is μ^*-measurable.

Proof. For (1), suppose $E \in \mathcal{A}$.

Set $A_1 = E$ and $A_n = \emptyset \in \mathcal{A}$ for $n \geq 2$. Then $(A_n)_{n=1}^{\infty}$ is disjoint and $E = \bigcup_{n=1}^{\infty} A_n$. So
$$\mu^*(E) \leq \sum_{n=1}^{\infty} \mu_0(A_n) = \mu_0(E).$$

Now suppose $E \in \mathcal{A}$ and $(A_n) \subseteq \mathcal{A}$ covers E. We must show that

$$\mu_0(E) \leq \sum_{n=1}^{\infty} \mu_0(A_n).$$

Disjointify the cover by setting

$$C_1 = A_1 \qquad \text{and} \qquad C_n = A_n \setminus \bigcup_{j=1}^{n-1} A_j$$

for $n \geq 2$. Then each $C_n \in \mathcal{A}$, the sets C_n are pairwise disjoint, and $C_n \subseteq A_n$. The sets $B_n = E \cap C_n$ therefore form a pairwise disjoint sequence in $\mathcal{A}$ whose union is E. Since μ_0 is a pre-measure, finite additivity shows that it is monotone: if $B \subseteq A$ in $\mathcal{A}$, then

$$\mu_0(A) = \mu_0(B) + \mu_0(A \setminus B) \geq \mu_0(B).$$

Consequently,

$$\mu_0(E) = \sum_{n=1}^{\infty} \mu_0(B_n) \leq \sum_{n=1}^{\infty} \mu_0(A_n).$$

Hence $\mu_0(E) \leq \mu^*(E)$, and $\mu_0(E) = \mu^*(E)$.

For (2), fix $A \in \mathcal{A}$, $E \subseteq X$, and $\varepsilon > 0$. We need to show that

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(E) + \varepsilon.$$

Choose $B_n \in \mathcal{A}$ such that $E \subseteq \bigcup_{n=1}^{\infty} B_n$ and

$$\sum_{n=1}^{\infty} \mu_0(B_n) \leq \mu^*(E) + \varepsilon.$$

Then

$$\begin{aligned}
\mu^*(E \cap A) + \mu^*(E \cap A^c) &\leq \mu^*\left(\left(\bigcup_{n=1}^{\infty} B_n\right) \cap A\right) + \mu^*\left(\left(\bigcup_{n=1}^{\infty} B_n\right) \cap A^c\right) \\
&\leq \sum_{n=1}^{\infty} (\mu^*(B_n \cap A) + \mu^*(B_n \cap A^c)) \\
&= \sum_{n=1}^{\infty} (\mu_0(B_n \cap A) + \mu_0(B_n \cap A^c))
\end{aligned}$$

$$\begin{aligned} &= \sum_{n=1}^{\infty} \mu_0(B_n) \\ &\leq \mu^*(E) + \varepsilon. \end{aligned}$$

Since this holds for every $E \subseteq X$, letting $\varepsilon \to 0$ shows that A is μ^*-measurable. ☐

Theorem 5.4 (Refinement of Carathéodory's extension theorem) Suppose $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra and $\mathcal{M}$ is the σ-algebra generated by $\mathcal{A}$. If μ_0 is a pre-measure on $\mathcal{A}$, then there exists a measure μ on $\mathcal{M}$ such that $\mu|_{\mathcal{A}} = \mu_0$. Moreover, if μ_0 is σ-finite, then μ is unique.

Proof. Let μ^* be the outer measure induced by μ_0, and let $\widetilde{\mathcal{M}}$ denote the σ-algebra of μ^*-measurable sets. By Carathéodory's theorem, $\widetilde{\mathcal{M}}$ is a σ-algebra, and by Lemma 5.3(2), we have $\mathcal{A} \subseteq \widetilde{\mathcal{M}}$. Hence

$$\mathcal{M} = \sigma(\mathcal{A}) \subseteq \widetilde{\mathcal{M}}.$$

Therefore

$$\mu := \mu^*|_{\mathcal{M}}$$

is a measure on $\mathcal{M}$, and Lemma 5.3(1) gives $\mu|_{\mathcal{A}} = \mu_0$.

Now let ν be another measure on $\mathcal{M}$ such that $\nu|_{\mathcal{A}} = \mu_0$.

Assume first that $\mu_0(X) < \infty$. Fix $E \in \mathcal{M}$ and $\varepsilon > 0$. By the definition of $\mu = \mu^*|_{\mathcal{M}}$, there exist sets $A_j \in \mathcal{A}$ such that

$$E \subseteq \bigcup_{j=1}^{\infty} A_j \qquad \text{and} \qquad \sum_{j=1}^{\infty} \mu(A_j) \leq \mu(E) + \varepsilon.$$

Since $\nu(A_j) = \mu(A_j)$ for every j,

$$\nu(E) \leq \nu\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \nu(A_j) = \sum_{j=1}^{\infty} \mu(A_j) \leq \mu(E) + \varepsilon.$$

Letting $\varepsilon \to 0$ yields $\nu(E) \leq \mu(E)$. Applying the same argument to E^c gives

$$\nu(E^c) \leq \mu(E^c).$$

Because $\nu(X) = \mu_0(X) = \mu(X) < \infty$, we obtain

$$\nu(E) = \nu(X) - \nu(E^c) \geq \mu(X) - \mu(E^c) = \mu(E).$$

Thus $\nu(E) = \mu(E)$ for every $E \in \mathcal{M}$, so $\nu = \mu$ in the finite case.

Now assume μ_0 is σ-finite. Choose sets $X_n \in \mathcal{A}$ such that

$$X = \bigcup_{n=1}^{\infty} X_n \qquad \text{and} \qquad \mu_0(X_n) < \infty \text{ for all } n.$$

Replacing X_n by

$$X_1, \qquad X_2 \setminus X_1, \qquad X_3 \setminus (X_1 \cup X_2), \ldots,$$

we may assume that the sets X_n are pairwise disjoint. For each n, define

$$\mathcal{A}_n := \{A \cap X_n : A \in \mathcal{A}\} \qquad \text{and} \qquad \mathcal{M}_n := \{E \cap X_n : E \in \mathcal{M}\}.$$

Then $\mathcal{A}_n$ is an algebra on X_n, $\mathcal{M}_n = \sigma(\mathcal{A}_n)$, and the restrictions of μ and ν to $\mathcal{M}_n$ both extend the finite pre-measure $\mu_0|_{\mathcal{A}_n}$. By the finite case, these restrictions agree on $\mathcal{M}_n$. Hence, for every $E \in \mathcal{M}$,

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E \cap X_n) = \sum_{n=1}^{\infty} \nu(E \cap X_n) = \nu(E).$$

Therefore the extension is unique when μ_0 is σ-finite. $\square$

Borel Measures

Definition 5.5 A **Borel measure** on $\mathbb{R}$ is a measure $\mu : \mathcal{B}_{\mathbb{R}} \to [0, \infty]$, where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra on $\mathbb{R}$.

Lecture 6 — Friday, January 15, 2016

Definition 6.1 An ***h*-interval** in $\mathbb{R}$ is a set of one of the forms

$$(a,b], \qquad (a,\infty), \qquad \text{and} \qquad \emptyset,$$

where $-\infty \le a < b < \infty$ in the first form and $-\infty \le a < \infty$ in the second.

Fact 6.2 Let

$$\mathcal{A} = \left\{ \bigcup_{i=1}^{n} I_i : I_1, \ldots, I_n \text{ are pairwise disjoint } h\text{-intervals}, n \in \mathbb{N} \right\}.$$

Then $\mathcal{A}$ is an algebra on $\mathbb{R}$.

Proof. First note that $\emptyset \in \mathcal{A}$, since $\emptyset$ is an h-interval.

We claim that if I, J are h-intervals, then $I \setminus J$ is a finite disjoint union of h-intervals (in fact, at most two). Indeed, write $I = (a,b]$ and $J = (c,d]$ with

$$-\infty \le a < b \le \infty \qquad \text{and} \qquad -\infty \le c < d \le \infty.$$

Then $I \setminus J$ is obtained by removing one interval from another, so it is one of the following: $\emptyset$, $(a,b]$, $(a, \min\{b,c\}]$, $(\max\{a,d\}, b]$, or the disjoint union $(a,c] \cup (d,b]$. Each piece is an h-interval.

Now let $E \in \mathcal{A}$ and let J be an h-interval. Write

$$E = \bigcup_{i=1}^{n} I_i$$

with I_i pairwise disjoint h-intervals. Then

$$E \setminus J = \bigcup_{i=1}^{n} (I_i \setminus J),$$

and each $I_i \setminus J$ is a finite disjoint union of h-intervals by the claim. Hence $E \setminus J \in \mathcal{A}$ after collecting all resulting pieces.

If now

$$F = \bigcup_{k=1}^{m} J_k \in \mathcal{A}$$

with J_k pairwise disjoint h-intervals, define

$$E_0 := E \qquad \text{and} \qquad E_k := E_{k-1} \setminus J_k, \quad \text{for } 1 \leq k \leq m.$$

By the previous paragraph, each $E_k \in \mathcal{A}$. Therefore $E \setminus F = E_m \in \mathcal{A}$. Since $\mathbb{R} = (-\infty, \infty]$ is an h-interval, $\mathbb{R} \in \mathcal{A}$. Thus for every $E \in \mathcal{A}$, $E^c = \mathbb{R} \setminus E \in \mathcal{A}$. Hence $\mathcal{A}$ is closed under complements.

Finally, if $E, F \in \mathcal{A}$, then $F \setminus E \in \mathcal{A}$, and

$$E \cup F = E \cup (F \setminus E)$$

is a disjoint union of two sets in $\mathcal{A}$. Hence $E \cup F \in \mathcal{A}$.

So $\mathcal{A}$ contains $\emptyset$, is closed under complements, and is closed under finite unions. Therefore $\mathcal{A}$ is an algebra on $\mathbb{R}$.

$\square$

Proposition 6.3 Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right-continuous. Extend F by

$$F(\infty) := \lim_{x \to \infty} F(x) \qquad \text{and} \qquad F(-\infty) := \lim_{x \to -\infty} F(x).$$

Then there exists a unique pre-measure μ_0 on $\mathcal{A}$ such that

$$\mu_0((a, b]) = F(b) - F(a)$$

for all $-\infty \leq a < b \leq \infty$, where $(a, \infty]$ denotes (a, ∞), and $\mu_0(\emptyset) = 0$.

Proof. Define $\mu_0 : \mathcal{A} \to [0, \infty]$ by $\mu_0(\emptyset) = 0$ and

$$\mu_0\left(\bigcup_{i=1}^{n} (a_i, b_i]\right) = \sum_{i=1}^{n} \big(F(b_i) - F(a_i)\big)$$

whenever the intervals $(a_i, b_i]$ are pairwise disjoint. To see that this is independent of the chosen representation, take the finite collection of all endpoints in two such representations and use them to form a common refinement. On each resulting half-open interval, both sums contribute the same increment of F; summing those increments proves that the two totals agree. Thus μ_0 is well defined. Finite additivity is immediate from the definition. It remains to prove countable additivity in the case where a disjoint union still belongs to $\mathcal{A}$.

Let $(I_n)_{n=1}^{\infty}$ be pairwise disjoint h-intervals such that $\bigcup_{n=1}^{\infty} I_n \in \mathcal{A}$. Each I_n lies in exactly one of the finitely many connected components of the union, so we may regroup the series component by component. It therefore suffices to consider the case in which

$$\bigcup_{n=1}^{\infty} I_n = (a, b]$$

is itself an h-interval. Write $I_n = (a_n, b_n]$. We must show that

$$\mu_0((a,b]) = \sum_{n=1}^{\infty} \mu_0((a_n, b_n]).$$

For every $N \geq 1$,

$$\mu_0((a,b]) \geq \mu_0\left(\bigcup_{n=1}^{N} (a_n, b_n]\right) = \sum_{n=1}^{N} \mu_0((a_n, b_n]),$$

so

$$\sum_{n=1}^{\infty} \mu_0((a_n, b_n]) \leq \mu_0((a,b]).$$

It remains to prove the reverse inequality.

Case 1: $a, b \in \mathbb{R}$. Let $\varepsilon > 0$. Choose $\delta > 0$ such that

$$F(a+\delta) - F(a) < \varepsilon.$$

For each $n \geq 1$, choose $\delta_n > 0$ such that

$$F(b_n + \delta_n) - F(b_n) < \frac{\varepsilon}{2^n}.$$

Then the open intervals $(a_n, b_n + \delta_n)$ cover $[a+\delta, b]$. By compactness, there exist indices $1, \ldots, N$ whose corresponding intervals still cover $[a+\delta, b]$. Reindexing if necessary, we may assume

$$a_{i+1} < b_i + \delta_i < b_{i+1} + \delta_{i+1} \quad \text{for } 1 \leq i \leq N-1.$$

Therefore

$$\begin{aligned}
\mu_0((a,b]) = F(b) - F(a) &\leq F(b) - F(a+\delta) + \varepsilon \\
&\leq F(b_N + \delta_N) - F(a_1) + \varepsilon \\
&= F(b_N + \delta_N) - F(a_N) + \sum_{i=1}^{N-1} \big(F(a_{i+1}) - F(a_i)\big) + \varepsilon \\
&\leq \sum_{i=1}^{N} \big(F(b_i + \delta_i) - F(a_i)\big) + \varepsilon \\
&\leq \sum_{i=1}^{N} \big(F(b_i) - F(a_i)\big) + \sum_{i=1}^{N} \frac{\varepsilon}{2^i} + \varepsilon \\
&\leq \sum_{i=1}^{N} \mu_0((a_i, b_i]) + 2\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$\mu_0((a,b]) \leq \sum_{n=1}^{\infty} \mu_0((a_n, b_n]).$$

Case 2: $a \in \mathbb{R}$ and $b = \infty$. Fix $M > a$ and set

$$J_i = (a_i, b_i] \cap (a, M].$$

The nonempty sets J_i are pairwise disjoint h-intervals whose union is $(a, M]$. Applying Case 1 to this truncated decomposition and using monotonicity gives

$$F(M) - F(a) = \sum_{i=1}^{\infty} \mu_0(J_i) \leq \sum_{i=1}^{\infty} \mu_0((a_i, b_i]).$$

Letting $M \to \infty$ yields the claim.

Case 3: $a = -\infty$. If $b < \infty$, intersect the intervals $(a_i, b_i]$ with $(M, b]$ and apply Case 1 as above. This gives

$$F(b) - F(M) \leq \sum_{i=1}^{\infty} \mu_0((a_i, b_i]),$$

and the desired inequality follows as $M \to -\infty$. If $b = \infty$ as well, apply Case 1 to the intersections with $(M, L]$, and then let $M \to -\infty$ and $L \to \infty$.

Thus μ_0 is a pre-measure on $\mathcal{A}$. Uniqueness is immediate because every element of $\mathcal{A}$ is a finite disjoint union of h-intervals. $\square$

Theorem 6.4

1. If $F : \mathbb{R} \to \mathbb{R}$ is increasing and right-continuous, then there exists a unique Borel measure μ on $\mathbb{R}$ such that
$$\mu((a, b]) = F(b) - F(a)$$
for all $a, b \in \mathbb{R}$ with $a \leq b$.

2. If μ is a Borel measure on $\mathbb{R}$ which is finite on compact sets, then there exists an increasing right-continuous function $F : \mathbb{R} \to \mathbb{R}$ such that
$$\mu((a, b]) = F(b) - F(a)$$
for all $a, b \in \mathbb{R}$ with $a \leq b$. If G is another such function, then $G - F$ is constant (i.e., F is unique up to an additive constant).

Proof. For (1), Proposition 6.3 yields a pre-measure μ_0 on $\mathcal{A}$ satisfying
$$\mu_0((a, b]) = F(b) - F(a).$$
By the Carathéodory extension theorem, μ_0 extends to a measure μ on $\mathcal{B}_{\mathbb{R}}$. To prove uniqueness, it suffices to show that μ_0 is σ-finite. This is immediate because
$$\mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n] \qquad \text{and} \qquad \mu_0((-n, n]) = F(n) - F(-n) < \infty.$$

For (2), define $F : \mathbb{R} \to \mathbb{R}$ by
$$F(x) = \begin{cases} \mu((0, x]) & x > 0 \\ 0 & x = 0 \\ -\mu((x, 0]) & x < 0 \end{cases},$$
which is the distribution function associated with μ. The function F is increasing. Moreover, for $0 < h \leq 1$,
$$F(x + h) - F(x) = \mu((x, x + h]).$$
The intervals on the right decrease to the empty set as $h \downarrow 0$ and are contained in the compact interval $[x, x+1]$, which has finite measure. Continuity from above therefore shows that $F(x+h) \to$

$F(x)$, so F is right-continuous.

Now let G be another increasing right-continuous function with

$$\mu((a,b]) = G(b) - G(a)$$

for all $a \leq b$. If $x > 0$, then

$$G(x) - G(0) = \mu((0,x]) = F(x).$$

If $x < 0$, then

$$G(0) - G(x) = \mu((x,0]) = -F(x),$$

so again $G(x) - F(x) = G(0)$. Therefore $G - F$ is constant on $\mathbb{R}$. □

Lecture 7 — Tuesday, January 19, 2016

2. Measurable Functions and Integration

Radon Measures

Definition 7.1 A **Radon measure** on $\mathbb{R}$ is a Borel measure μ such that $\mu(K) < \infty$ for every compact set $K \subseteq \mathbb{R}$.

By Theorem 6.4, Radon measures on $\mathbb{R}$ are in bijection with increasing right-continuous functions on $\mathbb{R}$, modulo additive constants. That should be quite useful.

Notation 7.2 Given a Radon measure μ on $\mathbb{R}$, define the outer measure $\mu^* : \mathcal{P}(\mathbb{R}) \to [0,\infty]$ by

$$\mu^*(E) = \inf\left\{\sum_{i=1}^{\infty} \mu(A_i) : A_i \in \mathcal{B}_{\mathbb{R}},\ E \subseteq \bigcup_{i=1}^{\infty} A_i\right\}.$$

Let $\mathcal{M}_\mu$ denote the μ^*-measurable sets, and set

$$\overline{\mu} := \mu^*|_{\mathcal{M}_\mu}.$$

Then $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}_\mu$ and $\overline{\mu}|_{\mathcal{B}_{\mathbb{R}}} = \mu$. The measure $\overline{\mu}$ is called the **completion** of μ.

$$\begin{tikzcd}
\mathcal{P}(\mathbb{R}) \arrow[dr, bend left, "\mu^*"] & \\
\mathcal{M}_\mu \arrow[r, "\overline{\mu}"] & {[0,\infty]} \\
\mathcal{B}_{\mathbb{R}} \arrow[ur, bend right, "\mu"'] &
\end{tikzcd}$$

FIGURE 4

The inclusions and their associated measures are summarized in Figure 4.

Usually we do not distinguish notationally between μ and its completion $\overline{\mu}$ unless the distinction is relevant.

Example 7.3 Take $F(x) = x$. The corresponding Radon measure m satisfies

$$m((a, b]) = b - a.$$

This is **Lebesgue measure** on $\mathbb{R}$. We write

$$\mathcal{L}_{\mathbb{R}} := \mathcal{M}_m,$$

and the sets in $\mathcal{L}_{\mathbb{R}}$ are called **Lebesgue measurable**.

Lemma 7.4 The σ-algebra generated by a countable family has cardinality at most $2^{\aleph_0}$.

Proof. Let $\mathcal{F} = \{A_n : n \geq 1\}$ be a countable family of subsets of a set X. Every element of $\mathcal{M}(\mathcal{F})$ can be described by a countable **Borel code**: the code records which generators are used and the countable sequence of complement and countable union operations used to form the set. There are only $2^{\aleph_0}$ such countable codes, since each code can be encoded by a subset of $\mathbb{N}$. Hence there is a surjection from a set of cardinality $2^{\aleph_0}$ onto $\mathcal{M}(\mathcal{F})$, and therefore

$$|\mathcal{M}(\mathcal{F})| \leq 2^{\aleph_0}.$$

□

Proposition 7.5 There exist Lebesgue measurable sets which are not Borel measurable.

Proof. Let C denote the standard Cantor set. Starting from $C_0 = [0, 1]$, form C_n by deleting the open middle third of every interval in C_{n-1}, and set

$$C = \bigcap_{n=0}^{\infty} C_n.$$

The Cantor set is closed, nowhere dense, and has Lebesgue measure $m(C) = 0$; indeed, C is covered by the 2^n intervals in C_n, each of length 3^{-n}, so

$$m(C) \leq 2^n \cdot 3^{-n} = (2/3)^n \to 0 \text{ as } n \to \infty.$$

Since C is a null set for m, every subset $E \subseteq C$ is m^*-measurable by completeness of the extended measure $\overline{m}$, and $\overline{m}(E) = 0$. Thus every subset of C is Lebesgue measurable.

Now, C is uncountable with $|C| = 2^{\aleph_0}$. Therefore,

$$|\mathcal{P}(C)| = 2^{|C|} = 2^{2^{\aleph_0}}.$$

On the other hand, there are only $2^{\aleph_0}$ many Borel subsets of $\mathbb{R}$. This is because the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ is generated by countably many sets (the open intervals with rational endpoints), and the σ-algebra generated by a countable family has cardinality at most $2^{\aleph_0}$ by Lemma 7.4; on the other hand, $\mathcal{B}_{\mathbb{R}}$ contains all singletons $\{x\}$ for $x \in \mathbb{R}$, so $|\mathcal{B}_{\mathbb{R}}| = 2^{\aleph_0}$.

Thus there are $|\mathcal{P}(C) \setminus \mathcal{B}(\mathbb{R})| = 2^{2^{\aleph_0}}$ sets that are Lebesgue measurable but not Borel measurable.

□

Example 7.6 Fix $a \in \mathbb{R}$ and define

$$F(x) = \begin{cases} 1, & x \geq a, \\ 0, & x < a. \end{cases}$$

Let δ_a denote the corresponding Radon measure. Then

$$\delta_a(E) = \begin{cases} 1, & a \in E, \\ 0, & a \notin E, \end{cases}$$

and in this case every subset of $\mathbb{R}$ is δ_a-measurable, so

$$\mathcal{M}_{\delta_a} = \mathcal{P}(\mathbb{R}).$$

This measure is called the **Dirac measure** at a.

Regularity of Radon Measures

Lemma 7.7 Let μ be a Radon measure. If $E \in \mathcal{M}_\mu$, then

$$\mu(E) = \inf\{\mu(U) \mid E \subseteq U, U \text{ open}\}.$$

Proof. If $\mu(E) = \infty$, the claimed identity is immediate: every open set containing E also has infinite measure. Assume $\mu(E) < \infty$.

We know that

$$\mu(E) = \inf\left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i]) \mid E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \right\}.$$

Fix $\varepsilon > 0$ and choose intervals $(a_i, b_i]$ with $E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i]$ and

$$\sum_{i=1}^{\infty} \mu((a_i, b_i]) < \mu(E) + \varepsilon.$$

Let $F : \mathbb{R} \to \mathbb{R}$ be the function corresponding to μ. For each i with $b_i < \infty$, choose $\delta_i > 0$ such that

$$F(b_i + \delta_i) - F(b_i) < \frac{\varepsilon}{2^i},$$

and set

$$J_i = \begin{cases} (a_i, b_i + \delta_i), & b_i < \infty, \\ (a_i, \infty), & b_i = \infty. \end{cases}$$

The set $U = \bigcup_{i=1}^{\infty} J_i$ is open and contains E. Moreover,

$$\begin{aligned}
\mu(U) &\leq \sum_{i=1}^{\infty} \mu(J_i) \\
&\leq \sum_{b_i<\infty} (F(b_i+\delta_i) - F(a_i)) + \sum_{b_i=\infty} (F(\infty) - F(a_i)) \\
&= \sum_{b_i<\infty} (F(b_i+\delta_i) - F(b_i)) + \sum_{i=1}^{\infty} (F(b_i) - F(a_i)) \\
&< \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} + \sum_{i=1}^{\infty} \mu((a_i, b_i]) \\
&< \mu(E) + 2\varepsilon.
\end{aligned}$$

Thus there is an open set containing E whose measure is less than $\mu(E) + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, this proves

$$\mu(E) = \inf\{\mu(U) : E \subseteq U,\ U \text{ open}\}.$$

□

Theorem 7.8 Let μ be a Radon measure on $\mathbb{R}$ and let $E \in \mathcal{M}_\mu$ be given. Then the following hold:

1. $\mu(E) = \inf\{\mu(U) \mid E \subseteq U, U \text{ open}\}$.
2. $\mu(E) = \sup\{\mu(K) \mid K \subseteq E, K \text{ compact}\}$.

These are called **regularity properties** of μ.

Figure 5 gives the geometric form of regularity: we can squeeze a measurable set between a compact core and an open neighbourhood while making both measure errors arbitrarily small.

Proof. (1) follows from Lemma 7.7.

For (2), first assume that E is bounded. Choose a compact interval K_0 with $E \subseteq K_0$. Since K_0 is compact, $\mu(K_0) < \infty$. By outer regularity, there is an open set $U \supseteq K_0 \setminus E$ such that

$$\mu(U) < \mu(K_0 \setminus E) + \varepsilon.$$

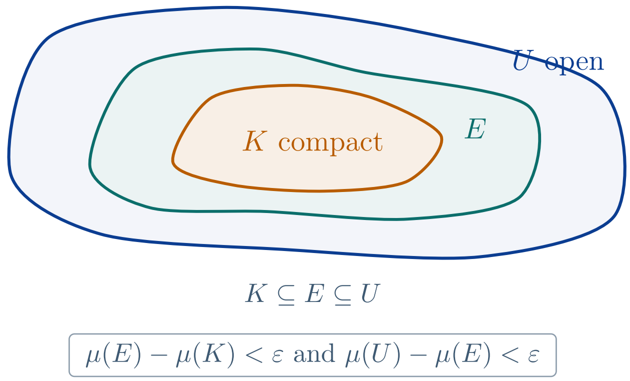

FIGURE 5

Set $K = K_0 \setminus U$. Then K is compact and $K \subseteq E$. Moreover,

$$\mu(K) = \mu(K_0) - \mu(K_0 \cap U) \geq \mu(K_0) - \mu(U) > \mu(K_0) - \mu(K_0 \setminus E) - \varepsilon = \mu(E) - \varepsilon.$$

Thus the supremum over compact subsets of E is at least $\mu(E)$. The reverse inequality is immediate from monotonicity, so (2) holds for bounded E.

Now let E be arbitrary. If $\mu(E) < \infty$, choose N large enough that

$$\mu(E \cap [-N, N]) > \mu(E) - \varepsilon/2$$

by continuity from below. By the bounded case, there is a compact set $K \subseteq E \cap [-N, N]$ with

$$\mu(K) > \mu(E \cap [-N, N]) - \varepsilon/2 > \mu(E) - \varepsilon.$$

If $\mu(E) = \infty$, then for every $M > 0$ there is an N such that $\mu(E \cap [-N, N]) > M + 1$. Applying the bounded case gives a compact set $K \subseteq E \cap [-N, N]$ with $\mu(K) > M$. Hence the supremum over compact subsets of E is infinite. This completes the proof. $\square$

Lecture 8 — Thursday, January 21, 2016

We assume the axiom of choice throughout.

Theorem 8.1 Let μ be a Radon measure on $\mathbb{R}$ and let $E \in \mathcal{M}_\mu$. Then the following statements are equivalent:

1. The set E is equal to a Borel set up to a null set.
2. There are a G_δ set V and a set $N \in \mathcal{M}_\mu$ with $\mu(N) = 0$ such that $E = V \setminus N$.
3. There are an F_σ set H and a set $N \in \mathcal{M}_\mu$ with $\mu(N) = 0$ such that $E = H \cup N$.

Proof. (2) *implies* (1). This is immediate because every G_δ set is Borel.

(3) *implies* (1). This is immediate because every F_σ set is Borel.

(1) *implies* (3). In fact, the following argument proves the conclusion directly for every $E \in \mathcal{M}_\mu$. Assume first that $\mu(E) < \infty$. By (2), choose compact sets $K_n \subseteq E$ such that

$$\mu(E) \leq \mu(K_n) + \frac{1}{n}.$$

Set $H = \bigcup_{n=1}^\infty K_n$. Then $H \subseteq E$ is an F_σ set and

$$\mu(H) \geq \mu(K_n) \geq \mu(E) - \frac{1}{n}$$

for every n. Hence $\mu(H) = \mu(E)$, so $N = E \setminus H$ is null and $E = H \cup N$.

For general E, apply the finite measure case to each set $E \cap [-k, k]$. Choose F_σ sets H_k and null sets N_k such that

$$E \cap [-k, k] = H_k \cup N_k$$

for every $k \in \mathbb{N}$. Then $H = \bigcup_{k=1}^\infty H_k$ is an F_σ set and $N = \bigcup_{k=1}^\infty N_k$ is null. Also

$$E = \bigcup_{k=1}^\infty (E \cap [-k, k]) = \bigcup_{k=1}^\infty (H_k \cup N_k) = H \cup N.$$

(1) *implies* (2). Apply the previous implication to E^c. Then $E^c = H \cup N$ for some F_σ set H and some null set N. Therefore

$$E = (H \cup N)^c = H^c \cap N^c = H^c \setminus N.$$

Since H is an F_σ set, H^c is a G_δ set, and N is null. $\square$

Remark 8.2 Theorems 7.8 and 8.1 remain valid for Radon measures on separable locally compact metric spaces. We will return to that generality later in the course.

Measurable Functions

Definition 8.3 Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. A function $f : X \to Y$ is called **measurable** if

$$f^{-1}(E) \in \mathcal{M}$$

for every $E \in \mathcal{N}$. We also say that f is **$(\mathcal{M}, \mathcal{N})$-measurable**.

Fact 8.4 Compositions of measurable functions are measurable.

Proof. Suppose $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable and $g : Y \to Z$ is $(\mathcal{N}, \mathcal{P})$-measurable. If $E \in \mathcal{P}$, then

$$(g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E)) \in \mathcal{M},$$

so $g \circ f$ is $(\mathcal{M}, \mathcal{P})$-measurable. $\square$

Proposition 8.5 Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces, and suppose that $\mathcal{N} = \sigma(\mathcal{E})$ for some family $\mathcal{E} \subseteq \mathcal{N}$. If $f : X \to Y$ satisfies

$$f^{-1}(E) \in \mathcal{M}$$

for every $E \in \mathcal{E}$, then f is measurable.

Proof. Define

$$\mathcal{F} := \{E \subseteq Y : f^{-1}(E) \in \mathcal{M}\}.$$

Then $\mathcal{F}$ is a σ-algebra. Indeed, $Y \in \mathcal{F}$ because $f^{-1}(Y) = X \in \mathcal{M}$. If $E \in \mathcal{F}$, then

$$f^{-1}(E^c) = X \setminus f^{-1}(E) \in \mathcal{M},$$

so $E^c \in \mathcal{F}$. If $E_n \in \mathcal{F}$ for all n, then

$$f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(E_n) \in \mathcal{M},$$

so $\bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$. By assumption, $\mathcal{E} \subseteq \mathcal{F}$. Since $\mathcal{F}$ is a σ-algebra containing $\mathcal{E}$, it also contains $\sigma(\mathcal{E}) = \mathcal{N}$. Therefore $f^{-1}(A) \in \mathcal{M}$ for every $A \in \mathcal{N}$, so f is measurable. □

Corollary 8.6 If X, Y are metric spaces and $f : X \to Y$ is continuous, then f is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.

Proof. If $U \subseteq Y$ is open, then $f^{-1}(U)$ is open and so $f^{-1}(U) \in \mathcal{B}_X$. Now apply Proposition 8.5 with $\mathcal{E} = \{U \subseteq Y : U \text{ is open}\}$. □

For functions with codomain $\mathbb{R}$ or $\mathbb{C}$, we always equip the codomain with its Borel σ-algebra.

Definition 8.7 A function $f : \mathbb{R} \to \mathbb{R}$ is **Borel measurable** if it is $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}})$-measurable. It is **Lebesgue measurable** if it is $(\mathcal{L}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}})$-measurable.

Warning If $f, g : \mathbb{R} \to \mathbb{R}$ are both Lebesgue measurable, then $f \circ g$ need not be Lebesgue measurable, even when g is continuous!

Fact 8.8 If $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable and $g : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, then $f \circ g$ is Lebesgue measurable.

Proof. Suppose $U \subseteq \mathbb{R}$ is open. $(f \circ g)^{-1}(U) = g^{-1}(f^{-1}(U))$. Since f is Borel, $f^{-1}(U)$ is Borel. So $g^{-1}(f^{-1}(U)) \in \mathcal{L}_{\mathbb{R}}$. □

Lemma 8.9 Let $(X, \mathcal{M})$ be given. If $f, g : X \to \mathbb{R}$ are measurable, then $f \times g : X \to \mathbb{R} \times \mathbb{R}$ is measurable, where $x \mapsto (f(x), g(x))$.

Proof. $\mathcal{B}_{\mathbb{R}\times\mathbb{R}}$ is generated by $\{(a,b)\times(c,d) \mid a,b,c,d\in\mathbb{R}\}$. Then

$$(f\times g)^{-1}((a,b)\times(c,d)) = f^{-1}((a,b))\cap g^{-1}((c,d))\in\mathcal{M}.$$

□

Proposition 8.10 If $f,g : X\to\mathbb{R}$ are measurable, then $f+g$ and fg are measurable as functions from X to $\mathbb{R}$.

Proof. Let $A:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be $(a,b)\mapsto a+b$ and $M:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be $(a,b)\mapsto ab$. A and M are both continuous. We have that $f+g=A\circ(f\times g)$ and $fg=M\circ(f\times g)$ are both measurable by Lemma 8.9 and Corollary 8.6. □

Lecture 9 — Friday, January 22, 2016

Notation 9.1 Let $\overline{\mathbb{R}}=[-\infty,\infty]$ with metric $d(x,y)=|\arctan(x)-\arctan(y)|$. Every open set is a countable union of intervals of the forms $[-\infty,a)$, $(a,\infty]$, and (a,b), where

$$-\infty\le a\le b\le\infty.$$

The Borel σ-algebra is generated by the sets $[a,\infty]$ with $a\in\mathbb{R}$.

Proposition 9.2 Let $(X,\mathcal{M})$ be a measurable space and let $\{f_j\}_{j=1}^{\infty}\subseteq\overline{\mathbb{R}}^X$ be measurable. Then $\sup f_j, \inf f_j, \limsup f_j, \liminf f_j$ are measurable.

Proof. Set $g=\sup_j f_j$ and fix $a\in\mathbb{R}$. We need to show $g^{-1}((a,\infty])\in\mathcal{M}$. If $x\in X$, then $g(x)>a$ if and only if $f_j(x)>a$ for some $j\in\mathbb{N}$. Now, $g^{-1}((a,\infty])=\bigcup_{j=1}^{\infty} f_j^{-1}((a,\infty])\in\mathcal{M}$. Note $\inf(f_j)=-\sup(-f_j)$ and

$$\limsup f_j=\inf_{j\ge1}\sup_{k\ge j} f_k$$

and

$$\liminf f_j=\sup_{j\ge1}\inf_{k\ge j} f_k.$$

Since measurability is preserved under negation, countable suprema, and countable infima, the remaining functions are measurable as well. $\square$

Proposition 9.3 If $f, g : X \to \mathbb{C}$ are measurable, then $f+g$, fg, $\overline{f}$, $\mathrm{Re}(f)$, and $\mathrm{Im}(f)$ are measurable. Moreover, if $(f_j)_{j=1}^{\infty}$ is a sequence of measurable functions converging pointwise to a function $f : X \to \mathbb{C}$, then f is measurable.

Proof. The maps

$$(z,w) \mapsto z+w, \qquad (z,w) \mapsto zw, \qquad z \mapsto \overline{z}, \qquad z \mapsto \mathrm{Re}(z), \qquad \text{and} \qquad z \mapsto \mathrm{Im}(z)$$

are continuous, hence Borel measurable. Therefore the first claim follows from the real-valued case together with the fact that

$$x \mapsto (f(x), g(x))$$

is measurable.

Now suppose $f_j \to f$ pointwise. Then $\mathrm{Re}(f_j) \to \mathrm{Re}(f)$ and $\mathrm{Im}(f_j) \to \mathrm{Im}(f)$ pointwise. By the real-valued part of this proposition, $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are measurable. Since

$$f = \mathrm{Re}(f) + i\,\mathrm{Im}(f),$$

the function f is measurable. $\square$

Definition 9.4 Let $E \subseteq X$. The **characteristic function** of E is the function $\mathbb{1}_E : X \to \mathbb{R}$ defined by

$$\mathbb{1}_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

Fact 9.5 $\mathbb{1}_E$ is measurable if and only if E is measurable.

Proof. *Forward implication.* If $\mathbb{1}_E$ is measurable, then

$$E = \mathbb{1}_E^{-1}(\{1\})$$

is measurable.

Reverse implication. Assume E is measurable, and let $B \subseteq \mathbb{R}$ be a Borel set. Then $\mathbb{1}_E^{-1}(B)$ is one of the sets

$$\emptyset, \qquad E, \qquad E^c, \qquad \text{and} \qquad X.$$

All four sets are measurable, so $\mathbb{1}_E$ is measurable. $\square$

Definition 9.6 A measurable function $\varphi : X \to \mathbb{C}$ is called a **simple function** if its image is a finite set.

Fact 9.7 If $\varphi, \psi : X \to \mathbb{C}$ are simple, then so are $\varphi + \psi$ and $\varphi \cdot \psi$.

Proof. If $\varphi(X) = \{a_1, \ldots, a_n\}$ and $\psi(X) = \{b_1, \ldots, b_m\}$, then

$$(\varphi + \psi)(X) \subseteq \{a_i + b_j : 1 \leq i \leq n, 1 \leq j \leq m\}$$

and

$$(\varphi \cdot \psi)(X) \subseteq \{a_i b_j : 1 \leq i \leq n, 1 \leq j \leq m\},$$

so $\varphi + \psi$ and $\varphi \cdot \psi$ are simple. $\square$

Theorem 9.8 Let $(X, \mathcal{M})$ be a measurable space and $f : X \to [0, \infty]$ be a measurable function. Then there are simple functions φ_n on X such that the following hold:

1. We have
$$0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq f.$$
2. We have $\varphi_n(x) \to f(x)$ for all $x \in X$.
3. If $E \subseteq X$ with $f|_E$ bounded, then $\varphi_n \to f$ uniformly.

Proof. For each $n \in \mathbb{N}$ and each integer k with $0 \leq k \leq n2^n - 1$, define

$$E_k^{(n)} := f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right) \qquad \text{and} \qquad F_n := f^{-1}([n, \infty]).$$

All of these sets are measurable because f is measurable. Now define

$$\varphi_n = \sum_{k=0}^{n\cdot 2^n-1} \frac{k}{2^n}\mathbb{1}_{E_k^{(n)}} + n\mathbb{1}_{F_n}.$$

Then each φ_n is a simple function and, for $n \geq 2$, $0 \leq \varphi_{n-1} \leq \varphi_n \leq f$.

If $f(x) \geq n$, then $\varphi_n(x) = n$. If $f(x) < n$, then there is a unique integer k with

$$\frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n},$$

and hence

$$\varphi_n(x) = \frac{k}{2^n}.$$

Therefore

$$0 \leq f(x) - \varphi_n(x) < \frac{1}{2^n}$$

whenever $f(x) < n$. It follows that $\varphi_n(x) \to f(x)$ for every $x \in X$.

The functions are increasing in n: each refinement replaces $\varphi_n(x)$ by a finer lower approximation to the same value $f(x)$, so

$$0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq f.$$

This proves (1) and (2).

For (3), suppose $f|_E$ is bounded, say $f(x) \leq M$ for all $x \in E$. If $n \geq M$, then the truncation term does not occur on E, and for every $x \in E$ we have

$$0 \leq f(x) - \varphi_n(x) < \frac{1}{2^n}.$$

Hence

$$\sup_{x\in E} |f(x) - \varphi_n(x)| \leq \frac{1}{2^n} \to 0,$$

so $\varphi_n \to f$ uniformly on E. $\square$

Integration

Notation 9.9 Let $(X, \mathcal{M}, \mu)$ be a measure space. Define

$$L^+ = L^+(X, \mathcal{M}, \mu) := \{f : X \to [0, \infty], f \text{ measurable}\}.$$

Definition 9.10 Let $\varphi : X \to [0, \infty)$ be simple. Write

$$\varphi = \sum_{j=1}^{n} a_j \mathbb{1}_{E_j},$$

where the E_j are measurable, disjoint, and $\bigcup_{j=1}^{n} E_j = X$. Define the **(Lebesgue) integral** of φ with respect to μ by

$$\int_X \varphi \, \mathrm{d}\mu = \sum_{j=1}^{n} a_j \mu(E_j).$$

Remark 9.11

1. Throughout, we adopt the convention $0 \cdot \infty = 0$. This just makes everything work.
2. We will often suppress some of the notation. All of the following are valid notations for the integral of φ:
$$\int_X \varphi \, \mathrm{d}\mu = \int \varphi \, \mathrm{d}\mu = \int_X \varphi = \int \varphi = \int_X \varphi(x) \, \mathrm{d}\mu(x).$$
3. If $A \in \mathcal{M}$ is measurable,
$$\int_A \varphi \, \mathrm{d}\mu = \int_X \varphi \mathbb{1}_A \, \mathrm{d}\mu.$$

Proposition 9.12 The integral of a simple function is well-defined. That is, if φ can be represented as a linear combination of characteristic functions in two different ways, then the two representations give the same value for the integral.

Proof. Suppose that
$$\varphi = \sum_{j=1}^{n} a_j \mathbb{1}_{E_j} = \sum_{k=1}^{m} b_k \mathbb{1}_{F_k}$$
with $(E_j), (F_k)$ being measurable partitions of X. We need to show
$$\sum_{j=1}^{n} a_j \mu(E_j) = \sum_{k=1}^{m} b_k \mu(F_k).$$
We have
$$\sum_{j=1}^{n} a_j \mu(E_j) = \sum_{j=1}^{n} a_j \mu\left(\bigcup_{k=1}^{m} (E_j \cap F_k)\right) = \sum_{j=1}^{n} \sum_{k=1}^{m} a_j \mu(E_j \cap F_k).$$
Similarly,
$$\sum_{k=1}^{m} b_k \mu(F_k) = \sum_{j=1}^{n} \sum_{k=1}^{m} b_k \mu(E_j \cap F_k).$$
We need
$$\sum_{j=1}^{n} \sum_{k=1}^{m} b_k \mu(E_j \cap F_k) = \sum_{j=1}^{n} \sum_{k=1}^{m} a_j \mu(E_j \cap F_k).$$
Fix j, k. If $\mu(E_j \cap F_k) = 0$, then $a_j \mu(E_j \cap F_k) = b_k \mu(E_j \cap F_k) = 0$. Otherwise, $E_j \cap F_k \neq \emptyset$, so fix $x \in E_j \cap F_k$. Since $x \in E_j$, $\varphi(x) = a_j$. Since $x \in F_k$, $\varphi(x) = b_k$. So $a_j = b_k$. $\square$

Proposition 9.13 For simple functions $\varphi, \psi \in L^+$ and $c \geq 0$, the properties hold:

1. The integral is **homogeneous** in the sense that
$$\int c\varphi \, \mathrm{d}\mu = c \int \varphi \, \mathrm{d}\mu.$$
2. The integral is **additive** in the sense that
$$\int (\varphi + \psi) \, \mathrm{d}\mu = \int \varphi \, \mathrm{d}\mu + \int \psi \, \mathrm{d}\mu.$$
3. The integral is **monotone** in the sense that if $\varphi \leq \psi$, then
$$\int \varphi \, \mathrm{d}\mu \leq \int \psi \, \mathrm{d}\mu.$$

4. The function $\nu : \mathcal{M} \to [0, \infty]$ defined by

$$\nu(E) = \int_E \varphi \, \mathrm{d}\mu$$

is a measure on X.

Remark 9.14 The identities (1) and (2) together imply that

$$\int_X \sum_{j=1}^{n} a_j \mathbb{1}_{E_j} \, \mathrm{d}\mu = \sum_{j=1}^{n} a_j \mu(E_j).$$

Lecture 10 — Tuesday, January 26, 2016

Proof of Proposition 9.13. (1) is immediate.

For (2), write

$$\varphi = \sum_{j=1}^{n} a_j \mathbb{1}_{E_j}$$

and

$$\psi = \sum_{k=1}^{m} b_k \mathbb{1}_{F_k}$$

where (E_j) and (F_k) are measurable partitions of X. Then

$$\begin{aligned}
\int (\varphi + \psi) &= \int \left(\sum_{j=1}^{n} a_j \mathbb{1}_{E_j} + \sum_{k=1}^{m} b_k \mathbb{1}_{F_k} \right) \\
&= \int \left(\sum_{j=1}^{n} \sum_{k=1}^{m} (a_j + b_k) \mathbb{1}_{E_j \cap F_k} \right) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{m} (a_j + b_k) \mu(E_j \cap F_k) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{m} a_j \mu(E_j \cap F_k) + \sum_{j=1}^{n} \sum_{k=1}^{m} b_k \mu(E_j \cap F_k)
\end{aligned}$$

$$\begin{aligned} &= \sum_{j=1}^{n} a_j \mu(E_j) + \sum_{k=1}^{m} b_k \mu(F_k) \\ &= \int \varphi + \int \psi. \end{aligned}$$

For (3), as above, write

$$\varphi = \sum_{j=1}^{n} a_j \mathbb{1}_{E_j}$$

and

$$\psi = \sum_{k=1}^{m} b_k \mathbb{1}_{F_k}.$$

Then

$$\int \varphi = \sum_{j=1}^{n} \sum_{k=1}^{m} a_j \mu(E_j \cap F_k)$$

and

$$\int \psi = \sum_{j=1}^{n} \sum_{k=1}^{m} b_k \mu(E_j \cap F_k).$$

It is enough to show that for each j, k, $a_j \mu(E_j \cap F_k) \leq b_k \mu(E_j \cap F_k)$. If $\mu(E_j \cap F_k) = 0$, this holds. Otherwise, $E_j \cap F_k \neq \emptyset$, so fix $x \in E_j \cap F_k$. Then

$$a_j = \varphi(x) \leq \psi(x) = b_k,$$

so the inequality holds.

For (4), let

$$\nu(E) = \int_E \varphi \, \mathrm{d}\mu.$$

We have $\nu(\emptyset) = 0$. Now fix disjoint $(A_n) \subseteq \mathcal{M}$ and set $A = \bigcup_{n=1}^{\infty} A_n$. Then

$$\begin{aligned} \nu(A) &= \int_A \varphi \, \mathrm{d}\mu \\ &= \int_X \varphi \mathbb{1}_A \, \mathrm{d}\mu \\ &= \sum_{j=1}^{n} a_j \mu(E_j \cap A) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{n} a_j \mu\left(E_j \cap \left(\bigcup_{n=1}^{\infty} A_n\right)\right) \\
&= \sum_{j=1}^{n} a_j \sum_{n=1}^{\infty} \mu(E_j \cap A_n) \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{n} a_j \mu(E_j \cap A_n) \\
&= \sum_{n=1}^{\infty} \nu(A_n).
\end{aligned}$$

Thus ν is a measure. $\square$

Definition 10.1 If $f : X \to [0,\infty]$ is measurable, define the **(Lebesgue) integral** of f with respect to μ by

$$\int f \,\mathrm{d}\mu = \sup\left\{\int \varphi \,\mathrm{d}\mu : 0 \leq \varphi \leq f, \varphi \text{ simple}\right\}.$$

Theorem 10.2 (Monotone convergence theorem) If $(f_n) \subseteq L^+$ with

$$0 \leq f_1 \leq f_2 \leq \dots,$$

then

$$\int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n.$$

This is essentially why the Lebesgue integral is defined the way it is.

Proof. Set $f = \lim_{n\to\infty} f_n$. Note

$$f_1 \leq f_2 \leq \dots \leq f.$$

Therefore

$$\lim_{n\to\infty} \int f_n \leq \int f.$$

Fix $\alpha \in (0,1)$ and a simple function φ with $0 \leq \varphi \leq f$. Define $E_n = \{x \in X : f_n(x) \geq \alpha\varphi(x)\}$. Note E_n is measurable, $E_1 \subseteq E_2 \subseteq \dots$, and $\bigcup_{n=1}^{\infty} E_n = X$. The function

$$\nu(A) = \int_A \varphi \,\mathrm{d}\mu$$

is a measure. By continuity from below,

$$\int_{E_n} \varphi \to \int_X \varphi.$$

Now,

$$\int f_n \geq \int_{E_n} f_n \geq \int_{E_n} \alpha\varphi = \alpha \int_{E_n} \varphi.$$

Letting $n \to \infty$,

$$\lim_{n\to\infty} \int f_n \geq \alpha \int \varphi.$$

Letting $\alpha \to 1$,

$$\lim_{n\to\infty} \int f_n \geq \int \varphi.$$

Thus

$$\lim_{n\to\infty} \int f_n \geq \int f.$$

$\square$

Theorem 10.3 If $(f_n)_{n=1}^{\infty} \subseteq L^+$, then

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n.$$

This is much nicer than linearity, since it allows us to exchange an *infinite* sum and an integral.

Proof. We first establish additivity for two nonnegative functions. Choose simple functions (φ_k) and (ψ_k) with $0 \leq \varphi_k \uparrow f_1$ and $0 \leq \psi_k \uparrow f_2$. Then $0 \leq \varphi_k + \psi_k \uparrow f_1 + f_2$, and

$$\int (f_1 + f_2) = \lim_k \int (\varphi_k + \psi_k) = \lim_k (\int \varphi_k + \int \psi_k) = \int f_1 + \int f_2,$$

where the first equality follows from the monotone convergence theorem. Induction now gives additivity for every finite sum. Finally, the partial sums satisfy

$$\sum_{n=1}^{N} f_n \uparrow \sum_{n=1}^{\infty} f_n$$

as $N \to \infty$, so

$$\begin{aligned}\int \sum_{n=1}^{\infty} f_n &= \lim_{N\to\infty} \int \sum_{n=1}^{N} f_n \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \int f_n \\ &= \sum_{n=1}^{\infty} \int f_n.\end{aligned}$$

The first equality again follows from the monotone convergence theorem. □

In general,

$$\lim_{n\to\infty} \int f_n \neq \int \lim_{n\to\infty} f_n.$$

Consider Lebesgue measure m on $\mathbb{R}$. Set $f_n = (1/n)\mathbb{1}_{[0,n]}$. Then

$$\int f_n = 1 \text{ for all } n \geq 1,$$

but $\lim_{n\to\infty} f_n = 0$ so

$$\int \lim_{n\to\infty} f_n = 0.$$

Theorem 10.4 (Fatou's lemma) If $(f_n) \subseteq L^+$, then

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n.$$

Proof. Fix $k \in \mathbb{N}$. Set $g_k = \inf_{n\geq k} f_n$. For $n \geq k$, $g_k \leq f_n$, so

$$\int g_k \leq \int f_n.$$

In particular,

$$\int g_k \leq \inf_{n\geq k} \int f_n.$$

Note

$$g_1 \leq g_2 \leq g_3 \leq \ldots$$

and $\lim_{k\to\infty} g_k = \liminf_{n\to\infty} f_n$. So, by the monotone convergence theorem,

$$\int \liminf_{n\to\infty} f_n = \lim_{k\to\infty} \int g_k \leq \liminf_{k\to\infty} \inf_{n\geq k} \int f_n = \liminf_{n\to\infty} \int f_n.$$

$\square$

Lecture 11 — Thursday, January 28, 2016

Definition 11.1 Let $(X, \mathcal{M}, \mu)$ be a measure space. Define

$$\mathcal{L}^1_{\mathbb{R}} = \mathcal{L}^1_{\mathbb{R}}(X, \mathcal{M}, \mu) := \left\{ f : X \to \mathbb{R} \mid f \text{ is measurable and } \int |f| < \infty \right\},$$

and similarly $\mathcal{L}^1_{\mathbb{C}}$.

Before passing to equivalence classes, we often write $\mathcal{L}^1$ for either $\mathcal{L}^1_{\mathbb{R}}$ or $\mathcal{L}^1_{\mathbb{C}}$ when the scalar field is clear from context. A function in $\mathcal{L}^1$ is called **integrable**.

Definition 11.2 If $f : X \to \mathbb{R}$ is a function, define

$$f^+ = \max\{f, 0\} \qquad \text{and} \qquad f^- = -\min\{f, 0\}.$$

Then $f^+, f^- \geq 0$, $f^+ - f^- = f$, and $f^+ + f^- = |f|$.

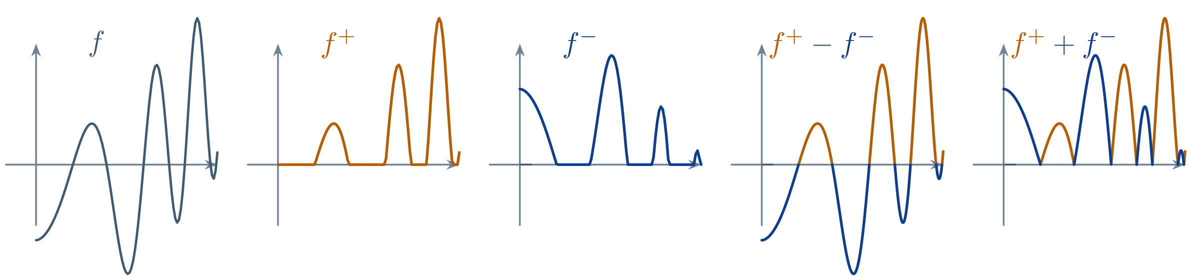

Figure 6

Figure 6 shows the positive and negative parts of f, together with their difference and sum.

Fact 11.3 If f is measurable, so are f^+ and f^-. Moreover, if $f \in \mathcal{L}^1_{\mathbb{R}}$, then $f^+, f^- \in \mathcal{L}^1_{\mathbb{R}}$ since

$$\int f^+, \int f^- \leq \int |f| < \infty.$$

f^+ and f^- are called the **positive and negative parts** of f.

Definition 11.4 If $f \in \mathcal{L}^1_{\mathbb{R}}$, define

$$\int f = \int f^+ - \int f^- \in \mathbb{R}.$$

Definition 11.5 If $f \in \mathcal{L}^1_{\mathbb{C}}$, define

$$\operatorname{Re}(f) = \frac{f + \bar{f}}{2} \qquad \text{and} \qquad \operatorname{Im}(f) = \frac{f - \bar{f}}{2i},$$

which are both $\mathbb{R}$-valued. Then $f = \operatorname{Re}(f) + i \operatorname{Im}(f)$.

Fact 11.6 If f is measurable, so are $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$. Moreover, $|\operatorname{Re}(f)| \leq |f|$ and $|\operatorname{Im}(f)| \leq |f|$. If $f \in \mathcal{L}^1_{\mathbb{C}}$, then $\operatorname{Re}(f), \operatorname{Im}(f) \in \mathcal{L}^1_{\mathbb{R}}$.

Proof. Since f is measurable, $\bar{f}$ is measurable, so $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ are measurable. We have

$$|\operatorname{Re}(f)| = \left|\frac{f + \bar{f}}{2}\right| \leq \frac{|f| + |\bar{f}|}{2} = |f|$$

and

$$|\operatorname{Im}(f)| = \left|\frac{f - \bar{f}}{2i}\right| \leq \frac{|f| + |\bar{f}|}{2} = |f|.$$

If $f \in \mathcal{L}^1_{\mathbb{C}}$, then monotonicity gives

$$\int |\operatorname{Re}(f)| \leq \int |f| < \infty$$

and

$$\int |\operatorname{Im}(f)| \leq \int |f| < \infty,$$

so $\operatorname{Re}(f), \operatorname{Im}(f) \in \mathcal{L}^1_{\mathbb{R}}$. $\square$

Definition 11.7 For $f \in \mathcal{L}^1_{\mathbb{C}}$, define

$$\int f = \int \mathrm{Re}(f) + i \int \mathrm{Im}(f).$$

Note that

$$\mathrm{Re}\Big(\int f\Big) = \int \mathrm{Re}(f) \qquad \text{and} \qquad \mathrm{Im}\Big(\int f\Big) = \int \mathrm{Im}(f).$$

Proposition 11.8 $\mathcal{L}^1_{\mathbb{R}}$ and $\mathcal{L}^1_{\mathbb{C}}$ are vector spaces, and

$$\int : \mathcal{L}^1 \to \mathbb{C}$$

(or $\mathcal{L}^1 \to \mathbb{R}$) is a linear functional on $\mathcal{L}^1$.

Proof. The set of all functions $X \to \mathbb{C}$ is a vector space, so it is enough to show that $\mathcal{L}^1$ is a subspace. The zero function belongs to $\mathcal{L}^1$. If $f, g \in \mathcal{L}^1$, then $f + g$ is measurable and

$$|f + g| \leq |f| + |g|,$$

so

$$\int |f + g| \leq \int |f| + \int |g| = \int |f| + \int |g| < \infty.$$

If $c \in \mathbb{C}$ and $f \in \mathcal{L}^1$, then

$$\int |cf| = \int |c| \cdot |f| = |c| \int |f| < \infty.$$

Thus $\mathcal{L}^1$ is a vector subspace.

We now prove linearity of the integral. First suppose $f, g \in \mathcal{L}^1_{\mathbb{R}}$, and set $h = f + g$. Since

$$h^+ - h^- = f^+ - f^- + g^+ - g^-,$$

we have

$$h^+ + f^- + g^- = h^- + f^+ + g^+.$$

All four functions belong to L^+, so integrating both sides gives

$$\int h^+ + \int f^- + \int g^- = \int h^- + \int f^+ + \int g^+.$$

Rearranging yields

$$\int h = \int h^+ - \int h^- = \int f^+ - \int f^- + \int g^+ - \int g^- = \int f + \int g.$$

Thus the integral is additive on $\mathcal{L}^1_{\mathbb{R}}$.

Now let $f, g \in \mathcal{L}^1_{\mathbb{C}}$. Using the real case for real and imaginary parts, we obtain

$$\begin{aligned}\int (f+g) &= \int \mathrm{Re}(f+g) + i \int \mathrm{Im}(f+g) \\ &= \int \mathrm{Re}(f) + \int \mathrm{Re}(g) + i \int \mathrm{Im}(f) + i \int \mathrm{Im}(g) \\ &= \int f + \int g.\end{aligned}$$

For scalar multiplication, let $c \in \mathbb{R}$ and $f \in \mathcal{L}^1_{\mathbb{R}}$. If $c \geq 0$, then

$$\int cf = c \int f$$

by definition of the integral. If $c < 0$, then

$$(-f)^+ = f^- \qquad \text{and} \qquad (-f)^- = f^+,$$

so $\int (-f) = -\int f$. Consequently,

$$\int cf = \int \big((-c)(-f)\big) = (-c) \int (-f) = c \int f.$$

Hence scalar linearity holds on $\mathcal{L}^1_{\mathbb{R}}$.

Finally, let $c = a + ib \in \mathbb{C}$ and let $f \in \mathcal{L}^1_{\mathbb{C}}$. Since

$$cf = (a\,\mathrm{Re}(f) - b\,\mathrm{Im}(f)) + i(a\,\mathrm{Im}(f) + b\,\mathrm{Re}(f)),$$

the real case gives

$$\begin{aligned}\int cf &= \int (a\,\mathrm{Re}(f) - b\,\mathrm{Im}(f)) + i \int (a\,\mathrm{Im}(f) + b\,\mathrm{Re}(f)) \\ &= a \int \mathrm{Re}(f) - b \int \mathrm{Im}(f) + i \left(a \int \mathrm{Im}(f) + b \int \mathrm{Re}(f) \right)\end{aligned}$$

$$
\begin{aligned}
&= (a+ib)\left(\int \mathrm{Re}(f) + i\int \mathrm{Im}(f)\right)\\
&= c\int f.
\end{aligned}
$$

Therefore the integral is linear on both $\mathcal{L}^1_{\mathbb{R}}$ and $\mathcal{L}^1_{\mathbb{C}}$. $\square$

Proposition 11.9 If $f \in \mathcal{L}^1$, then

$$\left|\int f\right| \leq \int |f|.$$

Proof. If $f \in \mathcal{L}^1_{\mathbb{R}}$, then

$$\left|\int f\right| = \left|\int f^+ - \int f^-\right| \leq \int f^+ + \int f^- = \int |f|.$$

Now assume f is $\mathbb{C}$-valued and

$$\int f \neq 0.$$

Define

$$\alpha := \frac{\overline{\int f}}{\left|\int f\right|}.$$

Then $|\alpha| = 1$, and

$$\alpha \int f = \frac{\overline{\int f}\int f}{\left|\int f\right|} = \frac{\left|\int f\right|^2}{\left|\int f\right|} = \left|\int f\right| \in \mathbb{R}.$$

Now,

$$\left|\int f\right| = \int \mathrm{Re}(\alpha f) \leq \int |\alpha f| = |\alpha| \int |f| = \int |f|.$$

$\square$

Proposition 11.10 Let $f \in \mathcal{L}^1$ be given. Then

$$\int |f| = 0$$

if and only if $f = 0$ almost everywhere.

Proof. *Forward implication.* Assume

$$\int |f| = 0.$$

For each $n \in \mathbb{N}$, set

$$E_n = \{x \in X : |f(x)| \geq 1/n\},$$

and let

$$E = \bigcup_{n=1}^{\infty} E_n.$$

Then $f = 0$ on E^c, so it suffices to prove that $\mu(E) = 0$. It is enough to show that $\mu(E_n) = 0$ for every n. Since

$$\mu(E_n) = \int \mathbb{1}_{E_n} \leq \int n|f| = n \int |f| = 0.$$

Thus $\mu(E_n) = 0$ for every n, and consequently $\mu(E) = 0$.

Reverse implication. Assume $f = 0$ almost everywhere. Let

$$N = \{x \in X : f(x) \neq 0\}.$$

Then $\mu(N) = 0$. If φ is any nonnegative simple function with $\varphi \leq |f|$, then $\varphi = 0$ on N^c and hence $\int \varphi = 0$. Taking the supremum over all such φ in the definition of the nonnegative integral gives $\int |f| = 0$. $\square$

Fact 11.11 Suppose $f_1, f_2, g_1, g_2 \in L^1$, $c \in \mathbb{C}$, $f_1 = f_2$ almost everywhere, and $g_1 = g_2$ almost everywhere. Then the following hold:

1. We have
$$\int f_1 = \int f_2.$$
2. $f_1 + g_1 = f_2 + g_2$ almost everywhere.

3. $cf_1 = cf_2$ almost everywhere.

Proof. For (1),

$$\left|\int f_1 - \int f_2\right| = \left|\int (f_1 - f_2)\right| \leq \int |f_1 - f_2| = 0,$$

since $f_1 - f_2 = 0$ almost everywhere, by Proposition 11.10.

For (2), we have

$$(f_1 + g_1) - (f_2 + g_2) = (f_1 - f_2) + (g_1 - g_2) = 0$$

almost everywhere, so $f_1 + g_1 = f_2 + g_2$ almost everywhere.

For (3), we have

$$cf_1 - cf_2 = c(f_1 - f_2) = 0$$

almost everywhere, so $cf_1 = cf_2$ almost everywhere. □

Definition 11.12 Let $\mathcal{L}^1(X, \mathcal{M}, \mu)$ denote the vector space of complex-valued integrable functions. Define

$$L^1 = L^1(X, \mathcal{M}, \mu) := \mathcal{L}^1(X, \mathcal{M}, \mu)/\sim,$$

where $f \sim g$ if and only if $f = g$ almost everywhere. We still write $f \in L^1$ to denote the equivalence class represented by a function $f : X \to \mathbb{C}$. The previous fact shows that L^1 is again a vector space, and the integral descends to a well-defined linear functional

$$\int : L^1 \to \mathbb{C}.$$

Here is an annoying fact: it can happen that $f = g$ almost everywhere and f is measurable, but g is not measurable. We will see an example of this later. This is why we need to pass to equivalence classes to get a well-defined integral on L^1.

Lecture 12 — Friday, January 29, 2016

Definition 12.1 Let

$$\mathcal{L}^1(X,\mathcal{M},\mu) := \{f : X \to \mathbb{C} \mid f \text{ is measurable and } \int |f| < \infty\}.$$

We define

$$L^1(X,\mathcal{M},\mu) := \mathcal{L}^1(X,\mathcal{M},\mu)/\sim,$$

where $f \sim g$ if and only if $f = g$ almost everywhere.

If $f, g : X \to \mathbb{C}$ with f measurable and $f = g$ almost everywhere, g is not necessarily measurable, unfortunately.

Example 12.2 Consider $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$. Let $C \subseteq [0,1]$ be the Cantor set. Then $C \in \mathcal{B}_{\mathbb{R}}$ and $m(C) = 0$. Also, there is a subset $A \subseteq C$ which is not Borel (that is, $A \notin \mathcal{B}_{\mathbb{R}}$). Then $\mathbb{1}_A|_{C^c} = 0$, so $\mathbb{1}_A = 0$ almost everywhere. However, $\mathbb{1}_A$ is not measurable.

Definition 12.3 The measure space $(X,\mathcal{M},\mu)$ is **complete** if whenever $E \in \mathcal{M}$ satisfies $\mu(E) = 0$ and $F \subseteq E$, we have $F \in \mathcal{M}$.

Proposition 12.4 If $(X,\mathcal{M},\mu)$ is a measure space, define

$$\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M}, \text{ there exists } N \in \mathcal{M} \text{ with } \mu(N) = 0 \text{ and } F \subseteq N\}.$$

Define $\bar{\mu} : \overline{\mathcal{M}} \to [0,\infty]$ by

$$\bar{\mu}(E \cup F) = \mu(E)$$

whenever $E \in \mathcal{M}$ and F is contained in a measurable null set. Then $(X, \overline{\mathcal{M}}, \bar{\mu})$ is a complete measure space. We call it the **completion** of $(X,\mathcal{M},\mu)$.

Proof. First, $\overline{\mathcal{M}}$ is a σ-algebra containing $\mathcal{M}$. The containment $\mathcal{M} \subseteq \overline{\mathcal{M}}$ follows by taking $F = \emptyset$. If $A = E \cup F$ with $E \in \mathcal{M}$ and $F \subseteq N$ for some measurable null set N, then

$$A^c = (E^c \cap N^c) \cup \big((E^c \cap N) \setminus F\big).$$

The first set is measurable, and the second set is contained in the measurable null set N, so $A^c \in \overline{\mathcal{M}}$. If $A_j = E_j \cup F_j$ with $E_j \in \mathcal{M}$ and $F_j \subseteq N_j$, where $\mu(N_j) = 0$, then

$$\bigcup_{j=1}^{\infty} A_j = \left(\bigcup_{j=1}^{\infty} E_j\right) \cup \left(\bigcup_{j=1}^{\infty} F_j\right),$$

and $\bigcup_{j=1}^{\infty} F_j$ is contained in the measurable null set $\bigcup_{j=1}^{\infty} N_j$. Hence $\bigcup_{j=1}^{\infty} A_j \in \overline{\mathcal{M}}$.

Next, $\bar{\mu}$ is well defined. If $E_1 \cup F_1 = E_2 \cup F_2$ with $E_i \in \mathcal{M}$ and $F_i \subseteq N_i$, where $\mu(N_i) = 0$, then

$$E_1 \Delta E_2 \subseteq N_1 \cup N_2,$$

so $\mu(E_1 \Delta E_2) = 0$. Therefore $\mu(E_1) = \mu(E_2)$.

To prove countable additivity, let $(A_j)_{j=1}^{\infty}$ be a pairwise disjoint sequence in $\overline{\mathcal{M}}$. Write $A_j = E_j \cup F_j$ with $E_j \in \mathcal{M}$ and $F_j \subseteq N_j$, where $\mu(N_j) = 0$. Since the sets A_j are disjoint and $E_j \subseteq A_j$, the sets E_j are disjoint. Hence

$$\bar{\mu}\left(\bigcup_{j=1}^{\infty} A_j\right) = \mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j) = \sum_{j=1}^{\infty} \bar{\mu}(A_j).$$

Finally, let $A \in \overline{\mathcal{M}}$ satisfy $\bar{\mu}(A) = 0$, and let $B \subseteq A$. Write $A = E \cup F$ with $E \in \mathcal{M}$ and $F \subseteq N$, where $\mu(N) = 0$. Then $\mu(E) = 0$, so $B \subseteq E \cup N$ is contained in a measurable null set. Thus $B \in \overline{\mathcal{M}}$, and the completed space is complete. □

Example 12.5 Let μ be a Radon measure on $\mathbb{R}$. The completion of $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$ is $(\mathbb{R}, \mathcal{M}_{\mu}, \mu)$. In particular, the completion of $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$ is $(\mathbb{R}, \mathcal{L}_{\mathbb{R}}, m)$.

Proposition 12.6 If $(X, \mathcal{M}, \mu)$ is a complete measure space, then the following hold:

1. If $f, g : X \to \mathbb{C}$, $f = g$ almost everywhere and f is measurable, then g is measurable too.
2. If $(f_n)_{n=1}^{\infty}$ is a sequence of measurable functions and $f_n \to f$ almost everywhere, then f is measurable.

Proof. For (1), there is a set $E \in \mathcal{M}$ with $\mu(E) = 0$ and $f|_{E^c} = g|_{E^c}$.

That is, $f \cdot \mathbb{1}_{E^c} = g \cdot \mathbb{1}_{E^c}$, so $g \cdot \mathbb{1}_{E^c}$ is measurable. Since

$$g = g \cdot \mathbb{1}_{E^c} + g \cdot \mathbb{1}_E,$$

it remains to show that $g \cdot \mathbb{1}_E$ is measurable. Let $B \subseteq \mathbb{C}$ be Borel. If $0 \notin B$, then

$$(g \cdot \mathbb{1}_E)^{-1}(B) \subseteq E,$$

which is measurable by completeness. If $0 \in B$, then $(g \cdot \mathbb{1}_E)^{-1}(B^c) \subseteq E$, so $(g \cdot \mathbb{1}_E)^{-1}(B^c)$ is measurable by completeness, and hence $(g \cdot \mathbb{1}_E)^{-1}(B)$ is measurable. Thus $g \cdot \mathbb{1}_E$ is measurable, and therefore g is measurable.

For (2), if $f_n \to f$ almost everywhere, there is some set $E \in \mathcal{M}$ such that $\mu(E) = 0$ and $f_n|_{E^c} \to f|_{E^c}$.

Each $f_n \cdot \mathbb{1}_{E^c}$ is measurable, and

$$f \cdot \mathbb{1}_{E^c} = \lim_{n\to\infty} f_n \cdot \mathbb{1}_{E^c}$$

pointwise. Hence $f \cdot \mathbb{1}_{E^c}$ is measurable. Since $f = f \cdot \mathbb{1}_{E^c}$ almost everywhere, (1) implies that f is measurable. $\square$

Theorem 12.7 Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(X, \overline{\mathcal{M}}, \bar{\mu})$ be its completion. Then the following hold:

1. $L^1(X, \mathcal{M}, \mu) \subseteq L^1(X, \overline{\mathcal{M}}, \bar{\mu})$.
2. For every $f \in L^1(X, \mathcal{M}, \mu)$, we have
$$\int f \, \mathrm{d}\mu = \int f \, \mathrm{d}\bar{\mu}.$$
3. If $g : X \to \mathbb{C}$ is $\overline{\mathcal{M}}$-measurable, there is some $f : X \to \mathbb{C}$ which is $\mathcal{M}$-measurable and $f = g$ almost everywhere (with respect to $\bar{\mu}$). In particular, we may identify $L^1(X, \mathcal{M}, \mu)$ with $L^1(X, \overline{\mathcal{M}}, \bar{\mu})$.

Proof. First, note $\mathcal{M} \subseteq \overline{\mathcal{M}}$, so every $\mathcal{M}$-measurable function is already $\overline{\mathcal{M}}$-measurable. If

$E \in \mathcal{M}$, then $\mu(E) = \bar{\mu}(E)$ by the way we defined $\bar{\mu}$. In particular,

$$\int \mathbb{1}_E \, \mathrm{d}\mu = \int \mathbb{1}_E \, \mathrm{d}\bar{\mu},$$

so if $\mathbb{1}_E \in L^1(X, \mathcal{M}, \mu)$, then $\mathbb{1}_E \in L^1(X, \overline{\mathcal{M}}, \bar{\mu})$. L^1 is a vector space, so that holds for linear combos too. So if $f \in L^1(X, \mathcal{M}, \mu)$ is simple, then $f \in L^1(X, \overline{\mathcal{M}}, \bar{\mu})$ too and

$$\int f \, \mathrm{d}\mu = \int f \, \mathrm{d}\bar{\mu}.$$

Now if $f \in L^1(X, \mathcal{M}, \mu)$ with $f \geq 0$, there are simple $\mathcal{M}$-measurable functions h_n with

$$h_1 \leq h_2 \leq \cdots \leq f$$

where $\lim_{n\to\infty} h_n = f$. Then

$$\int f \, \mathrm{d}\mu = \lim_{n\to\infty} \int h_n \, \mathrm{d}\mu = \lim_{n\to\infty} \int h_n \, \mathrm{d}\bar{\mu} = \int f \, \mathrm{d}\bar{\mu}.$$

Hence $f \in L^1(X, \overline{\mathcal{M}}, \bar{\mu})$. Now, every function in $L^1(X, \mathcal{M}, \mu)$ is a linear combo of positive functions.

The statements (1) and (2) both follow.

For (3), first consider $g = \mathbb{1}_A$ with $A \in \overline{\mathcal{M}}$. Choose $E \in \mathcal{M}$ such that $A \triangle E$ is contained in a measurable null set. Then $\mathbb{1}_A = \mathbb{1}_E$ almost everywhere, and $\mathbb{1}_E$ is $\mathcal{M}$-measurable.

Now let g be an arbitrary $\overline{\mathcal{M}}$-measurable complex-valued function. Choose simple $\overline{\mathcal{M}}$-measurable functions s_n such that $s_n \to g$ pointwise. By the characteristic function case, for each n there is an $\mathcal{M}$-measurable simple function t_n such that $s_n = t_n$ outside a measurable null set. Outside the countable union of these null sets, the sequence t_n converges to g. Define f to be the pointwise limit of t_n where this limit exists and define $f = 0$ elsewhere. Then f is $\mathcal{M}$-measurable and $f = g$ almost everywhere. This proves (3). $\square$

The Dominated Convergence Theorem

Theorem 12.8 (Dominated convergence theorem) Let $(f_n)_{n=1}^{\infty} \subseteq L^1$, and let $f : X \to \mathbb{C}$ be measurable. Suppose the following hold:

1. We have $f_n \to f$ almost everywhere.
2. There exists a function $g \in L^1$ with $|f_n| \le g$ for all n.

Then $f \in L^1$ and
$$\int f = \lim_{n\to\infty} \int f_n.$$

Example 12.9 Take $f_n = (1/n)\mathbb{1}_{[0,n]}$. Then $f_n \to 0$ but
$$\int f_n \to 1.$$
What is the problem? The sequence (f_n) is not dominated. Indeed, a dominating function would satisfy $g(x) \ge 1/\lceil x \rceil$ for $x > 0$, and that lower bound is not integrable on $[1, \infty)$.

Proof of Theorem 12.8. Since f is measurable and $|f| \le g$ almost everywhere,
$$\int |f| \le \int g < \infty.$$
Thus $f \in L^1$. It remains to prove convergence of the integrals.

Assume first that the functions are real-valued. Since $|f_n| \le g$, we have $g + f_n \ge 0$ and $g - f_n \ge 0$ for every n. By Fatou's lemma,
$$\int (g+f) = \int \lim_{n\to\infty} (g+f_n) \le \liminf_{n\to\infty} \int (g+f_n) = \int g + \liminf_{n\to\infty} \int f_n.$$
Similarly,
$$\int (g-f) \le \int g + \liminf_{n\to\infty} \int (-f_n) = \int g - \limsup_{n\to\infty} \int f_n.$$

Subtracting $\int g$ from both inequalities gives

$$\int f \leq \liminf_{n\to\infty} \int f_n$$

and

$$-\int f \leq -\limsup_{n\to\infty} \int f_n.$$

Therefore

$$\limsup_{n\to\infty} \int f_n \leq \int f \leq \liminf_{n\to\infty} \int f_n.$$

It follows that

$$\lim_{n\to\infty} \int f_n = \int f.$$

For complex-valued functions, apply the real-valued case to the real and imaginary parts. Since $|\operatorname{Re}(f_n)| \leq |f_n| \leq g$ and $|\operatorname{Im}(f_n)| \leq |f_n| \leq g$, both parts are dominated by g, and the result follows by linearity of the integral. $\square$

Lecture 13 — Tuesday, February 02, 2016

Theorem 13.1 If $(f_n)_{n=1}^\infty \subseteq L^1$ and

$$\sum_{n=1}^\infty \int |f_n| < \infty,$$

then

$$\sum_{n=1}^\infty f_n$$

converges almost everywhere and

$$\sum_{n=1}^\infty f_n \in L^1.$$

Moreover,

$$\int \sum_{n=1}^\infty f_n = \sum_{n=1}^\infty \int f_n.$$

Proof. First,

$$\int \sum_{n=1}^{\infty} |f_n| = \sum_{n=1}^{\infty} \int |f_n| < \infty$$

by the monotone convergence theorem. In particular,

$$\sum_{n=1}^{\infty} |f_n(x)| < \infty$$

for almost every x, and hence

$$\sum_{n=1}^{\infty} f_n(x)$$

converges almost everywhere. Moreover,

$$\int \left| \sum_{n=1}^{\infty} f_n \right| \leq \int \sum_{n=1}^{\infty} |f_n| < \infty,$$

so

$$\sum_{n=1}^{\infty} f_n \in L^1.$$

Let

$$g = \sum_{n=1}^{\infty} |f_n| \in L^1.$$

For every n,

$$|\sum_{k=1}^{n} f_k| \leq \sum_{k=1}^{n} |f_k| \leq g.$$

By the dominated convergence theorem,

$$\begin{aligned}
\int \sum_{n=1}^{\infty} f_n &= \int \lim_{n\to\infty} \sum_{k=1}^{n} f_k \\
&= \lim_{n\to\infty} \int \sum_{k=1}^{n} f_k \\
&= \sum_{n=1}^{\infty} \int f_n.
\end{aligned}$$

$\square$

3. Convergence and Product Integration

Convergence of Functions

Definition 13.2 We say that $f_n \to f$ **in L^1** if $f_n, f \in L^1$ and

$$\int |f_n - f| \to 0$$

as $n \to \infty$.

Definition 13.3 Let f_n, f be measurable. We say that $f_n \to f$ **in measure** if, for every $\varepsilon > 0$,

$$\mu(\{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\}) \to 0$$

as $n \to \infty$.

Example 13.4 Let $f_n \in L^1(\mathbb{R}, m)$ be given by $f_n(x) = \mathbb{1}_{[n,n+1]}$. Then $f_n \to 0$ pointwise. However, f_n does not converge to 0 in measure, since for $0 < \varepsilon \leq 1$ we have

$$m(\{x \in \mathbb{R} \mid |f_n(x)| \geq \varepsilon\}) = 1$$

for every n. The sequence also does not converge to 0 in L^1, since

$$\int |f_n| = 1.$$

In fact, (f_n) does not converge to anything in L^1. Nothing at all... NOTHING AT ALL.

Proposition 13.5 If $f_n \to f$ in L^1, then $f_n \to f$ in measure.

Proof. Fix $\varepsilon > 0$. Set $E_n = \{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\}$. We have that

$$\mu(E_n) = \int \mathbb{1}_{E_n} \leq \int \frac{1}{\varepsilon}|f_n - f|\mathbb{1}_{E_n} \leq \frac{1}{\varepsilon}\int |f_n - f| \to 0.$$

$\square$

Definition 13.6 A sequence of measurable functions $(f_n)_{n=1}^{\infty}$ is **Cauchy in measure** if, for every $\varepsilon > 0$,

$$\mu(\{x \in X \mid |f_n(x) - f_m(x)| \geq \varepsilon\}) \to 0$$

as $m, n \to \infty$. Equivalently, for every $\varepsilon > 0$ and every $\delta > 0$, there exists $N \in \mathbb{N}$ such that if $m, n \geq N$, then

$$\mu(\{x \in X \mid |f_n(x) - f_m(x)| \geq \varepsilon\}) < \delta.$$

Theorem 13.7 If (f_n) is Cauchy in measure, then there is a measurable function $f : X \to \mathbb{C}$ such that $f_n \to f$ in measure, and there is a subsequence (f_{n_k}) converging to f pointwise almost everywhere. Moreover, if $f_n \to g$ in measure, then $f = g$ almost everywhere, so limits in measure are unique almost everywhere.

Corollary 13.8 If $(f_n)_{n=1}^{\infty} \subseteq L^1$ and $f_n \to f$ in L^1, then there is a subsequence (f_{n_k}) with $f_{n_k} \to f$ pointwise almost everywhere.

Proof. Since $f_n \to f$ in L^1, the sequence converges to f in measure. Apply Theorem 13.7. $\square$

Proof of Theorem 13.7. Choose a subsequence (f_{n_k}) of (f_n) such that

$$\mu(\{x \in X \mid |f_{n_{k+1}}(x) - f_{n_k}(x)| \geq 2^{-k}\}) < 2^{-k}$$

for every k. Set

$$E_k = \{x \in X \mid |f_{n_{k+1}}(x) - f_{n_k}(x)| \geq 2^{-k}\} \qquad \text{and} \qquad F_j = \bigcup_{k=j}^{\infty} E_k.$$

Then

$$\mu(F_j) \leq \sum_{k=j}^{\infty} \mu(E_k) \leq \sum_{k=j}^{\infty} 2^{-k} = 2^{1-j}.$$

If $x \notin F_j$ and $k \geq j$, then

$$|f_{n_k}(x) - f_{n_j}(x)| \leq \sum_{i=j}^{k-1} |f_{n_{i+1}}(x) - f_{n_i}(x)| \leq \sum_{i=j}^{k-1} 2^{-i} \leq 2^{1-j}.$$

Set $F = \bigcap_{j=1}^{\infty} F_j$. Since $F_1 \supseteq F_2 \supseteq \cdots$ and $\mu(F_j) \to 0$, continuity from above gives $\mu(F) = 0$. If

$x \notin F$, then $x \notin F_j$ for some j, so the preceding estimate shows that $(f_{n_k}(x))_{k=1}^{\infty}$ is Cauchy in $\mathbb{C}$. Define

$$f(x) = \begin{cases} \lim_{k\to\infty} f_{n_k}(x), & x \notin F, \\ 0, & x \in F. \end{cases}$$

Then $f_{n_k} \to f$ pointwise almost everywhere, and f is measurable because

$$f = \lim_{k\to\infty} f_{n_k} \mathbb{1}_{F^c}$$

pointwise.

The subsequence also converges to f in measure. Indeed, given $\varepsilon > 0$ and $\delta > 0$, choose j with $2^{1-j} < \min\{\varepsilon, \delta\}$. For $k \geq j$,

$$\{x \in X \mid |f_{n_k}(x) - f(x)| \geq \varepsilon\} \subseteq F_j,$$

so

$$\mu(\{x \in X \mid |f_{n_k}(x) - f(x)| \geq \varepsilon\}) < \delta.$$

Finally, we show that the full sequence converges to f in measure. Fix $\varepsilon > 0$ and $\delta > 0$. Since (f_n) is Cauchy in measure, there exists N such that

$$\mu(\{x \in X \mid |f_n(x) - f_m(x)| \geq \varepsilon/2\}) < \delta/2$$

whenever $n, m \geq N$. Now choose k large enough that $n_k \geq N$ and

$$\mu(\{x \in X \mid |f_{n_k}(x) - f(x)| \geq \varepsilon/2\}) < \delta/2.$$

For every $n \geq N$,

$$\begin{aligned}\{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\} \subseteq \{x \in X \mid |f_n(x) - f_{n_k}(x)| \geq \varepsilon/2\} \\ \cup \{x \in X \mid |f_{n_k}(x) - f(x)| \geq \varepsilon/2\},\end{aligned}$$

so $\mu(\{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\}) < \delta$. Hence $f_n \to f$ in measure.

The uniqueness assertion follows from Proposition 14.1. □

Lecture 14 — Thursday, February 04, 2016

Proposition 14.1 If $f, g : X \to \mathbb{C}$ are measurable, $f_n \to f$ in measure, and $f_n \to g$ in measure, then $f = g$ almost everywhere.

Proof. Let $\varepsilon > 0$.

$$\{x \in X \mid |f(x) - g(x)| \geq \varepsilon\} \subseteq \{x \in X \mid |f - f_n| \geq \varepsilon/2\} \cup \{x \in X \mid |f_n - g| \geq \varepsilon/2\},$$

so

$$\mu(|f - g| \geq \varepsilon) \leq \mu(|f - f_n| \geq \varepsilon/2) + \mu(|f_n - g| \geq \varepsilon/2) \to 0$$

as $n \to \infty$. Thus $\mu(|f - g| \geq \varepsilon) = 0$. Now,

$$\{x \in X \mid f(x) \neq g(x)\} = \bigcup_{n=1}^{\infty} \{x \in X \mid |f(x) - g(x)| \geq 1/n\}.$$

Hence $\mu(f \neq g) = 0$, and so $f = g$ almost everywhere. □

Theorem 14.2 (Egoroff's theorem) Suppose $\mu(X) < \infty$. If $(f_n)_{n=1}^{\infty}$ is a sequence of measurable functions and $f_n \to f$ pointwise almost everywhere, then for every $\varepsilon > 0$, there is a measurable set $E \subseteq X$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on E^c.

Proof. For $k, n \in \mathbb{N}$, let

$$E_n(k) = \bigcup_{m=n}^{\infty} \{x \in X \mid |f_m(x) - f(x)| \geq 1/k\}.$$

For each fixed k, the set $\bigcap_{n=1}^{\infty} E_n(k)$ has measure zero because $f_n \to f$ pointwise almost everywhere. Also

$$E_1(k) \supseteq E_2(k) \supseteq E_3(k) \supseteq \cdots,$$

and $\mu(E_1(k)) \leq \mu(X) < \infty$. By continuity from above,

$$\mu(E_n(k)) \to 0$$

as $n \to \infty$. Choose $n(k) \in \mathbb{N}$ such that

$$\mu(E_{n(k)}(k)) < \frac{\varepsilon}{2^k},$$

and set

$$E = \bigcup_{k=1}^{\infty} E_{n(k)}(k).$$

Then $\mu(E) < \varepsilon$. If $x \notin E$, then $x \notin E_{n(k)}(k)$ for every $k \geq 1$. Therefore, whenever $m \geq n(k)$,

$$|f_m(x) - f(x)| < \frac{1}{k}.$$

It follows that

$$\sup_{x \in E^c} |f_m(x) - f(x)| \leq \frac{1}{k}$$

for all $m \geq n(k)$. Hence $f_n \to f$ uniformly on E^c. □

The main implications assembled so far are recorded in Figure 7. The dashed arrows require either a subsequence or the finite-measure hypothesis displayed beside them.

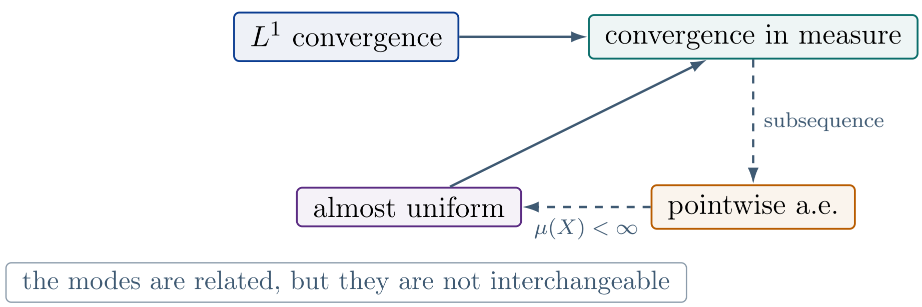

FIGURE 7

The conclusion of Egoroff's theorem can still hold when $\mu(X) = \infty$ under an integrable domination hypothesis. Suppose $f_n \to f$ pointwise almost everywhere and there is a function $g \in L^1$ such that $|f_n| \leq g$ for every n. Then $|f| \leq g$ almost everywhere. For each $j \geq 1$, the set

$$X_j = \{x \in X \mid g(x) > 1/(3j)\}$$

has finite measure because $\mu(X_j)/(3j) \leq \int g$. Applying Theorem 14.2 on X_j, choose a set $E_j \subseteq X_j$ with $\mu(E_j) < \varepsilon/2^j$ such that $f_n \to f$ uniformly on $X_j \setminus E_j$. Enlarge E_1 by the null set where $|f| \leq g$ fails, and set $E = \bigcup_{j=1}^{\infty} E_j$. Then $\mu(E) < \varepsilon$. For a fixed j, the convergence is uniform on $X_j \setminus E$, while on $X \setminus X_j$ we have $|f_n - f| \leq 2/(3j)$ for all n. Hence $f_n \to f$ uniformly on E^c.

Product Measures

Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. Our goal is to construct a measure space $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ and give meaning to the integral

$$\int_{X \times Y} f(x, y) \, \mathrm{d}(\mu \times \nu)(x, y).$$

Definition 14.3 Let $\mathcal{M} \otimes \mathcal{N}$ denote the σ-algebra generated by $\Sigma = \{A \times B \mid A \in \mathcal{M}, B \in \mathcal{N}\}$. The sets in Σ are called **measurable rectangles**.

Proposition 14.4 Let $(X, \mathcal{M})$, $(Y, \mathcal{N})$, and $(Z, \mathcal{P})$ be measurable spaces, and let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the **projection maps**. A function $f : Z \to X \times Y$ is measurable if and only if $\pi_X \circ f : Z \to X$ and $\pi_Y \circ f : Z \to Y$ are both measurable.

Proof. *Forward implication.* It is enough to show that the projection maps π_X and π_Y are measurable.

If $E \in \mathcal{M}$, then

$$\pi_X^{-1}(E) = E \times Y \in \mathcal{M} \otimes \mathcal{N},$$

so π_X is measurable. Similarly, π_Y is measurable.

Reverse implication. Let

$$\Sigma = \{E \times F : E \in \mathcal{M}, \ F \in \mathcal{N}\}.$$

For every rectangle $E \times F \in \Sigma$,

$$f^{-1}(E \times F) = (\pi_X \circ f)^{-1}(E) \cap (\pi_Y \circ f)^{-1}(F),$$

which is measurable by assumption. Since Σ generates $\mathcal{M} \otimes \mathcal{N}$, the generator criterion for measurability implies that f is measurable. $\square$

In general, there is no analogous statement for functions $X \times Y \to Z$.

Example 14.5 If X and Y are separable metric spaces (or second-countable topological spaces if we want), then $\mathcal{B}_X \otimes \mathcal{B}_Y = \mathcal{B}_{X\times Y}$. Here, the product metric is

$$d((x_1,y_1),(x_2,y_2)) = \sqrt{d(x_1,x_2)^2 + d(y_1,y_2)^2}.$$

When $X = Y = \mathbb{R}$, this is the usual metric on $\mathbb{R}^2$.

Proof. Consider the identity map

$$\mathrm{id} : (X \times Y, \mathcal{B}_{X\times Y}) \to (X \times Y, \mathcal{B}_X \otimes \mathcal{B}_Y).$$

The compositions $\pi_X \circ \mathrm{id}$ and $\pi_Y \circ \mathrm{id}$ are continuous, and hence they are $\mathcal{B}_{X\times Y}$-measurable. By the preceding proposition, id is $\mathcal{B}_{X\times Y}, \mathcal{B}_X \otimes \mathcal{B}_Y$-measurable. Therefore $\mathcal{B}_X \otimes \mathcal{B}_Y \subseteq \mathcal{B}_{X\times Y}$.

Reverse implication. Let $W \subseteq X \times Y$ be open. By separability, we may write

$$W = \bigcup_{n=1}^{\infty} U_n \times V_n,$$

where each $U_n \subseteq X$ and each $V_n \subseteq Y$ is open. Then $U_n \times V_n \in \mathcal{B}_X \otimes \mathcal{B}_Y$ for every n, so $W \in \mathcal{B}_X \otimes \mathcal{B}_Y$. Hence $\mathcal{B}_{X\times Y} \subseteq \mathcal{B}_X \otimes \mathcal{B}_Y$, and the two σ-algebras are equal. □

Our short-term goal is to construct a measure $\mu \times \nu$ on $(X \times Y, \mathcal{M} \otimes \mathcal{N})$ satisfying

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

for measurable rectangles $A \times B$ (think length times height). If μ and ν are σ-finite, then $\mu \times \nu$ will also be unique and σ-finite.

Lecture 15 — Friday, February 05, 2016

Definition 15.1 An **elementary family** on a set X is a set $\mathcal{E} \subseteq \mathcal{P}(X)$ such that the following hold:

1. The empty set belongs to $\mathcal{E}$.
2. If $E, F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$.
3. If $E \in \mathcal{E}$, then E^c can be written as $F_1 \cup F_2 \cup \cdots \cup F_n$, where $F_1, \ldots, F_n \in \mathcal{E}$ are disjoint.

Proposition 15.2 Let $\mathcal{E}$ be an elementary family on X, and let $\mathcal{A}(\mathcal{E})$ be the algebra generated by $\mathcal{E}$. If $A \in \mathcal{A}(\mathcal{E})$, then $A = E_1 \cup E_2 \cup \cdots \cup E_n$ for disjoint $E_i \in \mathcal{E}$.

Example 15.3 If $\mathcal{A}$ and $\mathcal{B}$ are algebras on X, Y respectively, then define $\mathcal{E} = \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. Then $\mathcal{E}$ is the elementary family we care about.

Proof. We verify the three properties in the definition. First,

$$\emptyset = \emptyset \times \emptyset \in \mathcal{E}.$$

Second, if $A, C \in \mathcal{A}$ and $B, D \in \mathcal{B}$, then

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D) \in \mathcal{E}.$$

Finally, for $A \in \mathcal{A}$ and $B \in \mathcal{B}$,

$$(A \times B)^c = (A^c \times B) \cup (A^c \times B^c) \cup (A \times B^c),$$

which is a finite disjoint union of elements of $\mathcal{E}$. Thus $\mathcal{E}$ is an elementary family. □

Proof of Proposition 15.2. Let $\mathcal{B}$ be the collection of finite disjoint unions of elements of $\mathcal{E}$. Then $\mathcal{E} \subseteq \mathcal{B} \subseteq \mathcal{A}(\mathcal{E})$. It is enough to show that $\mathcal{B}$ is an algebra, since then $\mathcal{B} = \mathcal{A}(\mathcal{E})$.

First let $E, F \in \mathcal{E}$. Then

$$E \cup F = E \cup (F \setminus E),$$

where E and $F \setminus E$ are disjoint. Since $F \setminus E = F \cap E^c$ and E^c is a finite disjoint union of elements of $\mathcal{E}$, the set $F \setminus E$ belongs to $\mathcal{B}$. Hence $E \cup F \in \mathcal{B}$.

By induction, if $A_1, \dots, A_n \in \mathcal{E}$, then $\bigcup_{j=1}^n A_j \in \mathcal{B}$. Indeed, assuming $\bigcup_{j=1}^{n-1} A_j \in \mathcal{B}$, we have

$$\bigcup_{j=1}^{n} A_j = A_n \cup \left(\bigcup_{j=1}^{n-1} A_j \setminus A_n \right),$$

and the two sets on the right can be written as a finite disjoint union of elements of $\mathcal{E}$. Therefore arbitrary finite unions of elements of $\mathcal{E}$ belong to $\mathcal{B}$. It follows that if $A, B \in \mathcal{B}$, then $A \cup B \in \mathcal{B}$.

It remains to check complements. Let $A \in \mathcal{B}$, and write

$$A = E_1 \cup \cdots \cup E_n$$

with $E_1, \dots, E_n \in \mathcal{E}$ disjoint. For each $1 \leq k \leq n$, write

$$E_k^c = F_1^k \cup \cdots \cup F_{m(k)}^k,$$

where the sets $F_1^k, \dots, F_{m(k)}^k$ belong to $\mathcal{E}$ and are disjoint for fixed k. Then

$$\begin{aligned} A^c &= \left(\bigcup_{k=1}^{n} E_k \right)^c \\ &= \bigcap_{k=1}^{n} E_k^c \\ &= \bigcap_{k=1}^{n} \left(\bigcup_{j=1}^{m(k)} F_j^k \right) \\ &= \bigcup_{1 \leq j_k \leq m(k)} \left(F_{j_1}^1 \cap F_{j_2}^2 \cap \cdots \cap F_{j_n}^n \right). \end{aligned}$$

The last expression is a finite union of elements of $\mathcal{E}$, and hence it belongs to $\mathcal{B}$. Thus $\mathcal{B}$ is an algebra, so $\mathcal{B} = \mathcal{A}(\mathcal{E})$. □

Definition 15.4 A **monotone class** on a set X is a set $\mathcal{C} \subseteq \mathcal{P}(X)$ such that the following hold:

1. If $(A_n)_{n=1}^\infty \subseteq \mathcal{C}$ with $A_1 \subseteq A_2 \subseteq \dots$, then $\bigcup_{n=1}^\infty A_n \in \mathcal{C}$.
2. If $(A_n)_{n=1}^\infty \subseteq \mathcal{C}$ with $A_1 \supseteq A_2 \supseteq \dots$, then $\bigcap_{n=1}^\infty A_n \in \mathcal{C}$.

Fact 15.5 If $\mathcal{A}$ is an algebra, then $\mathcal{A}$ is a monotone class if and only if $\mathcal{A}$ is a σ-algebra.

Proof. *Forward implication.* Given $(A_n)_{n=1}^{\infty} \subseteq \mathcal{A}$, the sets

$$B_k := \bigcup_{n=1}^{k} A_n$$

belong to $\mathcal{A}$ because $\mathcal{A}$ is an algebra, and $B_1 \subseteq B_2 \subseteq \cdots$. Since $\mathcal{A}$ is a monotone class,

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{k=1}^{\infty} \left(\bigcup_{n=1}^{k} A_n \right) \in \mathcal{A}.$$

Reverse implication. Every σ-algebra is obviously a monotone class. □

Theorem 15.6 (Monotone class lemma) Let $\mathcal{A}$ be an algebra, let $\mathcal{M}$ be the σ-algebra generated by $\mathcal{A}$, and let $\mathcal{C}$ be the monotone class generated by $\mathcal{A}$. Then $\mathcal{M} = \mathcal{C}$.

Proof. Since $\mathcal{M}$ is a monotone class containing $\mathcal{A}$, we immediately have $\mathcal{C} \subseteq \mathcal{M}$. It therefore remains to show that $\mathcal{C}$ is a σ-algebra.

Fix $F \in \mathcal{C}$ and define

$$\mathcal{C}(F) := \{E \in \mathcal{C} : E \cap F,\ E \setminus F,\ F \setminus E,\ E \cup F \in \mathcal{C}\}.$$

We first check that $\mathcal{C}(F)$ is a monotone class. If $E_n \uparrow E$ with every $E_n \in \mathcal{C}(F)$, then

$$E_n \cap F \uparrow E \cap F, \qquad E_n \setminus F \uparrow E \setminus F, \qquad F \setminus E_n \downarrow F \setminus E,$$

and $E_n \cup F \uparrow E \cup F$. Since $\mathcal{C}$ is a monotone class, all four limiting sets belong to $\mathcal{C}$, so $E \in \mathcal{C}(F)$. The same argument for $E_n \downarrow E$ reverses each of these arrows and again gives $E \in \mathcal{C}(F)$.

If $A \in \mathcal{A}$, then for every $B \in \mathcal{A}$ the sets

$$A \cap B, \qquad A \setminus B, \qquad B \setminus A, \qquad \text{and} \qquad A \cup B$$

all belong to $\mathcal{A} \subseteq \mathcal{C}$, so $A \in \mathcal{C}(B)$. Hence $\mathcal{A} \subseteq \mathcal{C}(B)$, and by minimality of $\mathcal{C}$ we obtain $\mathcal{C} \subseteq \mathcal{C}(B)$ for every $B \in \mathcal{A}$. Reversing the roles of A and B shows that $\mathcal{C} \subseteq \mathcal{C}(E)$ for every $E \in \mathcal{C}$.

Consequently, if $E, F \in \mathcal{C}$, then

$$E \cap F, \qquad E \setminus F, \qquad F \setminus E, \qquad \text{and} \qquad E \cup F$$

all belong to $\mathcal{C}$. In particular, $\mathcal{C}$ is closed under complements and finite unions.

Now let $(E_n)_{n=1}^{\infty} \subseteq \mathcal{C}$. Since $\mathcal{C}$ is closed under finite unions, the sets

$$B_k := \bigcup_{n=1}^{k} E_n$$

belong to $\mathcal{C}$, and $B_1 \subseteq B_2 \subseteq \cdots$. Because $\mathcal{C}$ is a monotone class,

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{k=1}^{\infty} B_k \in \mathcal{C}.$$

Therefore $\mathcal{C}$ is a σ-algebra. Since it contains $\mathcal{A}$, it must contain $\sigma(\mathcal{A}) = \mathcal{M}$. Hence $\mathcal{M} = \mathcal{C}$. $\square$

Lecture 16 — Tuesday, February 09, 2016

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. Define $\mathcal{E} = \{A \times B \mid A \in \mathcal{M}, B \in \mathcal{N}\}$. Let $\mathcal{A}$ be the algebra generated by $\mathcal{E}$, and let $\mathcal{M} \otimes \mathcal{N}$ be the σ-algebra generated by $\mathcal{E}$ (or by $\mathcal{A}$ if we like).

Remark 16.1

1. The elements of $\mathcal{E}$ are called the **measurable rectangles**.
2. Every set in $\mathcal{A}$ is a finite disjoint union of sets in $\mathcal{E}$.
3. By the monotone class lemma, the σ-algebra $\mathcal{M} \otimes \mathcal{N}$ is the monotone class generated by $\mathcal{A}$.

Question 16.2 The product measure construction raises two questions:

1. Why is E_x measurable?

2. Why is the function $x \mapsto \nu(E_x)$ measurable?

It turns out that the second property often fails without additional hypotheses.

Definition 16.3 Fix $E \subseteq X \times Y$. For $x \in X$, define

$$E_x = \{y \in Y \mid (x, y) \in E\}.$$

For $y \in Y$, define

$$E^y = \{x \in X \mid (x, y) \in E\}.$$

Given $f : X \times Y \to Z$, define $f_x : Y \to Z$ by $f_x(y) = f(x, y)$ and define $f^y : X \to Z$ by $f^y(x) = f(x, y)$.

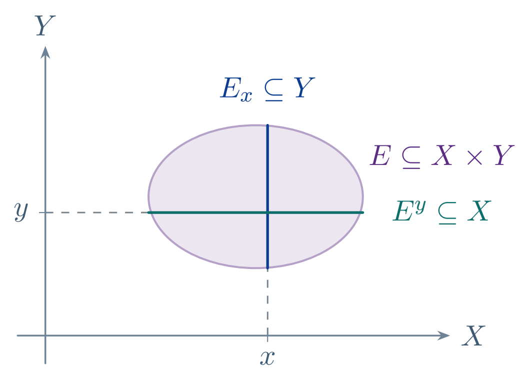

FIGURE 8

The vertical and horizontal sections of E are shown in Figure 8.

Proposition 16.4 Let X, Y, Z be measurable spaces. Assume $E \subseteq X \times Y$ and $f : X \times Y \to Z$ are measurable. Then the following hold:

1. For every $x \in X$, $E_x \subseteq Y$ and $f_x : Y \to Z$ are measurable.
2. For every $y \in Y$, $E^y \subseteq X$ and $f^y : X \to Z$ are measurable.

Proof. Fix $x \in X$, and define $j : Y \to X \times Y$ by $j(y) = (x, y)$. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the projection maps. To show that j is measurable, it is enough to show that

$\pi_X \circ j$ and $\pi_Y \circ j$ are measurable. The map $\pi_X \circ j$ is the constant map $y \mapsto x$, and $\pi_Y \circ j$ is the identity map $y \mapsto y$, so both are measurable. Hence j is measurable.

Now

$$E_x = \{y \in Y \mid (x, y) \in E\} = j^{-1}(E),$$

so E_x is measurable. Also, $f_x = f \circ j$, so f_x is measurable.

The proof of (2) is analogous. □

Theorem 16.5 For $E \in \mathcal{A}$, the functions $f_E : X \to [0, \infty]$ and $g_E : Y \to [0, \infty]$ defined by

$$f_E(x) := \nu(E_x) \qquad \text{and} \qquad g_E(y) := \mu(E^y)$$

are both measurable. Furthermore,

$$\int_X f_E \,\mathrm{d}\mu = \int_Y g_E \,\mathrm{d}\nu.$$

Proof. Let $\mathcal{C}$ be the collection of sets $E \in \mathcal{A}$ for which the theorem is true. We show that $\mathcal{C} = \mathcal{A}$.

Suppose $A \in \mathcal{M}$ and $B \in \mathcal{N}$, and set $E = A \times B \in \mathcal{E}$. Then $E_x = B$ if $x \in A$, while $E_x = \emptyset$ if $x \notin A$. Hence $f_E = \nu(B)\mathbb{1}_A$ is measurable. Similarly, $g_E = \mu(A)\mathbb{1}_B$ is measurable. Moreover,

$$\int_X f_E \,\mathrm{d}\mu = \int_X \nu(B)\mathbb{1}_A \,\mathrm{d}\mu = \mu(A)\nu(B)$$

and

$$\int_Y g_E \,\mathrm{d}\nu = \int_Y \mu(A)\mathbb{1}_B \,\mathrm{d}\nu = \mu(A)\nu(B),$$

so

$$\int_X f_E \,\mathrm{d}\mu = \int_Y g_E \,\mathrm{d}\nu.$$

Now suppose $E, F \in \mathcal{C}$ and $E \cap F = \emptyset$. For $x \in X$, we have $(E \cup F)_x = E_x \cup F_x$ and $E_x \cap F_x = \emptyset$. Hence

$$f_{E\cup F}(x) = \nu((E \cup F)_x) = \nu(E_x \cup F_x) = \nu(E_x) + \nu(F_x) = (f_E + f_F)(x).$$

Therefore $f_{E\cup F} = f_E + f_F$ is measurable. Similarly, $g_{E\cup F} = g_E + g_F$, so $g_{E\cup F}$ is measurable.

Finally,

$$\int_X f_{E\cup F}\,\mathrm{d}\mu = \int_X f_E\,\mathrm{d}\mu + \int_X f_F\,\mathrm{d}\mu = \int_Y g_E\,\mathrm{d}\nu + \int_Y g_F\,\mathrm{d}\nu = \int_Y g_{E\cup F}\,\mathrm{d}\nu.$$

Thus $E \cup F \in \mathcal{C}$. Since every set in $\mathcal{A}$ is a finite disjoint union of measurable rectangles by (2), it follows that $\mathcal{A} \subseteq \mathcal{C}$. Hence $\mathcal{C} = \mathcal{A}$. □

Definition 16.6 Define $\mu \times \nu : \mathcal{A} \to [0, \infty]$ by

$$(\mu \times \nu)(E) := \int_X f_E\,\mathrm{d}\mu = \int_Y g_E\,\mathrm{d}\nu.$$

By Theorem 16.5,

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

for every measurable rectangle $A \times B$.

Proposition 16.7 $\mu \times \nu$ is a premeasure on $\mathcal{A}$.

Proof. $(\mu \times \nu)(\emptyset) = 0$ is clear. Suppose $(A_n)_{n=1}^\infty$ is a disjoint sequence and $A = \bigcup_{n=1}^\infty A_n \in \mathcal{A}$. We need to show

$$(\mu \times \nu)(A) = \sum_{n=1}^\infty (\mu \times \nu)(A_n).$$

Define $f_n, f : X \to [0, \infty]$ by $f_n(x) = \nu((A_n)_x), f(x) = \nu(A_x)$. Note that $A_x = \left(\bigcup_{n=1}^\infty A_n\right)_x = \bigcup_{n=1}^\infty (A_n)_x$ and $(A_n)_x$ are all disjoint. Now

$$f(x) = \nu(A_x) = \nu\left(\bigcup_{n=1}^\infty (A_n)_x\right) = \sum_{n=1}^\infty \nu((A_n)_x) = \sum_{n=1}^\infty f_n(x),$$

so

$$f = \sum_{n=1}^\infty f_n.$$

So

$$(\mu \times \nu)(A) = \int f\,\mathrm{d}\mu = \int \sum_{n=1}^\infty f_n\,\mathrm{d}\mu = \sum_{n=1}^\infty \int f_n\,\mathrm{d}\mu = \sum_{n=1}^\infty (\mu \times \nu)(A_n).$$

So $\mu \times \nu$ is a premeasure. □

Definition 16.8 Define the **product measure** of μ and ν by $\mu \times \nu : \mathcal{M} \otimes \mathcal{N} \to [0, \infty]$ by

$$(\mu \times \nu)(A) = \inf \left\{ \sum_{n=1}^{\infty} (\mu \times \nu)(A_n) \mid A \subseteq \bigcup_{n=1}^{\infty} A_n, A_n \in \mathcal{A} \right\}.$$

By the Carathéodory extension theorem, $\mu \times \nu$ extends to a measure on $\mathcal{M} \otimes \mathcal{N}$. Moreover, if the premeasure $\mu \times \nu$ is σ-finite on $\mathcal{A}$, then this extension is unique.

Under the σ-finite hypothesis, the extension to a measure on $\mathcal{M} \otimes \mathcal{N}$ is unique by Theorem 5.4. We will show that if μ and ν are σ-finite, then the maps $\mathcal{M} \otimes \mathcal{N} \to [0, \infty]$,

$$E \mapsto \int \nu(E_x) \, \mathrm{d}\mu(x)$$

and

$$E \mapsto \int \mu(E^y) \, \mathrm{d}\nu(y)$$

are both measures and are equal to $\mu \times \nu$ on $\mathcal{A}$. Hence all three measures are equal on $\mathcal{M} \otimes \mathcal{N}$.

Lecture 17 — Thursday, February 11, 2016

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. Recall the product measure construction as follows. Set

$$\mathcal{E} = \{A \times B \mid A \in \mathcal{M}, B \in \mathcal{N}\},$$

let $\mathcal{A}$ be the algebra generated by $\mathcal{E}$, and let $\mathcal{M} \otimes \mathcal{N}$ be the σ-algebra generated by $\mathcal{A}$. There is a unique premeasure $\mu \times \nu : \mathcal{A} \to [0, \infty]$ satisfying

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B) \tag{17.1}$$

for all $A \in \mathcal{M}$ and $B \in \mathcal{N}$. The induced measure on $\mathcal{M} \otimes \mathcal{N}$ satisfies

$$(\mu \times \nu)(A) = \inf \left\{ \sum_{n=1}^{\infty} (\mu \times \nu)(A_n) \mid A \subseteq \bigcup_{n=1}^{\infty} A_n, \ A_n \in \mathcal{A} \right\}.$$

If μ and ν are σ-finite, then $\mu \times \nu$ is a σ-finite premeasure on $\mathcal{A}$. In this case, any measure λ on $\mathcal{M} \otimes \mathcal{N}$ satisfying (17.1) for all measurable rectangles satisfies $\lambda = \mu \times \nu$.

Theorem 17.1 Assume μ and ν are σ-finite. Then the following hold:

1. For every $E \in \mathcal{M} \otimes \mathcal{N}$, the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measurable.
2. The functions $\mathcal{M} \otimes \mathcal{N} \to [0, \infty]$,
$$E \mapsto \int \nu(E_x)\,\mathrm{d}\mu(x)$$
and
$$E \mapsto \int \mu(E^y)\,\mathrm{d}\nu(y)$$
are measures.
3. For every $E \in \mathcal{M} \otimes \mathcal{N}$,
$$(\mu \times \nu)(E) = \int \nu(E_x)\,\mathrm{d}\mu(x) = \int \mu(E^y)\,\mathrm{d}\nu(y).$$
4. For every $E \in \mathcal{M} \otimes \mathcal{N}$,
$$\int_X \nu(E_x)\,\mathrm{d}\mu(x) = \int_Y \mu(E^y)\,\mathrm{d}\nu(y).$$

We have already shown these assertions for $E \in \mathcal{A}$.

Proof. For (1), we only prove the measurability of $x \mapsto \nu(E_x)$.

Since ν is σ-finite, choose sets $Y_k \in \mathcal{N}$ such that

$$Y_1 \subseteq Y_2 \subseteq \cdots, \qquad \nu(Y_k) < \infty \text{ for every } k \in \mathbb{N}, \qquad \text{and} \qquad \bigcup_{k=1}^{\infty} Y_k = Y.$$

Fix $k \in \mathbb{N}$, and define

$$\mathcal{C}_k = \{E \in \mathcal{M} \otimes \mathcal{N} \mid x \mapsto \nu(E_x \cap Y_k) \text{ is measurable}\}.$$

Claim: $\mathcal{C}_k = \mathcal{M} \otimes \mathcal{N}$.

Verification: By Remark 16.1(3), it is enough to show that $\mathcal{C}_k$ is a monotone class and that $\mathcal{A} \subseteq \mathcal{C}_k$.

If $E \in \mathcal{A}$, then $E \cap (X \times Y_k) \in \mathcal{A}$ and

$$(E \cap (X \times Y_k))_x = E_x \cap Y_k.$$

Thus $x \mapsto \nu(E_x \cap Y_k)$ is measurable by Theorem 16.5, so $E \in \mathcal{C}_k$.

For $F \in \mathcal{M} \otimes \mathcal{N}$, write

$$f_{k,F}(x) := \nu(F_x \cap Y_k).$$

Suppose $(E_n)_{n=1}^\infty \subseteq \mathcal{C}_k$ with $E_1 \subseteq E_2 \subseteq \cdots$, and set $E = \bigcup_{n=1}^\infty E_n$. Then

$$(E_1)_x \cap Y_k \subseteq (E_2)_x \cap Y_k \subseteq \cdots$$

and

$$\bigcup_{n=1}^\infty ((E_n)_x \cap Y_k) = E_x \cap Y_k.$$

By continuity from below,

$$\nu((E_n)_x \cap Y_k) \to \nu(E_x \cap Y_k)$$

for every $x \in X$. Hence $f_{k,E_n} \to f_{k,E}$ pointwise. Since each f_{k,E_n} is measurable, $f_{k,E}$ is measurable, and therefore $E \in \mathcal{C}_k$.

Now suppose $(E_n)_{n=1}^\infty \subseteq \mathcal{C}_k$ with $E_1 \supseteq E_2 \supseteq \cdots$, and set $E = \bigcap_{n=1}^\infty E_n$. Then

$$(E_1)_x \cap Y_k \supseteq (E_2)_x \cap Y_k \supseteq \cdots$$

and

$$\bigcap_{n=1}^\infty ((E_n)_x \cap Y_k) = E_x \cap Y_k.$$

Since $\nu((E_1)_x \cap Y_k) \leq \nu(Y_k) < \infty$, continuity from above gives

$$\nu((E_n)_x \cap Y_k) \to \nu(E_x \cap Y_k)$$

for every $x \in X$. As before, $f_{k,E_n} \to f_{k,E}$ pointwise, so $f_{k,E}$ is measurable and $E \in \mathcal{C}_k$. Thus $\mathcal{C}_k$ is a monotone class containing $\mathcal{A}$, which proves the claim.

Now fix $E \in \mathcal{M} \otimes \mathcal{N}$. For every $k \in \mathbb{N}$, the function

$$g_k(x) := \nu(E_x \cap Y_k)$$

is measurable. Define $g : X \to [0, \infty]$ by $g(x) = \nu(E_x)$. Since $Y_1 \subseteq Y_2 \subseteq \cdots$ and $\bigcup_{k=1}^\infty Y_k = Y$, we have

$$E_x \cap Y_1 \subseteq E_x \cap Y_2 \subseteq \cdots \qquad \text{and} \qquad \bigcup_{k=1}^{\infty}(E_x \cap Y_k) = E_x.$$

Using continuity from below again, $g_k \to g$ pointwise. Hence g is measurable. This proves the asserted measurability of $x \mapsto \nu(E_x)$, and the argument for $y \mapsto \mu(E^y)$ is symmetric.

For (2), define $\lambda : \mathcal{M} \otimes \mathcal{N} \to [0, \infty]$ by

$$\lambda(E) = \int_X \nu(E_x)\,\mathrm{d}\mu(x).$$

Certainly $\lambda(\emptyset) = 0$. Now suppose $(E_n)_{n=1}^{\infty} \subseteq \mathcal{M} \otimes \mathcal{N}$ is a disjoint sequence. Set $E = \bigcup_{n=1}^{\infty} E_n$. The sets $(E_n)_x$ are disjoint and $E_x = \bigcup_{n=1}^{\infty}(E_n)_x$. Therefore

$$\nu(E_x) = \sum_{n=1}^{\infty} \nu((E_n)_x)$$

for every $x \in X$. By the monotone convergence theorem,

$$\begin{aligned}
\lambda(E) &= \int_X \nu(E_x)\,\mathrm{d}\mu(x) \\
&= \int_X \sum_{n=1}^{\infty} \nu((E_n)_x)\,\mathrm{d}\mu(x) \\
&= \sum_{n=1}^{\infty} \int_X \nu((E_n)_x)\,\mathrm{d}\mu(x) \\
&= \sum_{n=1}^{\infty} \lambda(E_n).
\end{aligned}$$

Thus λ is a measure. The proof for $E \mapsto \int_Y \mu(E^y)\,\mathrm{d}\nu(y)$ is identical.

For (3), note that for $A \in \mathcal{M}, B \in \mathcal{N}$, $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$. Also

$$\int_X \nu((A \times B)_x)\,\mathrm{d}\mu(x) = \mu(A)\nu(B)$$

because

$$\nu((A \times B)_x) = \begin{cases} \nu(B), & x \in A, \\ 0, & x \notin A. \end{cases}$$

Similarly,

$$\int_Y \mu((A \times B)^y)\,\mathrm{d}\nu(y) = \mu(A)\nu(B).$$

These three measures agree on $\mathcal{E}$, so they agree on $\mathcal{A}$. Since $\mu \times \nu$ is σ-finite, the uniqueness clause in Theorem 5.4 implies that these measures agree on $\mathcal{M} \otimes \mathcal{N}$.

For (4), note that

$$\int_Y \mathbb{1}_E(x,y)\,\mathrm{d}\nu(y) = \nu(E_x),$$

because $\mathbb{1}_E(x,y) = \mathbb{1}_{E_x}(y)$ when x is fixed. Similarly,

$$\int_X \mathbb{1}_E(x,y)\,\mathrm{d}\mu(x) = \mu(E^y).$$

The equality in (4) is therefore exactly the equality in (3). □

Lecture 18 — Friday, February 12, 2016

Theorem 18.1 (Fubini–Tonelli theorem) Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. Then the following hold:

1. (**Tonelli's theorem**) If $f : X \times Y \to [0, \infty]$ is $\mathcal{M} \otimes \mathcal{N}$-measurable, then

$$g : X \to [0,\infty], \qquad g(x) = \int_Y f(x,y)\,\mathrm{d}\nu(y)$$

and

$$h : Y \to [0,\infty], \qquad h(y) = \int_X f(x,y)\,\mathrm{d}\mu(x)$$

are measurable and nonnegative. Moreover,

$$\int_{X\times Y} f\,\mathrm{d}(\mu\times\nu) = \int_X g\,\mathrm{d}\mu = \int_Y h\,\mathrm{d}\nu.$$

2. (**Fubini's theorem**) If $f \in L^1(\mu \times \nu)$, then $f_x \in L^1(\nu)$ for almost every $x \in X$, and

$f^y \in L^1(\mu)$ for almost every $y \in Y$. In this case,

$$g(x) = \int_Y f(x,y)\,\mathrm{d}\nu(y)$$

and

$$h(y) = \int_X f(x,y)\,\mathrm{d}\mu(x)$$

are defined almost everywhere. After assigning the value 0 on the exceptional null sets, they are measurable, with $g \in L^1(\mu)$ and $h \in L^1(\nu)$. Moreover,

$$\int_{X\times Y} f\,\mathrm{d}(\mu\times\nu) = \int_X g\,\mathrm{d}\mu = \int_Y h\,\mathrm{d}\nu.$$

Proof. For (1), the result holds for characteristic functions. Suppose f is a simple function with $f \geq 0$. Write

$$f = \sum_{j=1}^{n} \lambda_j \mathbb{1}_{E_j},$$

where $\lambda_j \geq 0$. Now, the function

$$g(x) = \int_Y f(x,y)\,\mathrm{d}\nu(y) = \sum_{j=1}^{n} \lambda_j \int_Y \mathbb{1}_{E_j}(x,y)\,\mathrm{d}\nu(y).$$

Hence g is measurable since each

$$x \mapsto \int_Y \mathbb{1}_{E_j}(x,y)\,\mathrm{d}\nu(y) = \nu((E_j)_x)$$

is measurable. Similarly, h is measurable. Now,

$$\begin{aligned}\int_X g\,\mathrm{d}\mu &= \int_X \sum_{j=1}^{n} \lambda_j \nu((E_j)_x)\,\mathrm{d}\mu(x)\\ &= \sum_{j=1}^{n} \lambda_j \int_X \nu((E_j)_x)\,\mathrm{d}\mu(x)\\ &= \sum_{j=1}^{n} \lambda_j (\mu\times\nu)(E_j)\\ &= \int_{X\times Y} f\,\mathrm{d}(\mu\times\nu).\end{aligned}$$

Similarly,

$$\int_Y h \, \mathrm{d}\nu = \int_{X\times Y} f \, \mathrm{d}(\mu \times \nu).$$

Now suppose that $f : X \times Y \to [0, \infty]$ is an arbitrary measurable function. There exist simple functions $f_n : X \times Y \to [0, \infty]$ such that

$$0 \leq f_1 \leq f_2 \leq \cdots \leq f,$$

and $f_n \to f$ pointwise. Define

$$g_n(x) := \int_Y f_n(x, y) \, \mathrm{d}\nu(y) \qquad \text{and} \qquad h_n(y) := \int_X f_n(x, y) \, \mathrm{d}\mu(x).$$

Then

$$0 \leq g_1 \leq g_2 \leq \cdots \leq g,$$
$$0 \leq h_1 \leq h_2 \leq \cdots \leq h,$$

where

$$g(x) := \int_Y f(x, y) \, \mathrm{d}\nu(y) \qquad \text{and} \qquad h(y) := \int_X f(x, y) \, \mathrm{d}\mu(x).$$

By the monotone convergence theorem,

$$g(x) = \lim_{n\to\infty} g_n(x) \qquad \text{and} \qquad h(y) = \lim_{n\to\infty} h_n(y).$$

Hence g and h are measurable as pointwise limits of measurable functions. Applying the monotone convergence theorem once more,

$$\int_X g \, \mathrm{d}\mu = \int_X \left(\lim_{n\to\infty} g_n\right) \mathrm{d}\mu = \lim_{n\to\infty} \int_X g_n \, \mathrm{d}\mu = \lim_{n\to\infty} \int_{X\times Y} f_n \, \mathrm{d}(\mu \times \nu) = \int_{X\times Y} f \, \mathrm{d}(\mu \times \nu),$$

and similarly

$$\int_Y h \, \mathrm{d}\nu = \int_Y \left(\lim_{n\to\infty} h_n\right) \mathrm{d}\nu = \lim_{n\to\infty} \int_Y h_n \, \mathrm{d}\nu = \lim_{n\to\infty} \int_{X\times Y} f_n \, \mathrm{d}(\mu \times \nu) = \int_{X\times Y} f \, \mathrm{d}(\mu \times \nu).$$

For (2), suppose $f \in L^1(\mu \times \nu)$. Then

$$\int_{X\times Y} |f| \, \mathrm{d}(\mu \times \nu) < \infty.$$

Since $|f|$ is positive and measurable,

$$\int_X \left(\int_Y |f| \, \mathrm{d}\nu \right) \mathrm{d}\mu(x) = \int_{X \times Y} |f| \, \mathrm{d}(\mu \times \nu) < \infty.$$

So

$$\int_Y |f| \, \mathrm{d}\nu < \infty$$

for almost every $x \in X$, so $f_x \in L^1(Y, \nu)$ for almost every $x \in X$, and $f^y \in L^1(X, \mu)$ for almost every $y \in Y$. First assume that f is $\mathbb{R}$-valued.

$$g(x) = \int_Y f_x \, \mathrm{d}\nu = \int_Y (f_x)^+ \, \mathrm{d}\nu - \int_Y (f_x)^- \, \mathrm{d}\nu = \int_Y (f^+)_x \, \mathrm{d}\nu - \int_Y (f^-)_x \, \mathrm{d}\nu.$$

Now f^+ and f^- are positive and measurable, so g is measurable on the set where the section is integrable. This set is measurable and has full measure; define $g = 0$ on its complement. Also,

$$\int_X |g| \, \mathrm{d}\mu = \int_X \left| \int_Y f_x \, \mathrm{d}\nu \right| \mathrm{d}\mu(x) \leq \int_X \int_Y |f_x| \, \mathrm{d}\nu \, \mathrm{d}\mu(x) < \infty.$$

So $g \in L^1(X, \mu)$. Moreover,

$$\begin{aligned}
\int_X g \, \mathrm{d}\mu &= \int_X \int_Y f_x \, \mathrm{d}\nu \, \mathrm{d}\mu(x) \\
&= \int_{X \times Y} (f^+ - f^-) \, \mathrm{d}(\mu \times \nu) \\
&= \int_{X \times Y} f^+ \, \mathrm{d}(\mu \times \nu) - \int_{X \times Y} f^- \, \mathrm{d}(\mu \times \nu) \\
&= \int_{X \times Y} f \, \mathrm{d}(\mu \times \nu),
\end{aligned}$$

where the second equality uses Tonelli's theorem in (1). Similarly,

$$\int_Y h \, \mathrm{d}\nu = \int_{X \times Y} f \, \mathrm{d}(\mu \times \nu).$$

When f is $\mathbb{C}$-valued, apply the real-valued result to $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$. The results for

$$h(y) = \int_X f^y \, \mathrm{d}\mu$$

are identical. □

These theorems are normally used together. For example, to integrate $f : X \times Y \to \mathbb{C}$, we first prove that f is measurable (which is always very easy or very hard — there is no middle ground) and then prove that $f \in L^1(\mu \times \nu)$ by showing

$$\int |f| = \iint |f| \,\mathrm{d}\mu\,\mathrm{d}\nu < \infty,$$

and then use Fubini's theorem in (2) to write

$$\int_{X\times Y} f \,\mathrm{d}(\mu \times \nu) = \iint f \,\mathrm{d}\nu\,\mathrm{d}\mu.$$

Then integrate!

Remark 18.2 If μ and ν are complete measures, then $\mu \times \nu$ need not be complete. Take $X = Y = [0, 1]$, $\mathcal{M} = \mathcal{N} = \mathcal{L}_{\mathbb{R}}$, and $\mu = \nu = m$, where m is Lebesgue measure. Set $A = \{0\}$ and choose $B \subseteq [0, 1]$ with $B \notin \mathcal{L}_{\mathbb{R}}$. Then $A \times B$ is not in $\mathcal{L}_{\mathbb{R}} \otimes \mathcal{L}_{\mathbb{R}}$, but

$$A \times B \subseteq \{0\} \times [0, 1],$$

and the right-hand side has product measure zero.

The completion $(X \times Y, \overline{\mathcal{M} \otimes \mathcal{N}}, \overline{\mu \times \nu})$ repairs this issue, although the notation is not completely standard. The Fubini–Tonelli theorem holds in this completed setting with slight modifications: the sections f_x and f^y are only measurable for almost every x and almost every y. In particular, the completion of

$$\mathcal{L}_{\mathbb{R}} \otimes \cdots \otimes \mathcal{L}_{\mathbb{R}}$$

with respect to $m \times \cdots \times m$ is $\mathcal{L}_{\mathbb{R}^n}$, and the completed product measure is m^n, Lebesgue measure on $\mathbb{R}^n$.

Lecture 19 — Tuesday, February 23, 2016

4. Signed Measures, Differentiation, and Bounded Variation

Signed and Complex Measures

Definition 19.1 A **signed measure** ν on a measurable space $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that the following hold:

1. $\nu(\emptyset) = 0$.
2. At least one of $+\infty$ and $-\infty$ is *not* in the range of ν.
3. If $(E_n)_{n=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{M}$, then
$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \nu(E_n)$$
with convergence in $\mathbb{R} \cup \{\pm\infty\}$.

Remark 19.2 In (3), convergence of the series is part of the countable additivity requirement. Every measure in the previous sense is automatically a signed measure; in this context, such measures are often called **positive measures**.

Definition 19.3 A signed measure ν is **finite** if $\nu(E) \neq \pm\infty$ for every $E \in \mathcal{M}$.

Definition 19.4 A **complex measure** on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to \mathbb{C}$ such that, for every disjoint sequence $(E_n)_{n=1}^{\infty} \subseteq \mathcal{M}$,
$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \nu(E_n).$$

Remark 19.5
$$\nu(\emptyset) = \sum_{n=1}^{\infty} \nu(\emptyset) = \nu(\emptyset) + \nu(\emptyset) + \ldots,$$
and this series converges in $\mathbb{C}$ only when $\nu(\emptyset) = 0$. Thus, for complex measures, $\nu(\emptyset) = 0$ is a consequence of countable additivity rather than a separate assumption.

Every finite signed measure is a complex measure, but neither class contains the other in general.

Counting measure on $\mathbb{N}$ is a signed measure that is not complex-valued. Conversely,

$$\nu(E) = i \sum_{n \in E} 2^{-n}, \qquad E \subseteq \mathbb{N},$$

defines a complex measure whose values are not real, so it is not a signed measure.

Remark 19.6 Let X be a compact metric space, and let $M(X)$ denote the set of complex Borel measures on X. Then $M(X)$ is a $\mathbb{C}$-vector space with

$$(\mu + \nu)(E) = \mu(E) + \nu(E) \qquad \text{and} \qquad (a\mu)(E) = a\mu(E)$$

for all $\mu, \nu \in M(X)$, all $E \in \mathcal{B}_X$, and all $a \in \mathbb{C}$. Let

$$C(X) = \{f : X \to \mathbb{C} \mid f \text{ is continuous}\}.$$

Then $C(X)$ is a $\mathbb{C}$-vector space with pointwise addition and scalar multiplication. One version of the Riesz representation theorem says that $(C(X))^* \cong M(X)$ as Banach spaces. In particular, a measure $\mu \in M(X)$ corresponds to the functional

$$C(X) \to \mathbb{C}, f \mapsto \int f \,\mathrm{d}\mu$$

after integration with respect to a complex measure has been defined.

Example 19.7 The following are basic examples of signed and complex measures:

1. If μ_1 and μ_2 are positive measures and at least one is finite, then $\nu : \mathcal{M} \to [-\infty, \infty]$, $\nu(E) = \mu_1(E) - \mu_2(E)$ is a signed measure.
2. If ν_1 and ν_2 are finite signed measures, then $\nu : \mathcal{M} \to \mathbb{C}$, $\nu(E) = \nu_1(E) + i\nu_2(E)$ is a complex measure.
3. Suppose $f : X \to [-\infty, \infty]$ is measurable, μ is a positive measure, and either
$$\int f^+ \,\mathrm{d}\mu < \infty$$
or
$$\int f^- \,\mathrm{d}\mu < \infty,$$

holds. Then $\nu : \mathcal{M} \to [-\infty, \infty]$ defined by

$$\nu(E) = \int_E f \, \mathrm{d}\mu = \int_E f^+ \, \mathrm{d}\mu - \int_E f^- \, \mathrm{d}\mu$$

is a signed measure by the monotone convergence theorem.

4. If μ is a positive measure and $f \in L^1(\mu)$, then $\nu : \mathcal{M} \to \mathbb{C}$,

$$\nu(E) = \int_E f \, \mathrm{d}\mu$$

is a complex measure by the dominated convergence theorem.

All four examples in Example 19.7 have converses. In this sense, they describe the basic ways to build signed and complex measures.

Proposition 19.8 Let ν be a signed or complex measure. Then the following hold:

1. If $E_1 \subseteq E_2 \subseteq \dots$ are measurable, then $\nu(\bigcup_{n=1}^\infty E_n) = \lim_{n\to\infty} \nu(E_n)$.
2. If $E_1 \supseteq E_2 \supseteq \dots$ are measurable and $|\nu(E_1)| < \infty$, then $\nu(\bigcap_{n=1}^\infty E_n) = \lim_{n\to\infty} \nu(E_n)$.

Proof. For (1), set $A_1 = E_1$ and $A_n = E_n \setminus E_{n-1}$ for $n \geq 2$. These sets are disjoint, and countable additivity gives

$$\nu\left(\bigcup_{n=1}^\infty E_n\right) = \sum_{n=1}^\infty \nu(A_n).$$

If $\nu(E_1)$ is finite, the Nth partial sum is $\nu(E_N)$, so the displayed series proves the assertion. If $\nu(E_1) = +\infty$, then finite additivity forces $\nu(E_N) = +\infty$ for every N and also gives $\nu(\bigcup_n E_n) = +\infty$; the case $\nu(E_1) = -\infty$ is analogous.

For (2), let $F_n = E_1 \setminus E_n$. Then $F_1 \subseteq F_2 \subseteq \dots$,

$$\bigcup_{n=1}^\infty F_n = E_1 \setminus \left(\bigcap_{n=1}^\infty E_n\right),$$

and

$$\lim_{n\to\infty} \nu(F_n) = \nu\left(\bigcup_{n=1}^\infty F_n\right) = \nu(E_1) - \nu\left(\bigcap_{n=1}^\infty E_n\right).$$

Since $|\nu(E_1)| < \infty$, we may subtract from $\nu(E_1)$ to obtain

$$\lim_{n\to\infty} \nu(E_n) = \nu\left(\bigcap_{n=1}^{\infty} E_n\right).$$

$\square$

Definition 19.9 Let ν be a signed measure on $(X, \mathcal{M})$. A set $E \in \mathcal{M}$ is called a **positive set** for ν if $\nu(F) \geq 0$ for every $F \in \mathcal{M}$ with $F \subseteq E$. It is called a **negative set** for ν if $\nu(F) \leq 0$ for every such set F. Finally, E is called a **null set** for ν if $\nu(F) = 0$ for every such set F.

Theorem 19.10 (Hahn decomposition theorem) If ν is a signed measure on $(X, \mathcal{M})$, there is a positive set P and a negative set N both contained in X such that $X = P \cup N$ and $P \cap N = \emptyset$. If P', N' are two other such sets, then $P \Delta P' = N \Delta N'$ is null. The pair (P, N) is called a **Hahn decomposition** for ν.

Definition 19.11 Given two signed measures ν_1 and ν_2, we write $\nu_1 \perp \nu_2$ and say that they are **mutually singular** if there exist $E, F \in \mathcal{M}$ such that $X = E \cup F$, $E \cap F = \emptyset$, E is null for ν_2, and F is null for ν_1.

Theorem 19.12 (Jordan decomposition theorem) If ν is a signed measure on $\mathcal{M}$, there are unique positive measures ν^+, ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

Figure 9 places the two decompositions side by side: the Hahn theorem separates the underlying space, while the Jordan theorem separates the signed measure into mutually singular positive measures supported on those pieces.

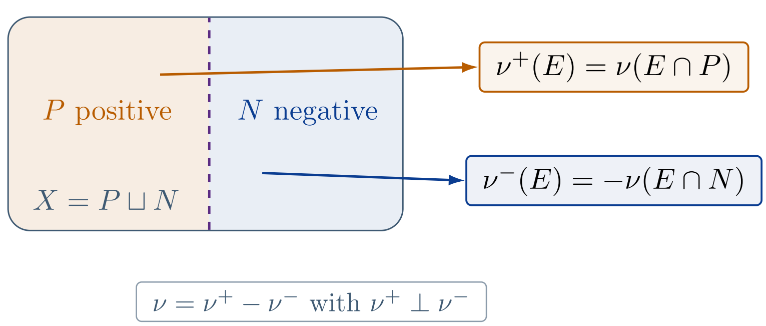

Figure 9

Lecture 20 — Thursday, February 25, 2016

For a signed measure ν on $(X, \mathcal{M})$, a set $E \in \mathcal{M}$ is positive, negative, or null according as every measurable set $F \subseteq E$ satisfies $\nu(F) \geq 0$, $\nu(F) \leq 0$, or $\nu(F) = 0$.

Example 20.1 Let $X = [-1, 1]$, let $\mathcal{M} = \mathcal{B}_{[-1,1]}$, and define

$$\nu(E) = -m([-1, 0] \cap E) + m([0, 1] \cap E),$$

where m is Lebesgue measure. Then ν is a signed measure. The set $[-1, 0)$ is negative and the set $(0, 1]$ is positive. There are also many sets $E \in \mathcal{M}$ with $\nu(E) = 0$ that are not null sets; for example,

$$\nu(X) = 0$$

even though X is not a null set.

Lemma 20.2 If $E \subseteq X$ is a positive set and $F \in \mathcal{M}$ with $F \subseteq E$, then F is positive. If $(E_n)_{n=1}^{\infty} \subseteq \mathcal{M}$ are positive sets, then $\bigcup_{n=1}^{\infty} E_n$ is a positive set. The same result holds for negative and null sets.

Proof. The first statement follows directly from the definition. For the second claim, set $F_n = E_n \setminus \left(\bigcup_{j=1}^{n-1} E_j\right)$. Then $F_n \in \mathcal{M}$, $F_n \subseteq E_n$ and so F_n is positive. Also, $F_n \cap F_j = \emptyset$ for $n \neq j$ and

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n.$$

Suppose $G \in \mathcal{M}$ and $G \subseteq \bigcup_{n=1}^{\infty} E_n$. Then $G = \bigcup_{n=1}^{\infty}(G \cap F_n)$ and the $G \cap F_n$'s are disjoint. Since $G \cap F_n \subseteq F_n$, $\nu(G \cap F_n) \geq 0$. Hence

$$\nu(G) = \sum_{n=1}^{\infty} \nu(G \cap F_n) \geq 0.$$

$\square$

Proof of Theorem 19.10. We may assume that $\nu(E) < \infty$ for all $E \in \mathcal{M}$; otherwise, replace ν by $-\nu$. This is legitimate because a signed measure cannot take both $+\infty$ and $-\infty$ as values.

Define

$$m = \sup\{\nu(E) \mid E \subseteq X \text{ is positive}\}.$$

For each $k \in \mathbb{N}$, fix a positive set $P_k \subseteq X$ such that $\nu(P_k) > m - 1/k$. Then

$$P := \bigcup_{k=1}^{\infty} P_k$$

is positive by the preceding lemma. For each $k \in \mathbb{N}$,

$$\nu(P) = \nu(P \setminus P_k) + \nu(P_k).$$

Since P is positive, $\nu(P \setminus P_k) \geq 0$, and therefore $\nu(P) \geq \nu(P_k)$ for all $k \in \mathbb{N}$. Letting $k \to \infty$ gives $\nu(P) \geq m$, and hence $\nu(P) = m$. In particular, $m < \infty$. Set $N = X \setminus P$. It remains to show that N is negative.

We use the following two facts:

1. If $E \in \mathcal{M}$, $E \subseteq N$, and E is positive, then E is null.
2. If $E \in \mathcal{M}$, $E \subseteq N$, and $\nu(E) > 0$, then there is a measurable subset $F \subseteq E$ with $\nu(F) > \nu(E)$.

To prove (1), suppose E is positive and not null. Then there is a measurable set $F \subseteq E$ with $\nu(F) > 0$. The sets P and F are disjoint, and $P \cup F$ is positive. Also,

$$\nu(P \cup F) = \nu(P) + \nu(F) = m + \nu(F) > m$$

because $m < \infty$. This contradicts the maximality of m.

To prove (2), suppose $E \in \mathcal{M}$, $E \subseteq N$, and $\nu(E) > 0$. If E were positive, then (1) would imply that E is null, contradicting $\nu(E) > 0$. Hence there is a measurable set $G \subseteq E$ with $\nu(G) < 0$. Set $F = E \setminus G$. Then $\nu(E) = \nu(F) + \nu(G) < \nu(F)$.

Now suppose, for a contradiction, that N is not negative. Choose $A_0 \subseteq N$ with $\nu(A_0) > 0$. Once A_{k-1} has been constructed, define

$$c_k = \sup\{\nu(B) - \nu(A_{k-1}) \mid B \subseteq A_{k-1} \text{ is measurable}\}.$$

By (2), $c_k > 0$. Let $n_k \in \mathbb{N}$ be the smallest positive integer such that $1/n_k < c_k$. Choose

$A_k \subseteq A_{k-1}$ with

$$\nu(A_k) - \nu(A_{k-1}) > \frac{1}{n_k}.$$

Finally, define $A = \bigcap_{k=1}^{\infty} A_k$. Then

$$\begin{aligned}\nu(A_k) &= \nu(A_{k-1}) + \nu(A_k) - \nu(A_{k-1}) > \nu(A_{k-2}) + \frac{1}{n_{k-1}} + \frac{1}{n_k} \\ &\geq \dots \\ &\geq \nu(A_0) + \sum_{j=1}^{k} \frac{1}{n_j}.\end{aligned}$$

Since $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ and $|\nu(A_0)| < \infty$, continuity from above gives $\nu(A_k) \to \nu(A)$. Therefore,

$$\sum_{j=1}^{\infty} \frac{1}{n_j} \leq \lim_{k \to \infty} \nu(A_k) = \nu(A) < \infty,$$

so the series converges, and in particular $n_k \to \infty$. Moreover,

$$\nu(A) \geq \sum_{j=1}^{\infty} \frac{1}{n_j} > 0.$$

By (2), there is a measurable set $B \subseteq A$ with $\nu(B) > \nu(A)$. Let $\delta = \nu(B) - \nu(A) > 0$. Since $n_k \to \infty$ and $\nu(A_{k-1}) \to \nu(A)$, we may choose k large enough that

$$\frac{1}{n_k - 1} < \frac{\delta}{2} \qquad \text{and} \qquad \nu(A_{k-1}) < \nu(A) + \frac{\delta}{2}.$$

Then $B \subseteq A \subseteq A_{k-1}$ and

$$\nu(B) - \nu(A_{k-1}) > \frac{\delta}{2} > \frac{1}{n_k - 1}.$$

Thus $c_k > 1/(n_k - 1)$, contradicting the minimality of n_k. Therefore N is negative.

It remains to prove uniqueness. Suppose (P, N) and (P', N') are Hahn decompositions. Then $P \setminus P' \subseteq P$, so $P \setminus P'$ is positive. Also $P \setminus P' \subseteq N'$, so $P \setminus P'$ is negative. Hence $P \setminus P'$ is null. Similarly, $P' \setminus P$ is null. Therefore

$$P \Delta P' = (P \setminus P') \cup (P' \setminus P)$$

is null. Since $N = X \setminus P$ and $N' = X \setminus P'$, we also have $N \Delta N' = P \Delta P'$. This proves uniqueness up to null sets. □

Lecture 21 — Friday, February 26, 2016

Definition 21.1 Given signed or complex measures ν_1 and ν_2, we write $\nu_1 \perp \nu_2$ and say that they are **mutually singular** if there exist $E, F \in \mathcal{M}$ such that $X = E \cup F$, $E \cap F = \emptyset$, E is null for ν_2, and F is null for ν_1.

Proof of Theorem 19.12. Let (P, N) be a Hahn decomposition for ν. For $E \in \mathcal{M}$, define

$$\nu^+(E) := \nu(E \cap P) \qquad \text{and} \qquad \nu^-(E) := -\nu(E \cap N).$$

Since P is positive and N is negative, $\nu^\pm$ are positive measures. Also

$$\nu(E) = \nu(E \cap P) + \nu(E \cap N) = \nu^+(E) - \nu^-(E).$$

Moreover, $\nu^-(P) = 0$ and $\nu^+(N) = 0$, so $\nu^+ \perp \nu^-$. For uniqueness, suppose $\nu = \mu^+ - \mu^-$, where $\mu^\pm$ are positive and $\mu^+ \perp \mu^-$. Choose disjoint sets $E, F \in \mathcal{M}$ with $E \cup F = X$, $\mu^-(E) = 0$, and $\mu^+(F) = 0$. Then every measurable subset of E has nonnegative ν-measure, so E is positive for ν; similarly, F is negative. Hence (E, F) is a Hahn decomposition. By uniqueness of Hahn decompositions up to ν-null sets from Theorem 19.10, the sets $E \Delta P$ and $F \Delta N$ are ν-null. Now, for $A \in \mathcal{M}$,

$$\mu^+(A) = \mu^+(A \cap E) = \nu(A \cap E) = \nu(A \cap P) = \nu^+(A),$$

where the middle equality uses $\mu^-(A \cap E) = 0$, and the last equality uses the fact that $E \Delta P$ is ν-null. Similarly, $\mu^-(A) = \nu^-(A)$. Therefore $\mu^\pm = \nu^\pm$. □

Definition 21.2 The pair (ν^+, ν^-) in Theorem 19.12 is called the **Jordan decomposition** of ν. The measures ν^+ and ν^- are the **positive and negative variations** of ν.

Theorem 21.3 If ν is a complex measure, then there exist unique signed measures $\mathrm{Re}(\nu)$ and $\mathrm{Im}(\nu)$ such that

$$\nu = \mathrm{Re}(\nu) + i\,\mathrm{Im}(\nu).$$

Proof. For $E \in \mathcal{M}$, define

$$\operatorname{Re}(\nu)(E) := \operatorname{Re}(\nu(E)) \qquad \text{and} \qquad \operatorname{Im}(\nu)(E) := \operatorname{Im}(\nu(E)).$$

Countable additivity of ν implies countable additivity of real and imaginary parts, so $\operatorname{Re}(\nu)$ and $\operatorname{Im}(\nu)$ are signed measures. Also,

$$\nu(E) = \operatorname{Re}(\nu)(E) + i \operatorname{Im}(\nu)(E) \quad \text{for all } E \in \mathcal{M}.$$

If $\nu = \sigma_1 + i\sigma_2$ with signed measures σ_1 and σ_2, then taking real and imaginary parts gives $\sigma_1 = \operatorname{Re}(\nu)$ and $\sigma_2 = \operatorname{Im}(\nu)$. Hence the decomposition is unique. □

Corollary 21.4 Every complex measure is a linear combination of (four) positive measures.

Proof. If ν is a complex measure, then by Theorem 21.3, $\nu = \operatorname{Re}(\nu) + i \operatorname{Im}(\nu)$. By Theorem 19.12, $\operatorname{Re}(\nu) = \operatorname{Re}(\nu)^+ - \operatorname{Re}(\nu)^-$ and $\operatorname{Im}(\nu) = \operatorname{Im}(\nu)^+ - \operatorname{Im}(\nu)^-$. Hence

$$\nu = \operatorname{Re}(\nu)^+ - \operatorname{Re}(\nu)^- + i \operatorname{Im}(\nu)^+ - i \operatorname{Im}(\nu)^-.$$

□

Definition 21.5 Let ν be a signed or complex measure on $(X, \mathcal{M})$. Define $|\nu| : \mathcal{M} \to [0, \infty]$ by

$$|\nu|(E) = \sup\left\{ \sum_{i=1}^{n} |\nu(E_i)| : E = \bigcup_{i=1}^{n} E_i,\ E_i \in \mathcal{M},\ E_i \cap E_j = \emptyset \text{ for } i \neq j \right\}.$$

The set function $|\nu|$ is called the **total variation** of ν. We will shortly prove that it is a positive measure.

Lemma 21.6 If ν is a signed measure, then $|\nu| = \nu^+ + \nu^-$.

Proof. Let (P, N) be a Hahn decomposition for ν. Fix $E \in \mathcal{M}$. Since $(P \cap E)$ and $(N \cap E)$ form a measurable partition of E,

$$|\nu|(E) \geq |\nu(P \cap E)| + |\nu(N \cap E)| = \nu^+(E) + \nu^-(E).$$

For the opposite inequality, let $E_1, \ldots, E_n$ be a measurable partition of E. Then

$$\sum_{i=1}^{n} |\nu(E_i)| = \sum_{i=1}^{n} |\nu^+(E_i) - \nu^-(E_i)| \leq \sum_{i=1}^{n} (\nu^+(E_i) + \nu^-(E_i)) = \nu^+(E) + \nu^-(E).$$

Taking the supremum over all finite measurable partitions of E gives $|\nu|(E) \leq \nu^+(E) + \nu^-(E)$. □

Lemma 21.7 For $E \in \mathcal{M}$,

$$|\nu|(E) = \sup\left\{\sum_{i=1}^{\infty} |\nu(E_i)| : E = \bigcup_{i=1}^{\infty} E_i,\ E_i \in \mathcal{M},\ E_i \cap E_j = \emptyset \text{ for } i \neq j\right\}.$$

where the supremum is taken over countable measurable partitions of E.

Note the difference between Definition 21.5 and Lemma 21.7: the former takes the supremum over finite partitions, while the latter takes the supremum over countable partitions.

Proof. Let $S_{\text{fin}}(E)$ denote the finite partition supremum in the definition of $|\nu|(E)$, and let $S_{\text{ct}}(E)$ denote the countable partition supremum in this lemma. We have $S_{\text{fin}}(E) \leq S_{\text{ct}}(E)$, since a finite partition can be extended to a countable partition by adding empty sets.

Conversely, fix $c < S_{\text{ct}}(E)$ and choose a countable measurable partition $(E_n)_{n=1}^{\infty}$ of E such that

$$\sum_{n=1}^{\infty} |\nu(E_n)| > c.$$

For some $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} |\nu(E_n)| > c.$$

Let $F = \bigcup_{n=N+1}^{\infty} E_n$. Then $E_1, \ldots, E_N, F$ form a finite measurable partition of E, and

$$|\nu(F)| + \sum_{n=1}^{N} |\nu(E_n)| > c,$$

so $S_{\text{fin}}(E) > c$. Since $c < S_{\text{ct}}(E)$ was arbitrary, $S_{\text{ct}}(E) \leq S_{\text{fin}}(E)$. □

Theorem 21.8 If ν is a signed or complex measure, then $|\nu|$ is a positive measure.

Proof. Fix pairwise disjoint sets $(E_n)_{n\geq 1} \subseteq \mathcal{M}$, and let $E = \bigcup_{n=1}^{\infty} E_n$. If $(F_k)_{k\geq 1}$ is a countable measurable partition of E, then countable additivity and Tonelli's theorem in (1) give

$$\begin{aligned}\sum_{k=1}^{\infty} |\nu(F_k)| \leq \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} |\nu(F_k \cap E_n)| &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\nu(F_k \cap E_n)| \\ &\leq \sum_{n=1}^{\infty} |\nu|(E_n),\end{aligned}$$

since $(F_k \cap E_n)_{k\geq 1}$ is a countable measurable partition of E_n for each n. Taking the supremum over all countable measurable partitions of E and using Lemma 21.7 gives

$$|\nu|(E) \leq \sum_{n=1}^{\infty} |\nu|(E_n).$$

For the reverse inequality, fix $N \in \mathbb{N}$ and choose numbers $c_n < |\nu|(E_n)$ for $1 \leq n \leq N$. For each such n, choose a countable measurable partition $(E_{n,k})_{k\geq 1}$ of E_n such that

$$\sum_{k=1}^{\infty} |\nu(E_{n,k})| > c_n.$$

The sets $E_{n,k}$ for $1 \leq n \leq N$ and $k \geq 1$, together with the remaining set $\bigcup_{n=N+1}^{\infty} E_n$, form a countable measurable partition of E. Therefore,

$$\sum_{n=1}^{N} c_n \leq \sum_{n=1}^{N}\sum_{k=1}^{\infty} |\nu(E_{n,k})| \leq |\nu|(E).$$

Taking the supremum over $c_n < |\nu|(E_n)$ gives

$$\sum_{n=1}^{N} |\nu|(E_n) \leq |\nu|(E).$$

Letting $N \to \infty$ gives

$$\sum_{n=1}^{\infty} |\nu|(E_n) \leq |\nu|(E).$$

Thus $|\nu|$ is countably additive. Since $|\nu|(\emptyset) = 0$ and $|\nu|(E) \geq 0$ for every $E \in \mathcal{M}$, it is a positive measure. □

Lecture 22 — Tuesday, March 01, 2016

Recall that if ν is a signed or complex measure on $(X, \mathcal{M})$, then its total variation is given by

$$|\nu|(E) = \sup\left\{\sum_{i=1}^{n} |\nu(E_i)| : E = \bigcup_{i=1}^{n} E_i,\ E_i \in \mathcal{M},\ E_i \cap E_j = \emptyset \text{ for } i \neq j\right\}.$$

By Theorem 21.8, $|\nu|$ is a positive measure on $(X, \mathcal{M})$. For signed measures, Lemma 21.6 identifies the total variation explicitly as $|\nu| = \nu^+ + \nu^-$.

Theorem 22.1 If ν is a complex measure, then $|\nu|$ is finite.

Proof. The real and imaginary parts of ν are finite signed measures, so their total variations are finite. If $E_1, \ldots, E_n$ partition X, then

$$\sum_{i=1}^{n} |\nu(E_i)| \leq \sum_{i=1}^{n} \big(|\operatorname{Re}(\nu)(E_i)| + |\operatorname{Im}(\nu)(E_i)|\big) \leq |\operatorname{Re}(\nu)|(X) + |\operatorname{Im}(\nu)|(X) < \infty.$$

Taking the supremum over finite partitions gives $|\nu|(X) < \infty$. □

Definition 22.2 For a complex or signed measure ν, define $L^1(\nu) = L^1(X, |\nu|)$. That is, $f \in L^1(\nu)$ if and only if

$$\int_X |f| \, \mathrm{d}|\nu| < \infty.$$

If ν is complex and $f \in L^1(\nu)$, then

$$\int f \, \mathrm{d}\nu = \int f \, \mathrm{d}\operatorname{Re}(\nu) + i \int f \, \mathrm{d}\operatorname{Im}(\nu).$$

Fact 22.3 If ν and ω are two complex measures and $f \in L^1(\nu) \cap L^1(\omega)$, then $f \in L^1(\nu+\omega)$, and
$$\int f \, \mathrm{d}(\nu+\omega) = \int f \, \mathrm{d}\nu + \int f \, \mathrm{d}\omega.$$

Proof. Since $|\nu+\omega| \le |\nu| + |\omega|$,
$$\int_X |f| \, \mathrm{d}|\nu+\omega| \le \int_X |f| \, \mathrm{d}|\nu| + \int_X |f| \, \mathrm{d}|\omega| < \infty,$$
so $f \in L^1(\nu+\omega)$. Also,
$$\int f \, \mathrm{d}(\nu+\omega) = \int f \, \mathrm{d}\nu + \int f \, \mathrm{d}\omega$$
by linearity of the integral with respect to positive measures. □

Definition 22.4 Let ν be a signed or complex measure, and let μ be a positive measure. We say that ν is **absolutely continuous with respect to $\boldsymbol{\mu}$** if, for every $E \in \mathcal{M}$, $\mu(E) = 0$ implies $\nu(E) = 0$. In this case, we write $\nu \ll \mu$.

Proposition 22.5 Let ν be a signed or complex measure, and let μ be a positive measure. Then $\nu \ll \mu$ if and only if $|\nu| \ll \mu$.

Proof. *Forward implication.* Suppose $\nu \ll \mu$, and let $E \in \mathcal{M}$ satisfy $\mu(E) = 0$. If $E_1, \ldots, E_n$ is a measurable partition of E, then $\mu(E_i) = 0$ and hence $\nu(E_i) = 0$ for every i. Therefore,
$$\sum_{i=1}^{n} |\nu(E_i)| = 0.$$
Taking the supremum over all finite measurable partitions of E gives $|\nu|(E) = 0$, so $|\nu| \ll \mu$.

Reverse implication. Suppose $|\nu| \ll \mu$. If $\mu(E) = 0$, then $|\nu|(E) = 0$. Since $|\nu(E)| \le |\nu|(E)$, we get $\nu(E) = 0$. Hence $\nu \ll \mu$. □

Example 22.6 Suppose μ is a positive measure and $f \in L^1(\mu)$. For $E \in \mathcal{M}$, define

$$\nu(E) = \int_E f \,\mathrm{d}\mu.$$

Then ν is a complex measure. Also, $\nu \ll \mu$. Note that the expression

$$\nu(E) = \int_E f \,\mathrm{d}\mu$$

is often written as $\mathrm{d}\nu = f\,\mathrm{d}\mu$.

Theorem 22.7 Let ν be a complex measure and let μ be a positive measure on $(X, \mathcal{M})$. Then $\nu \ll \mu$ if and only if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $E \in \mathcal{M}$,

$$\mu(E) < \delta \Longrightarrow |\nu|(E) < \varepsilon.$$

This is vaguely reminiscent of the ε-δ definition of continuity for functions. That is not a coincidence! In fact, if we view ν as a function from $\mathcal{M}$ to $\mathbb{C}$, then the condition in Theorem 22.7 is precisely the ε-δ definition of continuity of $|\nu|$ at the point $\emptyset$, where $\mathcal{M}$ is equipped with the pseudometric $d(E,F) = \mu(E \Delta F)$. Consequently, ν itself is continuous at $\emptyset$ with respect to this pseudometric because $|\nu(E)| \le |\nu|(E)$ for every $E \in \mathcal{M}$.

Proof. By Proposition 22.5, it suffices to prove the result when ν is a finite positive measure.

Forward implication. Assume $\nu \ll \mu$, and suppose the δ-ε condition fails. Then there exists $\varepsilon_0 > 0$ such that, for every $n \in \mathbb{N}$, there exists $E_n \in \mathcal{M}$ satisfying

$$\mu(E_n) < 2^{-n} \qquad \text{and} \qquad \nu(E_n) \ge \varepsilon_0.$$

Set

$$E = \limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k.$$

Since ν is finite, continuity from above gives

$$\nu(E) = \lim_{n\to\infty} \nu\left(\bigcup_{k=n}^{\infty} E_k\right) \ge \limsup_{n\to\infty} \nu(E_n) \ge \varepsilon_0.$$

On the other hand,

$$\mu\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \sum_{k=n}^{\infty} 2^{-k} = 2^{-n+1}.$$

Therefore,

$$\mu(E) = \lim_{n\to\infty} \mu\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \lim_{n\to\infty} 2^{-n+1} = 0.$$

Thus $\mu(E) = 0$ but $\nu(E) \geq \varepsilon_0$, contradicting $\nu \ll \mu$.

Reverse implication. Assume the δ-ε condition. If $E \in \mathcal{M}$ and $\mu(E) = 0$, then $\nu(E) < \varepsilon$ for every $\varepsilon > 0$. Hence $\nu(E) = 0$, so $\nu \ll \mu$. □

Corollary 22.8 Let μ be a positive measure and let $f \in L^1(X, \mu)$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $E \in \mathcal{M}$,

$$\mu(E) < \delta \implies \int_E |f| \, \mathrm{d}\mu < \varepsilon.$$

Proof. Define the finite positive measure η by

$$\eta(E) = \int_E |f| \, \mathrm{d}\mu.$$

Then $\eta \ll \mu$, so the result follows from Theorem 22.7. □

Lecture 23 — Thursday, March 03, 2016

Theorem 23.1 (Lebesgue–Radon–Nikodym theorem) Let μ be a σ-finite positive measure on $(X, \mathcal{M})$, and let ν be either a signed measure whose total variation is σ-finite or a complex measure. Then the following hold:

1. There exist unique measures λ and ρ of the same type as ν such that $\lambda \perp \mu$, $\rho \ll \mu$, and $\nu = \lambda + \rho$. In the signed case they are σ-finite.

2. If ν is signed, there exists a measurable function $f : X \to \mathbb{R}$ such that either
$$\int_X f^+ \, \mathrm{d}\mu < \infty$$
or
$$\int_X f^- \, \mathrm{d}\mu < \infty,$$
and $\mathrm{d}\rho = f \, \mathrm{d}\mu$. If f_1 and f_2 both have this property, then $f_1 = f_2$ μ almost everywhere.
3. If ν is a complex measure, there exists a unique $f \in L^1(X, \mu)$ with $\mathrm{d}\rho = f \, \mathrm{d}\mu$.

The decomposition separates the part of ν visible to μ from the part concentrated on a μ-null set, as summarized in Figure 10.

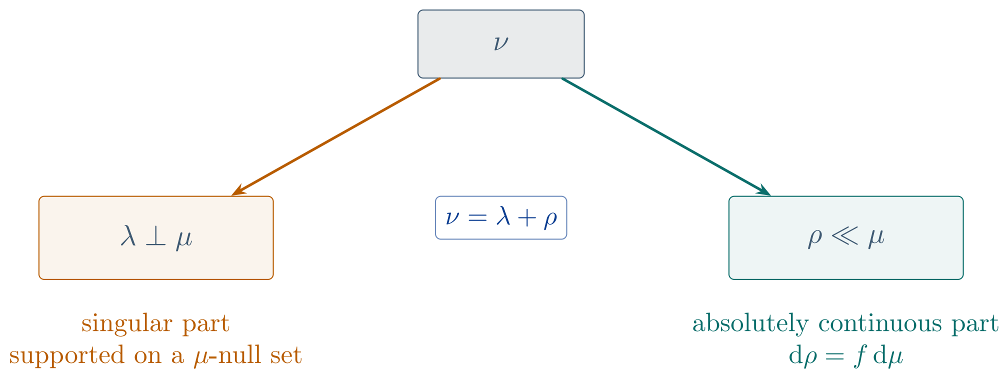

FIGURE 10

Lemma 23.2 Suppose μ and ν are finite positive measures. Either $\nu \perp \mu$, or there exist $\varepsilon > 0$ and $E \in \mathcal{M}$ such that $\mu(E) > 0$, $\nu(E) > 0$, and
$$\nu(F) \geq \varepsilon \mu(F)$$
for every $F \in \mathcal{M}$ with $F \subseteq E$. Equivalently, E is positive for the signed measure $\nu - \varepsilon\mu$.

Proof. For each $k \in \mathbb{N}$, let (P_k, N_k) be a Hahn decomposition for the signed measure $\nu - k^{-1}\mu$. Set
$$P = \bigcup_{k=1}^{\infty} P_k \qquad \text{and} \qquad N = \bigcap_{k=1}^{\infty} N_k.$$
If $\mu(P_k) > 0$ for some k, then P_k is positive for $\nu - k^{-1}\mu$, and the conclusion holds with $\varepsilon = 1/k$ and $E = P_k$. Thus assume $\mu(P_k) = 0$ for every k. Then $\mu(P) = 0$.

It remains to show that $\nu(N) = 0$. Since $N \subseteq N_k$ for every k and N_k is negative for $\nu - k^{-1}\mu$,

$$0 \leq \nu(N) \leq \frac{1}{k}\mu(N) \leq \frac{1}{k}\mu(X)$$

for every k. Letting $k \to \infty$ gives $\nu(N) = 0$. Therefore P and N witness $\nu \perp \mu$. □

Proof of Theorem 23.1. Assume first that μ, ν are finite, positive measures. Let

$$\mathcal{F} = \{f : X \to [0, \infty] : f \text{ is measurable, } \int_E f \, \mathrm{d}\mu \leq \nu(E) \text{for all } E \in \mathcal{M}\}.$$

The class $\mathcal{F}$ is nonempty because $0 \in \mathcal{F}$. If $f, g \in \mathcal{F}$, then $h = \max\{f, g\}$ also belongs to $\mathcal{F}$. Indeed, with $A = \{x \in X : f(x) \geq g(x)\}$, for every $E \in \mathcal{M}$ we have

$$\begin{aligned}
\int_E h \, \mathrm{d}\mu &= \int_{E \cap A} h \, \mathrm{d}\mu + \int_{E \setminus A} h \, \mathrm{d}\mu \\
&= \int_{E \cap A} f \, \mathrm{d}\mu + \int_{E \setminus A} g \, \mathrm{d}\mu \\
&\leq \nu(E \cap A) + \nu(E \setminus A) \\
&= \nu(E).
\end{aligned}$$

Also, if $f_1 \leq f_2 \leq \cdots$ are functions in $\mathcal{F}$ and $f = \lim_{n\to\infty} f_n$, then $f \in \mathcal{F}$, since the monotone convergence theorem gives

$$\int_E f \, \mathrm{d}\mu = \lim_{n\to\infty} \int_E f_n \, \mathrm{d}\mu \leq \nu(E).$$

Let

$$a = \sup\{\int_X f \, \mathrm{d}\mu : f \in \mathcal{F}\}.$$

Choose a sequence $(f_n)_{n=1}^{\infty} \subseteq \mathcal{F}$ such that

$$\int_X f_n \, \mathrm{d}\mu \to a.$$

Replacing f_n by $\max\{f_1, \ldots, f_n\}$, we may assume $f_1 \leq f_2 \leq \cdots$. Set $f = \lim_{n\to\infty} f_n$. By the preceding paragraph, $f \in \mathcal{F}$, and

$$\int_X f \, \mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu = a.$$

Define ρ by $\mathrm{d}\rho = f\,\mathrm{d}\mu$, that is,

$$\rho(E) = \int_E f\,\mathrm{d}\mu$$

for $E \in \mathcal{M}$, and set $\lambda = \nu - \rho$. Since $f \in \mathcal{F}$, the measure λ is positive. We have $\rho \ll \mu$ and $\nu = \lambda + \rho$.

It remains to show that $\lambda \perp \mu$. If not, the preceding lemma gives $\varepsilon > 0$ and $E \in \mathcal{M}$ such that $\mu(E) > 0$ and E is positive for $\lambda - \varepsilon\mu$. Then, for every $A \in \mathcal{M}$,

$$\int_A (f + \varepsilon \mathbb{1}_E)\,\mathrm{d}\mu = \rho(A) + \varepsilon\mu(A \cap E) \le \rho(A) + \lambda(A \cap E) \le \rho(A) + \lambda(A) = \nu(A).$$

Thus $f + \varepsilon\mathbb{1}_E \in \mathcal{F}$. But

$$\int (f + \varepsilon \cdot \mathbb{1}_E)\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu + \varepsilon\mu(E) > a.$$

This contradicts the definition of a, so $\lambda \perp \mu$.

Now assume that μ and ν are σ-finite positive measures. Choose a measurable partition $(Z_k)_{k=1}^{\infty}$ of X such that

$$X = \bigcup_{k=1}^{\infty} Z_k, \mu(Z_k) < \infty, \nu(Z_k) < \infty.$$

For each n, define finite positive measures

$$\mu_n(E) = \mu(E \cap Z_n) \qquad \text{and} \qquad \nu_n(E) = \nu(E \cap Z_n).$$

Applying the finite case gives $\lambda_n \perp \mu_n$, $\rho_n \ll \mu_n$, $\nu_n = \lambda_n + \rho_n$, and a nonnegative measurable function f_n supported on Z_n such that $\mathrm{d}\rho_n = f_n\,\mathrm{d}\mu_n$. Set

$$\lambda = \sum_{n=1}^{\infty} \lambda_n, \rho = \sum_{n=1}^{\infty} \rho_n, f = \sum_{n=1}^{\infty} f_n.$$

Then $\nu = \lambda + \rho$ by countable additivity on the partition (Z_n). Also $\rho \ll \mu$, because $\mu(E) = 0$ implies $\mu_n(E) = 0$ for every n, hence $\rho_n(E) = 0$ for every n. To see $\lambda \perp \mu$, for each n choose $A_n \in \mathcal{M}$ such that $\mu_n(A_n) = 0$ and $\lambda_n(X \setminus A_n) = 0$. Let $A = \bigcup_{n=1}^{\infty}(A_n \cap Z_n)$. Then $\mu(A) = 0$ and $\lambda(X \setminus A) = 0$, so $\lambda \perp \mu$.

Finally, for every $E \in \mathcal{M}$,

$$\int_E f\,\mathrm{d}\mu = \sum_{n=1}^{\infty} \int_E f_n\,\mathrm{d}\mu = \sum_{n=1}^{\infty} \rho_n(E \cap Z_n) = \rho(E)$$

by the monotone convergence theorem. Thus $\mathrm{d}\rho = f\,\mathrm{d}\mu$.

If ν is signed, apply the positive case to ν^+ and ν^-. If ν is complex, apply the signed case to $\mathrm{Re}(\nu)$ and $\mathrm{Im}(\nu)$. The uniqueness statements are proved in Theorem 24.2. □

Lecture 24 — Friday, March 04, 2016

In Theorem 23.1, we proved existence of the Lebesgue–Radon–Nikodym decomposition

$$\mathrm{d}\nu = \mathrm{d}\lambda + f\,\mathrm{d}\mu$$

when μ is σ-finite and positive and ν is signed or complex. We now prove uniqueness.

Lemma 24.1 Let ρ be a signed or complex measure, and let μ be a positive measure. If $\rho \perp \mu$ and $\rho \ll \mu$, then $\rho = 0$.

Proof. Choose disjoint measurable sets E and F with $E \cup F = X$, $\mu(E) = 0$, and $|\rho|(F) = 0$. Since $\rho \ll \mu$, Proposition 22.5 gives $|\rho| \ll \mu$, so $|\rho|(E) = 0$. Therefore $|\rho|(X) = 0$, and hence $\rho = 0$. □

Theorem 24.2 (Uniqueness in the Lebesgue–Radon–Nikodym theorem) Assume μ is a σ-finite positive measure and ν is a signed measure whose total variation is σ-finite or a complex measure. Suppose

$$\mathrm{d}\nu = \mathrm{d}\lambda_1 + f_1\,\mathrm{d}\mu = \mathrm{d}\lambda_2 + f_2\,\mathrm{d}\mu$$

are two decompositions as in Theorem 23.1. Then $\lambda_1 = \lambda_2$ and $f_1 = f_2$ μ-almost everywhere.

Proof. We first handle the case in which μ and $|\nu|$ are finite. Choose a μ-null set A_i supporting λ_i. Then the absolutely continuous part vanishes on A_i, while λ_i vanishes on A_i^c. Consequently, λ_i and $f_i\,\mathrm{d}\mu$ are restrictions of the finite measure ν to disjoint pieces, so both are finite. In the signed case, finiteness of $f_i\,\mathrm{d}\mu$ on the sets $\{f_i \geq 0\}$ and $\{f_i < 0\}$ shows that $\int f_i^+\,\mathrm{d}\mu$ and $\int f_i^-\,\mathrm{d}\mu$ are finite. In the complex case, $f_i \in L^1(\mu)$ is already part of Theorem 23.1. Thus $f_1, f_2 \in L^1(\mu)$ in either case.

Now,

$$\mathrm{d}(\lambda_1 - \lambda_2) = (f_2 - f_1)\,\mathrm{d}\mu.$$

The measure $\lambda_1 - \lambda_2$ is both mutually singular with respect to μ and absolutely continuous with respect to μ. By the preceding lemma, $\lambda_1 - \lambda_2 = 0$, so $\lambda_1 = \lambda_2$. Hence

$$\int_E (f_2 - f_1)\,\mathrm{d}\mu = 0$$

for every $E \in \mathcal{M}$. Set $g = f_2 - f_1$. If $g \neq 0$ on a set of positive measure, then one of the sets

$$\{\mathrm{Re}(g) > 0\}, \qquad \{\mathrm{Re}(g) < 0\}, \qquad \{\mathrm{Im}(g) > 0\}, \qquad \text{and} \qquad \{\mathrm{Im}(g) < 0\}$$

has positive measure. For instance, suppose $A = \{\mathrm{Re}(g) > 0\}$ has positive measure. Let

$$E_n = \{x \in X : \mathrm{Re}(g(x)) \geq 1/n\}.$$

Since $A = \bigcup_{n=1}^{\infty} E_n$, there exists n such that $\mu(E_n) > 0$. Then

$$\mathrm{Re}\left(\int_{E_n} g\,\mathrm{d}\mu\right) = \int_{E_n} \mathrm{Re}(g)\,\mathrm{d}\mu \geq \frac{1}{n}\mu(E_n) > 0,$$

contradicting $\int_{E_n} g\,\mathrm{d}\mu = 0$. The other three cases are identical. Therefore $g = 0$ μ almost everywhere, so $f_1 = f_2$ μ almost everywhere.

For the general case, choose a measurable partition $(X_n)_{n=1}^{\infty}$ of X such that

$$\mu(X_n) < \infty \qquad \text{and} \qquad |\nu|(X_n) < \infty$$

for every n. If

$$\mathrm{d}\nu = \mathrm{d}\lambda_1 + f_1\,\mathrm{d}\mu = \mathrm{d}\lambda_2 + f_2\,\mathrm{d}\mu,$$

then for every $E \in \mathcal{M}$,

$$\begin{aligned}\nu(E \cap X_n) &= \lambda_1(E \cap X_n) + \int_{E \cap X_n} f_1\,\mathrm{d}\mu \\ &= \lambda_2(E \cap X_n) + \int_{E \cap X_n} f_2\,\mathrm{d}\mu.\end{aligned}$$

By the finite case, $\lambda_1(E \cap X_n) = \lambda_2(E \cap X_n)$ and

$$\int_{E \cap X_n} f_1\,\mathrm{d}\mu = \int_{E \cap X_n} f_2\,\mathrm{d}\mu.$$

Therefore, for every $E \in \mathcal{M}$,

$$\lambda_1(E) = \sum_{n=1}^{\infty} \lambda_1(E \cap X_n) = \sum_{n=1}^{\infty} \lambda_2(E \cap X_n) = \lambda_2(E);$$

that is, $\lambda_1 = \lambda_2$. Also,

$$\int_E f_1 \mathbb{1}_{X_n} \,\mathrm{d}\mu = \int_E f_2 \mathbb{1}_{X_n} \,\mathrm{d}\mu,$$

so $f_1 \mathbb{1}_{X_n} = f_2 \mathbb{1}_{X_n}$ μ almost everywhere for every n. Hence

$$f_1 = \sum_{n=1}^{\infty} f_1 \mathbb{1}_{X_n} = \sum_{n=1}^{\infty} f_2 \mathbb{1}_{X_n} = f_2.$$

□

Definition 24.3 If μ and ν are σ-finite, with μ positive and ν signed or complex, then the decomposition $\nu = \lambda + \rho$, where $\lambda \perp \mu$ and $\rho \ll \mu$, is called the **Lebesgue decomposition** of ν with respect to μ. If $\nu \ll \mu$, the function f satisfying $\mathrm{d}\nu = f \,\mathrm{d}\mu$ is called the **Radon–Nikodym derivative** of ν with respect to μ. We often write

$$f = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \qquad \text{and} \qquad \mathrm{d}\nu = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \,\mathrm{d}\mu.$$

Proposition 24.4 If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, then $(\nu_1 + \nu_2) \ll \mu$ and

$$\frac{\mathrm{d}(\nu_1 + \nu_2)}{\mathrm{d}\mu} = \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} + \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu}$$

μ-almost everywhere.

Proof. The absolute continuity $(\nu_1 + \nu_2) \ll \mu$ follows immediately from $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$. It is enough to show that, for every $E \in \mathcal{M}$,

$$(\nu_1 + \nu_2)(E) = \int_E \left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} + \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu} \right) \mathrm{d}\mu.$$

But

$$(\nu_1 + \nu_2)(E) = \nu_1(E) + \nu_2(E) = \int_E \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} \,\mathrm{d}\mu + \int_E \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu} \,\mathrm{d}\mu = \int_E \left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} + \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu} \right) \mathrm{d}\mu.$$

The result follows from uniqueness of the Radon–Nikodym derivative. □

The plan now is to prove the corresponding formula for total variation: if ν is a complex measure, μ is a positive measure, and $\nu \ll \mu$, then $|\nu| \ll \mu$ and

$$\left|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right| = \frac{\mathrm{d}|\nu|}{\mathrm{d}\mu}$$

μ-almost everywhere. But it will take some work.

Lemma 24.5 If $\mathcal{U} \subseteq \mathbb{C}$ is open, then there are sequences $(z_n)_{n=1}^{\infty} \subseteq \mathbb{C}$ and $(r_n)_{n=1}^{\infty} \subseteq (0,\infty)$ such that, if

$$B_n = \{z \in \mathbb{C} : |z - z_n| \leq r_n\},$$

then each $B_n \subseteq \mathcal{U}$ and $\bigcup_{n=1}^{\infty} B_n = \mathcal{U}$.

Proof. Consider all closed balls

$$\overline{B}(q,r) = \{z \in \mathbb{C} : |z - q| \leq r\},$$

where $q \in \mathbb{Q} + i\mathbb{Q}$, $r \in \mathbb{Q}_{>0}$, and $\overline{B}(q,r) \subseteq \mathcal{U}$. This family is countable. It also covers $\mathcal{U}$: if $z \in \mathcal{U}$, choose $\delta > 0$ with $B_\delta(z) \subseteq \mathcal{U}$, then choose $q \in \mathbb{Q} + i\mathbb{Q}$ close enough to z and a rational r such that

$$|z - q| < r < \delta - |z - q|.$$

The resulting closed ball contains z and lies in $\mathcal{U}$. Enumerating these balls proves the claim. $\square$

Lemma 24.6 Suppose μ is a finite positive measure on $(X, \mathcal{M})$, and $f \in L^1(X,\mu)$. Assume $S \subseteq \mathbb{C}$ is closed, with

$$\frac{1}{\mu(E)} \int_E f \, \mathrm{d}\mu \in S$$

for every $E \in \mathcal{M}$ with $\mu(E) > 0$. Then $f(x) \in S$ for μ almost every $x \in X$.

Proof. If $S = \mathbb{C}$, there is nothing to prove. Otherwise fix $z \in \mathbb{C} \setminus S$ and $r > 0$ with $B = \{z' \in \mathbb{C} : |z' - z| \leq r\} \subseteq \mathbb{C} \setminus S$. Set $E = f^{-1}(B)$ and assume for a contradiction that $\mu(E) > 0$. For $x \in E$, $|f(x) - z| \leq r$. Now,

$$\left|\frac{1}{\mu(E)} \int_E f \, \mathrm{d}\mu - z\right| = \left|\frac{1}{\mu(E)} \int_E (f - z) \, \mathrm{d}\mu\right| \leq \frac{1}{\mu(E)} \int_E |f - z| \, \mathrm{d}\mu \leq r.$$

So

$$\frac{1}{\mu(E)} \int_E f \, \mathrm{d}\mu \in B \subseteq \mathbb{C} \setminus S,$$

a contradiction. So $\mu(f^{-1}(B)) = 0$ for all closed balls $B \subseteq \mathbb{C} \setminus S$. By Lemma 24.5, $\mu(f^{-1}(S^c)) = 0$. □

Proposition 24.7 If ν is a complex measure, then $\nu \ll |\nu|$ and

$$\left| \frac{\mathrm{d}\nu}{\mathrm{d}|\nu|} \right| = 1$$

$|\nu|$-almost everywhere.

Proof. The absolute continuity $\nu \ll |\nu|$ follows immediately from the inequality $|\nu(E)| \leq |\nu|(E)$. Let

$$h = \frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}.$$

We first show that $|h| \geq 1$ $|\nu|$ almost everywhere. Fix $n \in \mathbb{N}$ and set

$$E_n = \{x \in X : |h(x)| \leq 1 - 1/n\}.$$

Let $A_1, \ldots, A_k$ be a finite measurable partition of E_n. Then

$$\begin{aligned}
\sum_{i=1}^k |\nu(A_i)| &= \sum_{i=1}^k \left| \int_{A_i} h \, \mathrm{d}|\nu| \right| \\
&\leq \sum_{i=1}^k \int_{A_i} |h| \, \mathrm{d}|\nu| \\
&\leq \left(1 - \frac{1}{n}\right) \sum_{i=1}^k |\nu|(A_i) \\
&= \left(1 - \frac{1}{n}\right) |\nu|(E_n).
\end{aligned}$$

Taking the supremum over all finite measurable partitions of E_n gives

$$|\nu|(E_n) \leq \left(1 - \frac{1}{n}\right) |\nu|(E_n),$$

so $|\nu|(E_n) = 0$. Since $\{|h| < 1\} = \bigcup_{n=1}^\infty E_n$, we have $|h| \geq 1$ $|\nu|$ almost everywhere.

We next show that $|h| \leq 1$ $|\nu|$ almost everywhere. For every $E \in \mathcal{M}$ with $|\nu|(E) > 0$,

$$|\frac{1}{|\nu|(E)} \int_E h \, \mathrm{d}|\nu|| = |\frac{\nu(E)}{|\nu|(E)}| \leq 1.$$

By Lemma 24.6, applied with $S = \{z \in \mathbb{C} : |z| \leq 1\}$ and the measure $|\nu|$, we get $|h| \leq 1$ $|\nu|$ almost everywhere. Therefore $|h| = 1$ $|\nu|$ almost everywhere. □

Lecture 25 — Tuesday, March 08, 2016

Theorem 25.1 Let ν be a complex measure and let μ be a positive, σ-finite measure on $(X, \mathcal{M})$ such that $\nu \ll \mu$. Then $|\nu| \ll \mu$ and

$$\left|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right| = \frac{\mathrm{d}|\nu|}{\mathrm{d}\mu}$$

μ almost everywhere.

Proof. By Proposition 22.5, $\nu \ll \mu$ implies $|\nu| \ll \mu$. By Proposition 24.7, there is a measurable function $h : X \to \mathbb{C}$ with $|h| = 1$ $|\nu|$ almost everywhere and

$$\mathrm{d}\nu = h \, \mathrm{d}|\nu|.$$

Redefining h on a $|\nu|$-null set if necessary, we may assume $|h| = 1$ everywhere. Since $|\nu| \ll \mu$,

$$\mathrm{d}\nu = h \, \mathrm{d}|\nu| = h \, \frac{\mathrm{d}|\nu|}{\mathrm{d}\mu} \, \mathrm{d}\mu.$$

Therefore,

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = h \, \frac{\mathrm{d}|\nu|}{\mathrm{d}\mu}$$

μ almost everywhere. Taking absolute values and using $\frac{\mathrm{d}|\nu|}{\mathrm{d}\mu} \geq 0$ μ almost everywhere gives the desired identity. □

Differentiation and Bounded Variation

Definition 25.2 A family of Borel sets $(E_r)_{r>0}$ **shrinks nicely** to $x \in \mathbb{R}$ if the following hold:

1. We have $E_r \subseteq (x-r, x+r)$ for every $r > 0$.
2. There exists $\alpha > 0$, independent of r, such that $m(E_r) > \alpha r$ for every $r > 0$.

In particular,

$$0 \leq \frac{m(E_r)}{m(x-r, x+r)} \leq 1.$$

Theorem 25.3 (Lebesgue differentiation theorem) Suppose $f \in L^1(\mathbb{R}, m)$. For m-almost everywhere $x \in \mathbb{R}$,

$$\lim_{r\to 0^+} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| \, \mathrm{d}m(y) = 0$$

and

$$\lim_{r\to 0^+} \frac{1}{m(E_r)} \int_{E_r} f(y) \, \mathrm{d}m(y) = f(x)$$

for all $(E_r)_{r>0}$ shrinking nicely to $x \in \mathbb{R}$.

Proof. We prove the stated form from the usual interval form of the Lebesgue differentiation theorem from PMATH 450. There is a null set $N \subseteq \mathbb{R}$ such that, for every $x \in \mathbb{R} \setminus N$,

$$\lim_{r\to 0^+} \frac{1}{2r} \int_{x-r}^{x+r} |f(y) - f(x)| \, \mathrm{d}m(y) = 0.$$

Fix such an x, and let $(E_r)_{r>0}$ be any family shrinking nicely to x. By definition, there exists $\alpha > 0$ such that $E_r \subseteq (x-r, x+r)$ and $m(E_r) > \alpha r$ for every $r > 0$. Therefore

$$0 \leq \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| \, \mathrm{d}m(y) \leq \frac{1}{\alpha r} \int_{x-r}^{x+r} |f(y) - f(x)| \, \mathrm{d}m(y).$$

The right-hand side is

$$\frac{2}{\alpha} \cdot \frac{1}{2r} \int_{x-r}^{x+r} |f(y) - f(x)| \, \mathrm{d}m(y),$$

which tends to 0 as $r \to 0^+$. This proves the first limit. For the second limit,

$$\left|\frac{1}{m(E_r)}\int_{E_r} f(y)\,\mathrm{d}m(y) - f(x)\right| \leq \frac{1}{m(E_r)}\int_{E_r} |f(y)-f(x)|\,\mathrm{d}m(y),$$

and the right-hand side tends to 0 by the first part. □

Here is an important consequence of the Lebesgue differentiation theorem. Assume $f \in L^1(\mathbb{R}, m)$. Define $F : \mathbb{R} \to \mathbb{C}$ by

$$F(x) = \int_{(-\infty,x]} f(y)\,\mathrm{d}m(y).$$

Set $E_r = [x, x+r]$ for $r > 0$. Note that

$$f(x) = \lim_{r\to 0^+} \frac{1}{m(E_r)}\int_{E_r} f(y)\,\mathrm{d}m(y) = \lim_{r\to 0^+} \frac{F(x+r)-F(x)}{r}$$

for almost every $x \in \mathbb{R}$. For the left-hand derivative, take $h > 0$ and use the intervals $(x-h, x]$. Then

$$f(x) = \lim_{h\to 0^+} \frac{F(x)-F(x-h)}{h}$$

for almost every $x \in \mathbb{R}$. Therefore F' exists almost everywhere and $F' = f$ almost everywhere. This is essentially the fundamental theorem of calculus for Lebesgue integrals (or at least the first half). As for the second half, we will just have to wait and see.

Lemma 25.4 (Vitali covering lemma) Suppose $\mathcal{C}$ is a collection of open intervals in $\mathbb{R}$ and let $\mathcal{U} = \bigcup_{I\in\mathcal{C}} I$. If $m(\mathcal{U}) < \infty$, then there exist pairwise disjoint intervals $J_1, \ldots, J_m \in \mathcal{C}$ with

$$\sum_{j=1}^{m} m(J_j) > \frac{1}{3} m(\mathcal{U}).$$

Proof. Fix $\varepsilon > 0$. Choose a compact set $K \subseteq \mathcal{U}$ with

$$m(K) > m(\mathcal{U}) - \varepsilon.$$

Since $\mathcal{C}$ covers $\mathcal{U}$, there exist $I_1, \ldots, I_n \in \mathcal{C}$ such that

$$K \subseteq \bigcup_{i=1}^{n} I_i.$$

Choose J_1 from $\{I_1, \ldots, I_n\}$ so that $m(J_1) \geq m(I_i)$ for all i. Having chosen pairwise disjoint intervals $J_1, \ldots, J_{k-1}$, choose J_k from the remaining intervals that are disjoint from $J_1, \ldots, J_{k-1}$ and have maximal length among all such intervals, if such an interval exists. Continue until no more choices are possible. This produces pairwise disjoint intervals $J_1, \ldots, J_m$.

For each k, let J_k' denote the interval with the same center as J_k and three times the radius. We claim that

$$K \subseteq \bigcup_{k=1}^{m} J_k'.$$

It is enough to show that every I_i is contained in one of the J_k'. If $I_i = J_k$ for some k, there is nothing to prove. Otherwise, by maximality of the family $J_1, \ldots, J_m$, the interval I_i must intersect at least one of them. Let k be the smallest index such that $I_i \cap J_k \neq \emptyset$. Then

$$I_i \cap J_1 = I_i \cap J_2 = \cdots = I_i \cap J_{k-1} = \emptyset$$

and therefore the greedy choice of J_k gives

$$m(I_i) \leq m(J_k).$$

Since I_i intersects J_k and has length at most that of J_k, it follows that $I_i \subseteq J_k'$. This proves the claim.

Consequently,

$$m(K) \leq \sum_{k=1}^{m} m(J_k') = 3 \sum_{k=1}^{m} m(J_k).$$

Since $m(K) > m(\mathcal{U}) - \varepsilon$, we obtain

$$\sum_{k=1}^{m} m(J_k) > \frac{1}{3}\big(m(\mathcal{U}) - \varepsilon\big).$$

Letting $\varepsilon \to 0^+$ gives the result. $\square$

Proposition 25.5 If ν is a complex Borel measure on $\mathbb{R}$ with $\nu \perp m$, then for m almost everywhere $x \in \mathbb{R}$,

$$\lim_{r \to 0^+} \frac{\nu(E_r)}{m(E_r)} = 0$$

for any family $(E_r)_{r>0}$ shrinking nicely to $x \in \mathbb{R}$.

Corollary 25.6 Let ν be a complex Borel measure on $\mathbb{R}$. Let $\mathrm{d}\nu = \mathrm{d}\lambda + f\,\mathrm{d}m$ where $\lambda \perp m$ and $f \in L^1(\mathbb{R}, m)$. Then for m almost everywhere $x \in \mathbb{R}$,

$$\lim_{r\to 0^+} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

for any family $(E_r)_{r>0}$ shrinking nicely to $x \in \mathbb{R}$.

Proof.

$$\begin{aligned}\lim_{r\to 0^+} \frac{\nu(E_r)}{m(E_r)} &= \lim_{r\to 0^+} \frac{\lambda(E_r)}{m(E_r)} + \lim_{r\to 0^+} \frac{1}{m(E_r)} \int_{E_r} f\,\mathrm{d}m \\ &= 0 + f(x)\end{aligned}$$

for m almost everywhere $x \in \mathbb{R}$, by Proposition 25.5 and the Lebesgue differentiation theorem. $\square$

Proof of Proposition 25.5. By considering $\mathrm{Re}(\nu)^{\pm}$ and $\mathrm{Im}(\nu)^{\pm}$, we may assume without loss of generality that ν is positive. Since $\nu \perp m$, there exists a Borel set $A \subseteq \mathbb{R}$ such that $\nu(\mathbb{R} \setminus A) = 0$ and $m(A) = 0$.

If $(E_r)_{r>0}$ shrinks nicely to x, then for some $\alpha > 0$ independent of r we have $m(E_r) \geq \alpha r$ and $E_r \subseteq (x - r, x + r)$. Therefore

$$\frac{\nu(E_r)}{m(E_r)} \leq \frac{\nu(x - r, x + r)}{\alpha r}.$$

Thus it suffices to prove that

$$\frac{\nu(x - r, x + r)}{r} \to 0$$

for m almost everywhere $x \in \mathbb{R}$.

For each $k \in \mathbb{N}$, define

$$D_k = \left\{ x \in \mathbb{R} : \limsup_{r\to 0^+} \frac{\nu(x - r, x + r)}{r} > \frac{1}{k} \right\}.$$

It is enough to show that $m(D_k) = 0$ for every k. Fix k, $\varepsilon > 0$, and $M \in \mathbb{N}$. By inner regularity, choose a compact set $K \subseteq A$ such that

$$\nu(\mathbb{R} \setminus K) < \varepsilon.$$

In particular, $m(K) = 0$.

For every $x \in (D_k \cap [-M, M]) \setminus K$, choose $r_x > 0$ so small that $r_x < 1$, the interval

$$I_x = (x - r_x, x + r_x)$$

is disjoint from K, and

$$\frac{\nu(I_x)}{r_x} > \frac{1}{k}.$$

Set

$$V = \bigcup_{x \in (D_k \cap [-M,M]) \setminus K} I_x.$$

The set V lies in $(-M-1, M+1)$, so it has finite Lebesgue measure. Applying Lemma 25.4 gives finitely many pairwise disjoint intervals $I_1, \ldots, I_N$ from this collection such that

$$\sum_{i=1}^{N} m(I_i) > \frac{1}{3} m(V).$$

Each I_i is disjoint from K, so $\nu(I_i) = \nu(I_i \setminus K)$. Hence

$$\frac{1}{3} m(V) < \sum_{i=1}^{N} 2r_i < 2k \sum_{i=1}^{N} \nu(I_i) \leq 2k\, \nu(\mathbb{R} \setminus K) < 2k\varepsilon.$$

Therefore

$$m(D_k \cap [-M, M]) \leq m(K) + m(V) < 6k\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $m(D_k \cap [-M, M]) = 0$. Letting $M \to \infty$ gives $m(D_k) = 0$, as required. $\square$

Lecture 26 — Thursday, March 10, 2016

Functions of Bounded Variation

Definition 26.1 Let $F : \mathbb{R} \to \mathbb{C}$ be a function. Define $T_F : \mathbb{R} \to [0, \infty]$ by

$$T_F(x) = \sup \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : -\infty < x_0 < \cdots < x_n = x \right\}.$$

The value $T_F(x)$ is the **total variation of F up to x**. It measures how much oscillation the function has on $(-\infty, x]$.

Proposition 26.2 T_F is increasing.

Proof. If $x < y$, fix

$$x_0 < \cdots < x_n = x.$$

Then

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \leq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + |F(y) - F(x)| \leq T_F(y).$$

□

Observe that Proposition 26.2 implies that $T_F(\infty) = \lim_{x\to\infty} T_F(x)$.

Definition 26.3 The quantity $T_F(\infty)$ is called the **total variation** of F. We say that F has **bounded variation** if $T_F(\infty) < \infty$. We denote

$$BV := \{F : \mathbb{R} \to \mathbb{C} : T_F(\infty) < \infty\}.$$

Definition 26.4 Given $F : [a, b] \to \mathbb{C}$, the total variation of F on $[a, b]$ is

$$\sup \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : a = x_0 < \cdots < x_n = b \right\}.$$

Moreover, we denote by $BV([a, b])$ the set of functions $F : [a, b] \to \mathbb{C}$ such that the total variation is finite.

For a real-valued F, Figure 11 depicts the sum associated with one partition. Refining the partition may increase this sum; the supremum over all partitions is the total variation.

Proposition 26.5 If $F \in BV$ and $G = F|_{[a,b]}$, then $G \in BV([a, b])$, and the total variation

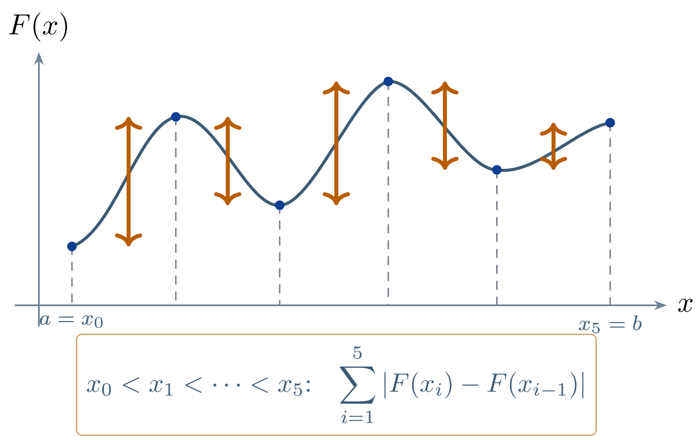

FIGURE 11

of G is $T_F(b) - T_F(a)$. Conversely, if $G \in BV([a,b])$, define $F : \mathbb{R} \to \mathbb{C}$ by

$$F(x) = \begin{cases} G(a), & x < a, \\ G(x), & a \leq x \leq b, \\ G(b), & b < x. \end{cases}$$

Then $F \in BV$ and F has the same total variation as G, since extending functions by constants does not add any variation.

We use these facts to regard $BV([a,b])$ as a subspace of BV, despite the mild abuse of notation.

Proof. Let $V_{[a,b]}(G)$ denote the total variation of G on $[a,b]$.

First suppose that $F \in BV$ and $G = F|_{[a,b]}$. Fix a partition

$$a = x_0 < x_1 < \cdots < x_n = b.$$

If

$$y_0 < y_1 < \cdots < y_m = a$$

is any partition ending at a, then concatenating the two partitions gives

$$\sum_{j=1}^{m} |F(y_j) - F(y_{j-1})| + \sum_{i=1}^{n} |G(x_i) - G(x_{i-1})| \leq T_F(b).$$

Taking the supremum over all such partitions ending at a gives

$$T_F(a) + \sum_{i=1}^{n} |G(x_i) - G(x_{i-1})| \le T_F(b).$$

Hence $V_{[a,b]}(G) \le T_F(b) - T_F(a)$.

For the reverse inequality, let $\varepsilon > 0$. Choose a partition

$$z_0 < z_1 < \cdots < z_N = b$$

such that

$$\sum_{j=1}^{N} |F(z_j) - F(z_{j-1})| > T_F(b) - \varepsilon.$$

Inserting the point a into this partition, if necessary, can only increase the corresponding sum. The part of the resulting partition lying at or below a contributes at most $T_F(a)$, and the part from a to b contributes at most $V_{[a,b]}(G)$. Therefore

$$T_F(b) - \varepsilon < T_F(a) + V_{[a,b]}(G).$$

Letting $\varepsilon \to 0^+$ gives $T_F(b) - T_F(a) \le V_{[a,b]}(G)$, so the two quantities are equal.

Conversely, suppose that $G \in BV([a,b])$ and extend it to $F : \mathbb{R} \to \mathbb{C}$ by constants outside $[a,b]$. Any variation of F on $(-\infty, \infty)$ occurs on the interval $[a,b]$, since F is constant on $(-\infty, a]$ and on $[b, \infty)$. Thus every partition for F has total variation at most $V_{[a,b]}(G)$, while every partition of $[a,b]$ is also a partition for the extended function. Hence

$$T_F(\infty) = V_{[a,b]}(G) < \infty.$$

Therefore $F \in BV$, and F has the same total variation as G. $\square$

Example 26.6 Let $F : \mathbb{R} \to \mathbb{R}$ be given by $F(x) = \sin x$. This function has infinite total variation on $\mathbb{R}$, so $F \notin BV$. However, $F \in BV([a,b])$ for every finite interval $[a,b]$.

Example 26.7 Let $F(x) = \sin(1/x)$ for $x > 0$. By choosing partition points at successive maxima and minima of F, we see that $F \notin BV([0,\varepsilon])$ for every $\varepsilon > 0$. On the other hand, $F \in BV([\varepsilon, \infty))$ for every $\varepsilon > 0$.

Example 26.8 Let $F(x) = x\sin(1/x)$ for $x > 0$, and set $F(0) = 0$. Then F is continuous, and indeed uniformly continuous, on $[0, b]$ for every $b > 0$. However, the successive oscillations have sizes comparable to $1/n$ as $x \to 0$, so the total variation diverges and $F \notin BV([0, b])$ for every $b > 0$.

Proposition 26.9 If $F : \mathbb{R} \to \mathbb{R}$ is bounded and increasing, then F has bounded variation. Moreover, $T_F(x) = F(x) - F(-\infty)$.

Proof. Since F is bounded and increasing, the limits $F(-\infty)$ and $F(\infty)$ exist and are finite. Fix $x \in \mathbb{R}$. If

$$x_0 < \cdots < x_n = x,$$

then the sum telescopes:

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = F(x_n) - F(x_0) = F(x) - F(x_0).$$

Since $F(-\infty) \le F(x_0)$, this gives

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \le F(x) - F(-\infty).$$

Taking the supremum over all partitions ending at x, we obtain

$$T_F(x) \le F(x) - F(-\infty).$$

Conversely, let $\varepsilon > 0$. By the definition of $F(-\infty)$, there exists $x_0 < x$ such that

$$F(x_0) < F(-\infty) + \varepsilon.$$

Using the partition $x_0 < x$, we get

$$T_F(x) \ge F(x) - F(x_0) > F(x) - F(-\infty) - \varepsilon.$$

Letting $\varepsilon \to 0^+$ gives $T_F(x) \ge F(x) - F(-\infty)$. Hence

$$T_F(x) = F(x) - F(-\infty)$$

for every $x \in \mathbb{R}$. Finally,

$$T_F(\infty) = F(\infty) - F(-\infty) < \infty,$$

so F has bounded variation. $\square$

Proposition 26.10 If $F : [a, b] \to \mathbb{C}$ is differentiable on (a, b) and F' is bounded on (a, b), then $F \in BV[a, b]$.

Proof. Fix $M > 0$ with $|F'(x)| \leq M$ for $x \in (a, b)$. Given a partition

$$a = x_0 < \cdots < x_n = b,$$

for each i choose $\alpha_i \in \mathbb{C}$ with $|\alpha_i| = 1$ and

$$\alpha_i\big(F(x_i) - F(x_{i-1})\big) = |F(x_i) - F(x_{i-1})|.$$

Apply the real mean value theorem to the real-valued function $\mathrm{Re}(\alpha_i F)$. For some $c_i \in (x_{i-1}, x_i)$,

$$|F(x_i) - F(x_{i-1})| = (x_i - x_{i-1})\,\mathrm{Re}(\alpha_i F'(c_i)) \leq M(x_i - x_{i-1}).$$

Summing over i gives total variation at most $M(b - a)$, so $F \in BV[a, b]$. $\square$

Proposition 26.11 If $F, G \in BV$ and $a \in \mathbb{C}$, then $F + G \in BV$ and $aF \in BV$. In particular, BV is a vector space.

Proof. If

$$x_0 < \cdots < x_n = x,$$

then

$$\sum_{i=1}^{n} |(F+G)(x_i) - (F+G)(x_{i-1})| \leq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + \sum_{i=1}^{n} |G(x_i) - G(x_{i-1})| \leq T_F(x) + T_G(x).$$

Taking the supremum over all partitions ending at x gives

$$T_{F+G}(x) \leq T_F(x) + T_G(x).$$

In particular, $T_{F+G}(\infty) \leq T_F(\infty) + T_G(\infty) < \infty$, so $F + G \in BV$. Similarly, if $a = 0$, then aF is the zero function, which is in BV. If $a \neq 0$, then

$$\sum_{i=1}^{n} |(aF)(x_i) - (aF)(x_{i-1})| = |a| \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \leq |a| T_F(x).$$

Taking the supremum over all partitions ending at x gives

$$T_{aF}(x) \leq |a| T_F(x).$$

In particular, $T_{aF}(\infty) \leq |a| T_F(\infty) < \infty$, so $aF \in BV$. Thus BV is closed under addition and scalar multiplication, so BV is a vector space. $\square$

Proposition 26.12 For a real-valued $F \in BV$, there are unique bounded, increasing functions $P_F, N_F : \mathbb{R} \to \mathbb{R}$ such that

$$P_F - N_F = F \qquad \text{and} \qquad P_F + N_F = T_F.$$

These functions are called the **positive and negative variations** of F, and $F = P_F - N_F$ is called the **Jordan decomposition** of F.

Proof. Uniqueness follows immediately from

$$P_F = \frac{1}{2}(T_F + F) \qquad \text{and} \qquad N_F = \frac{1}{2}(T_F - F).$$

For existence, define P_F and N_F by these formulas. It remains to show that they are bounded and increasing. Fix $x < y$ and $\varepsilon > 0$. Choose a partition

$$x_0 < \cdots < x_n = x$$

with

$$T_F(x) \leq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + \varepsilon.$$

Then

$$\begin{aligned} 2P_F(y) - 2P_F(x) &= T_F(y) + F(y) - T_F(x) - F(x) \\ &\geq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + |F(y) - F(x)| + F(y) - F(x) \end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{n}|F(x_i)-F(x_{i-1})|-\varepsilon\\
&=|F(y)-F(x)|+F(y)-F(x)-\varepsilon\\
&\geq -\varepsilon.
\end{aligned}
$$

Letting $\varepsilon \to 0^+$ gives $P_F(y) \geq P_F(x)$. The same argument applied to $T_F - F$ gives $N_F(y) \geq N_F(x)$. Finally, P_F and N_F are bounded because F and T_F are bounded. □

Proposition 26.13 $F \in BV$ if and only if $\operatorname{Re}(F), \operatorname{Im}(F) \in BV$.

Proof. *Forward implication.* If $F \in BV$, then for every partition,

$$\sum_i |\operatorname{Re}(F(x_i)) - \operatorname{Re}(F(x_{i-1}))| \leq \sum_i |F(x_i) - F(x_{i-1})|,$$

and the same inequality holds for the imaginary part. Hence $\operatorname{Re}(F), \operatorname{Im}(F) \in BV$.

Reverse implication. If $\operatorname{Re}(F), \operatorname{Im}(F) \in BV$, then

$$|F(x_i) - F(x_{i-1})| \leq |\operatorname{Re}(F(x_i)) - \operatorname{Re}(F(x_{i-1}))| + |\operatorname{Im}(F(x_i)) - \operatorname{Im}(F(x_{i-1}))|.$$

Taking suprema over partitions shows $F \in BV$. □

Lecture 27 — Friday, March 11, 2016

Proposition 27.1 For $F : \mathbb{R} \to \mathbb{C}$, the following are equivalent:

1. We have $F \in BV$.
2. The function F is a $\mathbb{C}$-linear combination of bounded increasing functions $\mathbb{R} \to \mathbb{R}$.
3. There exist bounded increasing functions $F_1, F_2, F_3, F_4 : \mathbb{R} \to \mathbb{R}$ such that $F = F_1 + iF_2 - F_3 - iF_4$.

Proof. (1) *implies* (3). By Proposition 26.13, the functions $\mathrm{Re}(F)$ and $\mathrm{Im}(F)$ both belong to BV. Applying Proposition 26.12 to these real-valued functions, there exist bounded increasing functions $F_1, F_2, F_3, F_4 : \mathbb{R} \to \mathbb{R}$ such that

$$\mathrm{Re}(F) = F_1 - F_3 \qquad \text{and} \qquad \mathrm{Im}(F) = F_2 - F_4.$$

Therefore

$$F = \mathrm{Re}(F) + i\,\mathrm{Im}(F) = F_1 + iF_2 - F_3 - iF_4,$$

so (3) holds.

(3) *implies* (2). This is immediate, since the representation in (3) expresses F as a $\mathbb{C}$-linear combination of bounded increasing functions $\mathbb{R} \to \mathbb{R}$.

(2) *implies* (1). Write

$$F = a_1 G_1 + \cdots + a_n G_n,$$

where $a_1, \dots, a_n \in \mathbb{C}$ and each $G_j : \mathbb{R} \to \mathbb{R}$ is bounded and increasing. By Proposition 26.9, each G_j belongs to BV. Since BV is a vector space by Proposition 26.11, it follows that $F \in BV$. ☐

Theorem 27.2 Given $F \in BV$, the set of discontinuities of F is countable (or finite).

Remark 27.3 Given a finite or infinite sequence of distinct points $(x_n)_{n=1}^N$, where N may be finite or infinite, define $F : \mathbb{R} \to \mathbb{C}$ by

$$F(x) = \begin{cases} 2^{-n} & x = x_n \\ 0 & x \neq x_n \text{ for any } n \end{cases}.$$

Then $F \in BV$ and the discontinuities of F are exactly $\{x_1, x_2, \dots\}$.

Proof of Theorem 27.2. By Proposition 27.1(2), it suffices to prove the result when F is increasing. Then

$$F(x^+) = \lim_{y \to x^+} F(y) \qquad \text{and} \qquad F(x^-) = \lim_{y \to x^-} F(y)$$

exist for every $x \in \mathbb{R}$, and $F(x^-) \leq F(x^+)$. Moreover, F is continuous at x if and only if $F(x^-) = F(x^+)$. Let

$$S = \{x \in \mathbb{R} : F(x^-) \neq F(x^+)\}.$$

For each $x \in S$, choose a rational number

$$q(x) \in (F(x^-), F(x^+)).$$

If $x, y \in S$ and $x < y$, then monotonicity gives $F(x^+) \le F(y^-)$, so the intervals $(F(x^-), F(x^+))$ and $(F(y^-), F(y^+))$ are disjoint. Hence $q(x) \ne q(y)$. Thus $q : S \to \mathbb{Q}$ is injective, so S is countable. □

Lemma 27.4 If $F \in BV$, then $F(x^+)$ exists for all x. If $G(x) = F(x^+)$, then G is right continuous and $G \in BV$.

Proof. It suffices to prove the claim for bounded increasing functions and then apply it to the linear-combination representation in Proposition 27.1(2). Assume, then, that F is increasing and bounded. The limit

$$G(x) = F(x^+) = \inf_{t>x} F(t)$$

exists for every x. Fix $x \in \mathbb{R}$ and $\varepsilon > 0$. Choose $z > x$ such that $F(z) - G(x) < \varepsilon$. If $x < y < z$, then

$$0 \le G(y) - G(x) \le F(z) - G(x) < \varepsilon.$$

Hence G is right-continuous. It is also bounded and increasing, so $G \in BV$ by Proposition 26.9. Linear combinations preserve right-continuity and bounded variation, which proves the general case. □

Combining Proposition 25.5 with the Lebesgue differentiation theorem, if μ is a complex Borel measure on $\mathbb{R}$ and

$$\mathrm{d}\mu = \mathrm{d}\lambda + f\,\mathrm{d}m, \qquad \lambda \perp m, \qquad \text{and} \qquad f \in L^1(\mathbb{R}, m),$$

then

$$\lim_{r \to 0^+} \frac{\mu(E_r)}{m(E_r)} = f(x)$$

for m-almost every $x \in \mathbb{R}$ and every family $(E_r)_{r>0}$ shrinking nicely to x, for example $E_r = (x - r, x + r]$.

Theorem 27.5 If $F \in BV$ and $G(x) = F(x^+)$, then F and G are differentiable almost everywhere, and $F' = G'$ almost everywhere.

Proof. By the Jordan decomposition reduction in Proposition 27.1(3), it suffices to prove the result when F is bounded and increasing. Then G is bounded, increasing, and right-continuous, so there is a finite positive Borel measure ν on $\mathbb{R}$ such that

$$\nu((a,b]) = G(b) - G(a).$$

For $r > 0$,

$$\frac{G(x+r) - G(x)}{r} = \frac{\nu(x, x+r]}{m(x, x+r]},$$

and the analogous formula for $r < 0$ uses the interval $(x+r, x]$. By the Lebesgue differentiation theorem, these one-sided quotients converge to the same value for m-almost every x. Hence G' exists almost everywhere.

It remains to show that $H' = 0$ almost everywhere for $H = G - F$. The function H is supported on the discontinuity set of F, which is countable by the preceding theorem. Let $(x_n)_{n=1}^N$ enumerate the set $\{x \in \mathbb{R} : H(x) \neq 0\}$, where N may be finite or infinite. We first show that

$$\sum_{n=1}^{N} H(x_n) < \infty.$$

If $N < \infty$, there is nothing to prove. Otherwise, fix $N_0 \in \mathbb{N}$ and choose points

$$a_0 < a_1 < a_2 < \cdots < a_{N_0}$$

such that each interval (a_i, a_{i+1}) contains exactly one of $x_1, \ldots, x_{N_0}$. If $x_j \in (a_i, a_{i+1})$, then

$$H(x_j) = F(x_j^+) - F(x_j) \leq F(a_{i+1}) - F(a_i).$$

Therefore,

$$\sum_{j=1}^{N_0} H(x_j) \leq \sum_{i=1}^{N_0} (F(a_{i+1}) - F(a_i)) = F(a_{N_0}) - F(a_0) < F(\infty) - F(-\infty).$$

Letting $N_0 \to \infty$ gives

$$\sum_{n=1}^{\infty} H(x_n) < \infty.$$

Define a finite positive Borel measure λ by

$$\lambda(E) = \sum_{x \in E} H(x)$$

where the sum is over the countable support of H. Then, for every $r \neq 0$,

$$\left|\frac{H(x+r)-H(x)}{r}\right| \leq 2\frac{\lambda([x-|r|,x+|r|])}{m([x-|r|,x+|r|])}.$$

The measure λ is singular with respect to Lebesgue measure, so the singular part of the Lebesgue differentiation theorem implies that the right-hand side tends to 0 for m almost every x. Thus $H' = 0$ almost everywhere, and consequently $F' = G'$ almost everywhere. □

Example 27.6 Let $f_0(x) = x$ on $[0,1]$, and define f_{n+1} recursively by

$$f_{n+1}(x) = \begin{cases} \frac{1}{2}f_n(3x), & 0 \leq x \leq \frac{1}{3}, \\ \frac{1}{2}, & \frac{1}{3} \leq x \leq \frac{2}{3}, \\ \frac{1}{2} + \frac{1}{2}f_n(3x-2), & \frac{2}{3} \leq x \leq 1. \end{cases}$$

Then $f = \lim_{n\to\infty} f_n$ exists and is continuous and increasing. Thus $f \in BV$ and f' exists almost everywhere. In fact, $f' = 0$ almost everywhere, while $f(0) = 0$ and $f(1) = 1$. This is the infamous **Cantor function**, and it shows that an increasing function can have derivative zero almost everywhere without being constant. More generally, poorly behaved functions can still have very nice derivatives (almost everywhere).

Lecture 28 — Tuesday, March 15, 2016

Proposition 28.1 If $F : \mathbb{R} \to \mathbb{R}$ is right-continuous and of bounded variation, then $T_F(-\infty) = 0$ and T_F is right-continuous, where

$$T_F(x) = \sup\left\{\sum_{j=1}^{n} |F(t_j) - F(t_{j-1})| : t_0 < \cdots < t_n = x\right\}.$$

Proof. To see that $T_F(-\infty) = 0$, fix $\varepsilon > 0$. Let $x_0 < \cdots < x_n$ be in $\mathbb{R}$ with

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \geq T_F(\infty) - \varepsilon.$$

Fix $y < x_0$. We claim that $T_F(y) < \varepsilon$. Let

$$t_0 < \cdots < t_k = y$$

be given. Then

$$\begin{aligned}\sum_{j=1}^{k} |F(t_j) - F(t_{j-1})| &= \left(\sum_{j=1}^{k} |F(t_j) - F(t_{j-1})| + |F(x_0) - F(y)| + \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \right) \\ &\qquad - |F(x_0) - F(y)| - \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \\ &\leq T_F(\infty) - 0 - (T_F(\infty) - \varepsilon) \\ &= \varepsilon.\end{aligned}$$

For right-continuity, fix $x \in \mathbb{R}$ and set

$$L = \lim_{y \downarrow x} T_F(y),$$

which exists because T_F is increasing. Consider any partition of $[x, y]$, and let t be its first point after x. The first increment is $|F(t) - F(x)|$, while all remaining increments sum to at most the variation on $[t, y]$, namely $T_F(y) - T_F(t)$. Taking the supremum over partitions and using $T_F(t) \geq L$ gives

$$T_F(y) - T_F(x) \leq \sup_{x < t \leq y} |F(t) - F(x)| + T_F(y) - L.$$

As $y \downarrow x$, the supremum tends to 0 by right-continuity of F, and $T_F(y) \to L$. Hence $L \leq T_F(x)$. Monotonicity gives the reverse inequality, so $L = T_F(x)$ and T_F is right-continuous at x. □

Definition 28.2 A function $F : \mathbb{R} \to \mathbb{C}$ has **normalized bounded variation**, written $F \in NBV$, if $F \in BV$, F is right-continuous, and $F(-\infty) = 0$.

Fact 28.3 If $F \in BV$, define $G : \mathbb{R} \to \mathbb{C}$ by $G(x) = F(x^+) - F(-\infty)$. Then $G \in NBV$ and $F' = G'$ almost everywhere.

Proof. The function G is right-continuous and satisfies $G(-\infty) = 0$. Also, G is of bounded variation because $G(x) - G(y) = F(x^+) - F(y^+)$ for $x > y$. Finally, $G' = F'$ almost everywhere by Theorem 27.5. □

Theorem 28.4 If μ is a complex Borel measure on $\mathbb{R}$ and

$$F(x) = \mu(-\infty, x],$$

then $F \in NBV$. Conversely, if $F \in NBV$, then there exists a unique complex Borel measure μ_F such that

$$F(x) = \mu_F(-\infty, x]$$

for all $x \in \mathbb{R}$. Moreover, the correspondence $F \leftrightarrow \mu_F$ is linear; if F is real-valued with Jordan decomposition $F = P_F - N_F$, then

$$\mu_F = \mu_{P_F} - \mu_{N_F} \qquad \text{and} \qquad |\mu_F| = \mu_{T_F}.$$

Hence complex Borel measures on $\mathbb{R}$ are in vector space isomorphism with NBV.

Proof. *Forward implication.* We already know the result for positive measures: if σ is a positive Borel measure on $\mathbb{R}$, then the function

$$x \mapsto \sigma(-\infty, x]$$

is increasing, right-continuous, vanishes at $-\infty$, and therefore belongs to NBV.

Now write the complex measure μ as

$$\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4),$$

where each μ_j is a positive Borel measure. Set

$$F_j(x) = \mu_j(-\infty, x].$$

Each F_j lies in NBV, so

$$F = F_1 - F_2 + i(F_3 - F_4)$$

is of bounded variation, is right-continuous, and satisfies $F(-\infty) = 0$. Hence $F \in NBV$.

Reverse implication. Let $F \in NBV$. Apply the Jordan decomposition separately to the real and imaginary parts of F:

$$\text{Re}(F) = P_1 - N_1 \qquad \text{and} \qquad \text{Im}(F) = P_2 - N_2,$$

where P_1, N_1, P_2, N_2 are increasing right-continuous functions vanishing at $-\infty$. For each of these functions there exists a unique positive Borel measure whose distribution function is the given function. Denote these measures by $\sigma_{P_1}, \sigma_{N_1}, \sigma_{P_2}, \sigma_{N_2}$, and set

$$\mu_F = \sigma_{P_1} - \sigma_{N_1} + i(\sigma_{P_2} - \sigma_{N_2}).$$

Then μ_F is a complex Borel measure and, by construction,

$$\mu_F(-\infty, x] = F(x)$$

for all $x \in \mathbb{R}$.

Uniqueness follows because the half-lines $(-\infty, x]$ generate $\mathcal{B}_{\mathbb{R}}$. The linearity of the correspondence is immediate from the construction. For real-valued F, the identities involving P_F, N_F, and T_F follow from the corresponding statements for the Jordan decomposition of signed measures. □

Question 28.5 For $F \in NBV$, when do we have $\mu_F \perp m$ or $\mu_F \ll m$?

If $F \in NBV$, write $\mathrm{d}\mu_F = \mathrm{d}\lambda + f\,\mathrm{d}m$ for $\lambda \perp m$ and $f \in L^1(\mathbb{R}, m)$. The preceding differentiation result gives $F' = f$ almost everywhere, so

$$\mathrm{d}\mu_F = \mathrm{d}\lambda + F'\,\mathrm{d}m.$$

Corollary 28.6 $\mu_F \perp m$ if and only if $F' = 0$ almost everywhere with respect to m. Also, $\mu_F \ll m$ if and only if

$$F(x) = \int_{-\infty}^{x} F'(t)\,\mathrm{d}t$$

for all $x \in \mathbb{R}$.

Proof. By the preceding discussion, there is a measure $\lambda \perp m$ such that

$$\mathrm{d}\mu_F = \mathrm{d}\lambda + F'\,\mathrm{d}m.$$

By uniqueness in the Lebesgue–Radon–Nikodym theorem, we have $\mu_F \perp m$ if and only if the absolutely continuous part $F'\,\mathrm{d}m$ is zero. This holds if and only if $F' = 0$ almost everywhere with respect to m. This proves the first claim.

For the second claim, if $\mu_F \ll m$, then again by uniqueness in Theorem 24.2 we must have $\lambda = 0$, so $\mathrm{d}\mu_F = F' \, \mathrm{d}m$. Hence, for every $x \in \mathbb{R}$,

$$F(x) = \mu_F(-\infty, x] = \int_{(-\infty,x]} F'(t) \, \mathrm{d}t = \int_{-\infty}^{x} F'(t) \, \mathrm{d}t.$$

Conversely, suppose that

$$F(x) = \int_{-\infty}^{x} F'(t) \, \mathrm{d}t$$

for every $x \in \mathbb{R}$. Define a complex Borel measure ν by

$$\nu(E) = \int_E F'(t) \, \mathrm{d}t.$$

Then $\nu \ll m$, and for every $x \in \mathbb{R}$,

$$\nu(-\infty, x] = \int_{-\infty}^{x} F'(t) \, \mathrm{d}t = F(x) = \mu_F(-\infty, x].$$

By the uniqueness in Theorem 28.4, it follows that $\nu = \mu_F$. Therefore $\mu_F \ll m$. □

Definition 28.7 A function $F : \mathbb{R} \to \mathbb{C}$ is **uniformly continuous** if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|F(x) - F(y)| < \varepsilon$ whenever $x, y \in \mathbb{R}$ and $|x - y| < \delta$.

A function $F : \mathbb{R} \to \mathbb{C}$ is **absolutely continuous** if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, whenever $(a_1, b_1), \dots, (a_n, b_n)$ are disjoint intervals,

$$\sum_{j=1}^{n} (b_j - a_j) < \delta \implies \sum_{j=1}^{n} |F(b_j) - F(a_j)| < \varepsilon.$$

Theorem 28.8 For $F \in NBV$, F is absolutely continuous if and only if $\mu_F \ll m$.

Proof. *Forward implication.* Suppose $\mu_F \ll m$. By Theorem 22.7, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$m(E) < \delta \implies |\mu_F|(E) < \varepsilon$$

for every Borel set $E \subseteq \mathbb{R}$. Let $(a_1, b_1), \dots, (a_n, b_n)$ be disjoint intervals such that

$$a_1 < b_1 \le a_2 < b_2 \le \cdots < b_n,$$

and set

$$E = \bigcup_{j=1}^{n} (a_j, b_j].$$

If $\sum_{j=1}^{n}(b_j - a_j) < \delta$, then $m(E) < \delta$, and therefore

$$\sum_{j=1}^{n} |F(b_j) - F(a_j)| = \sum_{j=1}^{n} |\mu_F((a_j, b_j])| \le |\mu_F|(E) < \varepsilon.$$

Thus F is absolutely continuous.

Reverse implication. Suppose that F is absolutely continuous. Fix $E \in \mathcal{B}_{\mathbb{R}}$ with $m(E) = 0$. Let $\varepsilon > 0$ be given, and choose $\delta > 0$ such that

$$\sum_{j=1}^{n} |b_j - a_j| < \delta \Longrightarrow \sum_{j=1}^{n} |F(b_j) - F(a_j)| < \varepsilon$$

for every finite disjoint family of intervals. Choose an open set $U \supseteq E$ with $m(U) < \delta$. Write U as a countable disjoint union of open intervals,

$$U = \bigcup_{k=1}^{\infty} (a_k, b_k).$$

Consider any finite family of pairwise disjoint half-open intervals $(c_j, d_j]$ whose union lies in U. Their total length is at most $m(U) < \delta$, so absolute continuity gives

$$\sum_{j} |\mu_F((c_j, d_j])| = \sum_{j} |F(d_j) - F(c_j)| < \varepsilon.$$

Taking suprema first over interval partitions inside each component of U and then over finite collections of components gives $|\mu_F|(U) \le \varepsilon$. Hence $|\mu_F|(E) \le |\mu_F|(U) \le \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $|\mu_F|(E) = 0$, so $\mu_F \ll m$. $\square$

Lemma 28.9 If F is absolutely continuous on $[a, b]$, then $F \in BV[a, b]$.

Proof. Apply the definition of absolute continuity with $\varepsilon = 1$ to obtain $\delta > 0$. Choose points

$$a = t_0 < t_1 < \cdots < t_n = b$$

such that

$$t_j - t_{j-1} < \delta$$

for every $j = 1, \dots, n$.

Now let

$$a = x_0 < x_1 < \cdots < x_m = b$$

be any partition of $[a, b]$. Insert the points $t_1, \dots, t_{n-1}$ into this partition. By the triangle inequality, this can only increase the sum

$$\sum_{i=1}^{m} |F(x_i) - F(x_{i-1})|.$$

It is therefore enough to consider the refined partition. For each $j = 1, \dots, n$, the subintervals of the refined partition contained in $[t_{j-1}, t_j]$ are disjoint and have total length $t_j - t_{j-1} < \delta$. Hence absolute continuity gives

$$\sum |F(u_k) - F(u_{k-1})| < 1$$

for the part of the refined partition lying in $[t_{j-1}, t_j]$. Summing over j yields

$$\sum_{i=1}^{m} |F(x_i) - F(x_{i-1})| \leq n.$$

Since this bound is independent of the partition, the total variation of F on $[a, b]$ is finite. Therefore $F \in BV[a, b]$. □

Theorem 28.10 (Fundamental theorem of calculus for absolutely continuous functions) If

$$-\infty < a < b < \infty$$

and $F : [a, b] \to \mathbb{C}$, then the following are equivalent:

1. The function F is absolutely continuous.
2. There exists a function $f \in L^1([a, b], m)$ such that

$$F(x) - F(a) = \int_a^x f(t)\,\mathrm{d}t$$

for every $x \in [a, b]$.

3. The derivative F' exists almost everywhere, $F' \in L^1([a,b],m)$, and

$$F(x) - F(a) = \int_a^x F'(t)\,\mathrm{d}t$$

for every $x \in [a,b]$.

Proof. (1) *implies* (3). This follows by extending F constantly outside $[a,b]$, applying the preceding results for functions in NBV to the normalized representative, and then restricting back to $[a,b]$.

(3) *implies* (2). This is immediate by taking $f = F'$.

(2) *implies* (1). Given $\varepsilon > 0$, the absolute continuity of the integral gives $\delta > 0$ such that

$$m(E) < \delta \Longrightarrow \int_E |f|\,\mathrm{d}m < \varepsilon.$$

If $(a_1,b_1),\ldots,(a_n,b_n)$ are disjoint subintervals of $[a,b]$ with $\sum_{j=1}^n (b_j - a_j) < \delta$, then

$$\sum_{j=1}^n |F(b_j) - F(a_j)| \le \sum_{j=1}^n \int_{a_j}^{b_j} |f|\,\mathrm{d}m = \int_{\bigcup_{j=1}^n (a_j,b_j)} |f|\,\mathrm{d}m < \varepsilon.$$

Thus F is absolutely continuous. □

Lecture 29 — Thursday, March 17, 2016

5. Functional Analysis and Representation Theorems

Functional Analysis and L_p Spaces

Let us start by recalling some real analysis.

Definition 29.1 A norm on a complex vector space X is a function $\|\cdot\| : X \to [0, \infty)$ such that the following hold:

1. $\|x\| = 0$ if and only if $x = 0$.
2. $\|ax\| = |a| \cdot \|x\|$ for every $a \in \mathbb{C}$ and $x \in X$.
3. $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in X$.

A norm induces a metric $d : X \times X \to [0, \infty)$ where $d(x, y) = \|x - y\|$. A pair $(X, \|\cdot\|)$ is called a **normed space**. It is a **Banach space** if X is **complete** with respect to $\|\cdot\|$, (that is, every Cauchy sequence converges).

Example 29.2 The following are standard examples:

1. The space $\mathbb{C}^n$ is a Banach space with the norm
$$\|(x_i)\| = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}.$$
2. If Ω is a compact metric space, then
$$C(\Omega) = \{f : \Omega \to \mathbb{C} : f \text{ is continuous}\}$$
is a Banach space with the norm
$$\|f\|_\infty = \max_{x \in \Omega} |f(x)|.$$
3. If (X, μ) is a measure space, then $L^1(X, \mu)$ is a Banach space with
$$\|f\|_1 = \int |f| \,\mathrm{d}\mu.$$
Completeness of L^1 is nontrivial.
4. For $1 \leq p < \infty$,
$$L^p(X, \mu) = \{f : X \to \mathbb{C} : f \text{ is measurable}, \int |f|^p \,\mathrm{d}\mu < \infty\}$$

with
$$\|f\|_p = (\int |f|^p \, \mathrm{d}\mu)^{1/p}$$
is a Banach space. Proving the triangle inequality for $\|\cdot\|_p$ is nontrivial and relies on the convexity of the function $t \mapsto t^p$.

5. Define
$$\mathcal{L}^\infty(X,\mu) = \{f : X \to \mathbb{C} : f \text{ is measurable and bounded}\}.$$
For $f, g \in \mathcal{L}^\infty(X,\mu)$, write $f \sim g$ if $f = g$ almost everywhere, and set
$$L^\infty(X,\mu) = \mathcal{L}^\infty(X,\mu)/\sim.$$
Thus $f \in L^\infty(X,\mu)$ if and only if there exists $M \geq 0$ such that $|f| \leq M$ almost everywhere. This is a Banach space. The norm is
$$\|f\|_\infty = \inf\{M \geq 0 : |f| \leq M \text{ almost everywhere}\}.$$
For example, if $f(x) = 0$ for $x \neq 1/2$ and $f(1/2) = 1$, then $\|f\|_\infty = 0$ with respect to Lebesgue measure.

6. Let Ω be a compact metric space, and let $M(\Omega)$ be the set of complex Borel measures on Ω. If $\|\mu\| = |\mu|(\Omega)$, then $M(\Omega)$ is a Banach space.

Definition 29.3 If X is a normed space, given a linear functional $f : X \to \mathbb{C}$, define
$$\|f\| = \sup\{|f(x)| : x \in X, \|x\| \leq 1 \text{ (or } \|x\| = 1)\}.$$
We say that f is **bounded** if $\|f\| < \infty$. The **dual** of X is
$$X^* = \{f : X \to \mathbb{C} : f \text{ is linear and bounded}\}.$$

The dual space X^* is always a Banach space.

Definition 29.4 If X and Y are Banach spaces, we say that X and Y are **isometrically isomorphic** if there exists a bijective linear map $T : X \to Y$ such that $\|Tx\|_Y = \|x\|_X$ for every $x \in X$.

Theorem 29.5 (Riesz representation theorem) Let (X, μ) be a measure space and let $1 \leq p < \infty$ with $1/p + 1/q = 1$. Under suitable hypotheses, the dual of $L^p(X, \mu)$ is represented by $L^q(X, \mu)$ through the map $f \mapsto \ell_f$, where $\ell_f : L^p \to \mathbb{C}$ is given by

$$\ell_f(g) = \int fg \, \mathrm{d}\mu.$$

In particular, for σ-finite measure spaces we have $(L^1(X, \mu))^* \cong L^\infty(X, \mu)$, while $(L^\infty(X, \mu))^*$ is generally much larger than $L^1(X, \mu)$.

The dual pairing in this theorem is organized in Figure 12: a function in L^q becomes a bounded linear functional on L^p by integration against its argument.

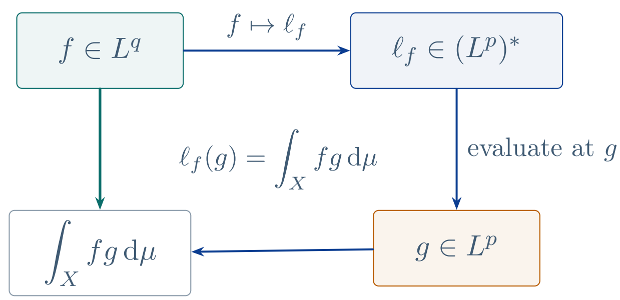

FIGURE 12

Theorem 29.6 (Riesz representation theorem) If Ω is a compact metric space, $(C(\Omega))^* \cong M(\Omega)$. A measure μ in $M(\Omega)$ corresponds to the functional $C(\Omega) \to \mathbb{C}$,

$$f \mapsto \int f \, \mathrm{d}\mu.$$

The Riesz in Theorem 29.5 and the Riesz in Theorem 29.6 were different Rieszes. The first theorem is due to Frigyes Riesz, while the second is due to his brother Marcel Riesz.

We will prove these theorems later on. For the time being, (X, μ) is a measure space and μ is a positive measure.

Definition 29.7 For $1 \leq p < \infty$, define

$$L^p(X, \mu) = \left\{ f : X \to \mathbb{C} : f \text{ is measurable}, \int |f|^p < \infty \right\}.$$

where $f = g$ in L^p if and only if $f = g$ almost everywhere. Define the ***L^p*-norm** of f by

$$\|f\|_p = \left(\int |f|^p \,\mathrm{d}\mu\right)^{1/p}.$$

Is the L^p-norm a legitimate norm? We have $\|0\|_p = 0$. If $\|f\|_p = 0$, then $|f|^p = 0$ almost everywhere, so $f = 0$ in L^p. For $a \in \mathbb{C}$,

$$\|af\|_p = \left(\int |af|^p \,\mathrm{d}\mu\right)^{1/p} = |a| \left(\int |f|^p \,\mathrm{d}\mu\right)^{1/p} = |a| \cdot \|f\|_p.$$

It remains to prove the triangle inequality.

Definition 29.8 A function $f : \mathbb{R} \to \mathbb{R}$ is **convex** if, for all $x, y \in \mathbb{R}$ and $0 \le t \le 1$, $f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$.

Proposition 29.9 A function $f : \mathbb{R} \to \mathbb{R}$ is convex if and only if, for all $s < t < u$ in $\mathbb{R}$,

$$\frac{f(t) - f(s)}{t - s} \le \frac{f(u) - f(t)}{u - t}$$

.

Proof. Write $t = \lambda s + (1-\lambda)u$, where

$$\lambda = \frac{u - t}{u - s}.$$

The convexity inequality for this choice is equivalent, after rearranging, to

$$\frac{f(t) - f(s)}{t - s} \le \frac{f(u) - f(t)}{u - t}.$$

The converse follows by reversing this algebra. □

Corollary 29.10 If f is differentiable, then f is convex if and only if f' is increasing.

Proof. Apply Proposition 29.9 to the secant slopes and use the mean value theorem to compare those slopes with values of f'. □

Lemma 29.11 (Young's inequality) If $0 \leq \lambda \leq 1$ and $a, b \geq 0$, then $a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b$.

Proof. If $\lambda \in \{0,1\}$, the result is equality. Now assume $0 < \lambda < 1$. If $a = 0$ or $b = 0$, the result is immediate; otherwise $a, b > 0$. Since $\exp : \mathbb{R} \to \mathbb{R}$ is convex,

$$\begin{aligned} a^\lambda b^{1-\lambda} &= \exp\left(\log\left(a^\lambda b^{1-\lambda}\right)\right) \\ &= \exp\left(\log\left(a^\lambda\right) + \log\left(b^{1-\lambda}\right)\right) \\ &= \exp(\lambda \log(a) + (1-\lambda)\log(b)) \\ &\leq \lambda \exp(\log(a)) + (1-\lambda)\exp(\log(b)) \\ &= \lambda a + (1-\lambda)b. \end{aligned}$$

□

Lecture 30 — Friday, March 18, 2016

Theorem 30.1 (Hölder's inequality) Let (X, μ) be a measure space, and let $1 \leq p, q \leq \infty$ with $1/p + 1/q = 1$. If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$.

Proof. *Case* $p = 1, q = \infty$: then $|fg| \leq |f| \cdot \|g\|_\infty$ almost everywhere, and hence

$$\|fg\|_1 \leq \int |f| \cdot \|g\|_\infty \, \mathrm{d}\mu = \|f\|_1 \|g\|_\infty.$$

The case $p = \infty$, $q = 1$ is identical.

Case $1 < p < \infty$: If either norm is zero, then the corresponding function is zero almost everywhere and the result is immediate. Otherwise, both norms are finite by hypothesis, so we may replace f by $f/\|f\|_p$ and g by $g/\|g\|_q$ and assume $\|f\|_p = \|g\|_q = 1$. By Young's inequality with $\lambda = 1/p$,

$$|fg| = (|f|^p)^{1/p}(|g|^q)^{1/q} \leq \frac{1}{p}|f|^p + \frac{1}{q}|g|^q.$$

Integrating gives

$$\int |fg| \, \mathrm{d}\mu \leq \frac{1}{p}\int |f|^p + \frac{1}{q}\int |g|^q = \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_p \|g\|_q.$$

□

Theorem 30.2 (Minkowski's inequality) For $1 \leq p \leq \infty$, and $f, g : X \to \mathbb{C}$ measurable, $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

Proof. When $p = \infty$, $|f+g| \leq \|f\|_\infty + \|g\|_\infty$ almost everywhere, so $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$. When $p = 1$,

$$\|f+g\|_1 = \int |f+g| \leq \int (|f| + |g|) = \|f\|_1 + \|g\|_1.$$

When $1 < p < \infty$, the result is immediate if either $\|f\|_p$ or $\|g\|_p$ is infinite. Otherwise $f, g \in L^p$, and the elementary bound

$$|f+g|^p \leq 2^{p-1}(|f|^p + |g|^p)$$

shows that $f+g \in L^p$. If $\|f+g\|_p = 0$ there is nothing to prove. Otherwise,

$$|f+g|^p = |f+g| \cdot |f+g|^{p-1} \leq |f| \cdot |f+g|^{p-1} + |g| \cdot |f+g|^{p-1}.$$

Thus

$$\|f+g\|_p^p \leq \int |f| \cdot |f+g|^{p-1} + \int |g| \cdot |f+g|^{p-1}.$$

By Hölder's inequality, the right-hand side is bounded above by

$$\begin{aligned} \|f\|_p \||f+g|^{p-1}\|_q + \|g\|_p \||f+g|^{p-1}\|_q &= (\|f\|_p + \|g\|_p)\left(\int |f+g|^{(p-1)q}\right)^{1/q} \\ &= (\|f\|_p + \|g\|_p)\left(\int |f+g|^p\right)^{1/q} \\ &= (\|f\|_p + \|g\|_p)\|f+g\|_p^{p/q}. \end{aligned}$$

Dividing by $\|f+g\|_p^{p/q}$ gives $\|f+g\|_p \leq \|f\|_p + \|g\|_p$. □

With the triangle inequality finally established, we have the following corollary.

Corollary 30.3 L^p is a normed space.

We will show later that L^p is a Banach space.

Proposition 30.4 Let (X, μ) be a finite measure space. If $1 \le p < \infty$ and $1/p + 1/q = 1$, then the function $\ell : L^q \to (L^p)^*$ given by

$$\ell(f)(g) = \int fg \, \mathrm{d}\mu$$

for $f \in L^q$ and $g \in L^p$ is a linear injection satisfying $\|\ell(f)\| \le \|f\|_q$. Moreover, every $\omega \in (L^p)^*$ induces a complex measure

$$\nu(E) = \omega(\mathbb{1}_E).$$

Proof. We first show that $\ell(f) \in (L^p)^*$ for every $f \in L^q$. If $g \in L^p$, then $fg \in L^1$ by Hölder's inequality, so $\ell(f)(g)$ is defined. Moreover, for each $g \in L^p$,

$$|\ell(f)(g)| \le \int |fg| \, \mathrm{d}\mu \le \|f\|_q \|g\|_p.$$

Thus $\|\ell(f)\| \le \|f\|_q < \infty$, so $\ell(f) \in (L^p)^*$.

The map ℓ is linear. Indeed, if $f_1, f_2 \in L^q$ and $\lambda \in \mathbb{C}$, then for every $g \in L^p$,

$$\begin{aligned} \ell(\lambda f_1 + f_2)(g) &= \int (\lambda f_1 + f_2) g \, \mathrm{d}\mu \\ &= \lambda \int f_1 g \, \mathrm{d}\mu + \int f_2 g \, \mathrm{d}\mu \\ &= \lambda \ell(f_1)(g) + \ell(f_2)(g) \\ &= (\lambda \ell(f_1) + \ell(f_2))(g). \end{aligned}$$

Hence $\ell(\lambda f_1 + f_2) = \lambda \ell(f_1) + \ell(f_2)$.

We next show that ℓ is injective. If $f \in L^q$ and $\ell(f) = 0$, then for each measurable $E \subseteq X$, the function $\mathbb{1}_E$ belongs to L^p because (X, μ) is finite. Therefore,

$$0 = \ell(f)(\mathbb{1}_E) = \int \mathbb{1}_E f \, \mathrm{d}\mu = \int_E f \, \mathrm{d}\mu.$$

Thus

$$\int_E f \, \mathrm{d}\mu = 0$$

for every measurable $E \subseteq X$, so $f = 0$ almost everywhere and $\ker \ell = \{0\}$.

It remains to construct the asserted measure. Fix $\omega \in (L^p)^*$. For each measurable $E \subseteq X$, define

$$\nu(E) = \omega(\mathbb{1}_E).$$

We first verify that ν is a complex measure. If $E, F \subseteq X$ are disjoint measurable sets, then

$$\nu(E \cup F) = \omega(\mathbb{1}_{E \cup F}) = \omega(\mathbb{1}_E + \mathbb{1}_F) = \omega(\mathbb{1}_E) + \omega(\mathbb{1}_F) = \nu(E) + \nu(F)$$

so ν is finitely additive. If $E_1 \supseteq E_2 \supseteq \cdots$ are measurable and $E = \bigcap_{n=1}^{\infty} E_n$, then $\mu(E_n \setminus E) \to 0$ because $\mu(X) < \infty$. Also $\mathbb{1}_{E_n} - \mathbb{1}_E \to 0$ pointwise and $|\mathbb{1}_{E_n} - \mathbb{1}_E| \leq 1$. By the dominated convergence theorem,

$$\|\mathbb{1}_{E_n} - \mathbb{1}_E\|_p = \left(\int |\mathbb{1}_{E_n} - \mathbb{1}_E|^p \,\mathrm{d}\mu\right)^{1/p} \xrightarrow{n\to\infty} 0.$$

Thus $\mathbb{1}_{E_n} \to \mathbb{1}_E$ in L^p. Since ω is continuous, $\omega(\mathbb{1}_{E_n}) \to \omega(\mathbb{1}_E)$, so $\nu(E_n) \to \nu(E)$. Finite additivity together with continuity from above proves that ν is a complex measure. □

Lecture 31 — Tuesday, March 22, 2016

Fact 31.1 If $1 \leq p, r < \infty$, then

$$\||f|^r\|_p = \|f\|_{pr}^r.$$

Proof.

$$\||f|^r\|_p = \left(\int |f|^{rp} \,\mathrm{d}\mu\right)^{1/p} = \left(\int |f|^{pr} \,\mathrm{d}\mu\right)^{1/p} = \|f\|_{pr}^r.$$

□

Proposition 31.2 If (X, μ) is a finite measure space and $1 \leq p \leq q \leq \infty$, then $L^q \subseteq L^p$. In fact, for $f \in L^q$, we have

$$\|f\|_p \leq \|f\|_q \mu(X)^{1/p - 1/q}.$$

Proof. If $p = q$, the assertion is immediate. Suppose $p < q < \infty$ and $f \in L^q$. Note that

$$\frac{p}{q} + \frac{q-p}{q} = 1,$$

so

$$\|f\|_p^p = \int |f|^p \cdot 1 = \int (|f|^p \cdot 1) \leq \||f|^p\|_{q/p} \|1\|_{q/(q-p)}$$

by Hölder's inequality. The product on the right is

$$\left(\int |f|^q \, \mathrm{d}\mu \right)^{p/q} \mu(X)^{(q-p)/q} = \|f\|_q^p \mu(X)^{1-p/q}.$$

Thus $\|f\|_p \leq \|f\|_q \mu(X)^{1/p-1/q}$. When $q = \infty$,

$$\|f\|_p = \left(\int |f|^p \, \mathrm{d}\mu \right)^{1/p} \leq \left(\int \|f\|_\infty^p \, \mathrm{d}\mu \right)^{1/p} = \|f\|_\infty \mu(X)^{1/p}.$$

□

Proposition 31.3 If (X, μ) is a measure space and $1 \leq p < \infty$, then

$$\operatorname{span}\{\mathbb{1}_E : E \subseteq X \text{ measurable and } \mu(E) < \infty\}$$

is dense in L^p. When $p = \infty$,

$$\operatorname{span}\{\mathbb{1}_E : E \subseteq X \text{ measurable}\}$$

is dense in L^∞.

Proof. If $p = \infty$, the simple approximation theorem gives simple functions s_n with

$$\|f - s_n\|_\infty \to 0$$

for every $f \in L^\infty$. Each simple function belongs to the span of characteristic functions of measurable sets, so the claim follows.

Now suppose $1 \leq p < \infty$ and let $f \in L^p$. There exist simple functions f_n such that $f_n \to f$ almost everywhere and

$$|f_n| \leq |f|$$

for all n. Each f_n belongs to L^p. Write it in canonical form as

$$\sum_{i=1}^{N_n} c_{n,i} \mathbb{1}_{E_{n,i}}.$$

Thus the sets $E_{n,i}$ are pairwise disjoint and the coefficients $c_{n,i}$ are distinct and nonzero. In particular,

$$|c_{n,i}|^p \mathbb{1}_{E_{n,i}} \leq |f_n|^p,$$

so

$$|c_{n,i}|^p \mu(E_{n,i}) \leq \int |f_n|^p \, \mathrm{d}\mu < \infty.$$

Therefore $\mu(E_{n,i}) < \infty$, and each f_n lies in the stated span.

Finally,

$$|f - f_n|^p \to 0$$

almost everywhere, and

$$|f - f_n|^p \leq (|f| + |f_n|)^p \leq (2|f|)^p = 2^p |f|^p \in L^1.$$

By the dominated convergence theorem, $\|f - f_n\|_p \to 0$. Hence the stated span is dense in L^p. $\square$

Theorem 31.4 If (X, μ) is a finite measure space and $1 \leq p < \infty$ and $1/p + 1/q = 1$, then $(L^p)^* \cong L^q$.

Proof. Recall that in Proposition 30.4, we had $\ell : L^q \to (L^p)^*$ defined by

$$(\ell(f))(g) = \int fg$$

for $f \in L^q$ and $g \in L^p$. We already know that ℓ is linear and injective, and that $\|\ell(f)\| \leq \|f\|_q$ for all $f \in L^q$. It remains to prove surjectivity and the reverse inequality.

Let $\omega \in (L^p)^*$. For each measurable set $E \subseteq X$, define

$$\nu(E) = \omega(\mathbb{1}_E).$$

We already verified that ν is a measure. If $\mu(E) = 0$, then $\mathbb{1}_E = 0$ in $L^p(X, \mu)$, so $\nu(E) = \omega(\mathbb{1}_E) =$

0. Thus $\nu \ll \mu$. By the Radon–Nikodym theorem, there exists a measurable function f such that

$$\mathrm{d}\nu = f\,\mathrm{d}\mu.$$

Define $\alpha : X \to \mathbb{C}$ by

$$\alpha(x) = \begin{cases} \dfrac{\overline{f(x)}}{|f(x)|}, & f(x) \neq 0 \\ 0, & f(x) = 0 \end{cases}.$$

Then $\alpha f = |f|$. If h is a simple function, then

$$\omega(h) = \int h\,\mathrm{d}\nu = \int hf\,\mathrm{d}\mu.$$

By density of simple functions in L^∞ and continuity of both sides with respect to the L^∞ norm on a finite measure space, it follows that

$$\omega(h) = \int hf\,\mathrm{d}\mu$$

for every $h \in L^\infty$.

We now show that $f \in L^q$. First suppose $p = 1$, so $q = \infty$. For every measurable set E, take $h = \alpha \mathbb{1}_E$. Then

$$\int_E |f|\,\mathrm{d}\mu = |\omega(h)| \leq \|\omega\|\,\mu(E).$$

If $\{|f| > \|\omega\| + \varepsilon\}$ had positive measure for some $\varepsilon > 0$, this inequality would be impossible. Hence

$$\|f\|_\infty \leq \|\omega\|.$$

Now suppose $1 < p < \infty$, so $q < \infty$. Set

$$E_n = \{x \in X : |f(x)| \leq n\} \qquad \text{and} \qquad g_n = \mathbb{1}_{E_n}\alpha|f|^{q-1}.$$

Since $|g_n| \leq n^{q-1}$ and $\mu(X) < \infty$, we have $g_n \in L^\infty \subseteq L^p$. Moreover,

$$\int_{E_n} |f|^q\,\mathrm{d}\mu = |\omega(g_n)| \leq \|\omega\|\,\|g_n\|_p = \|\omega\| \left(\int_{E_n} |f|^{(q-1)p}\,\mathrm{d}\mu\right)^{1/p} = \|\omega\| \left(\int_{E_n} |f|^q\,\mathrm{d}\mu\right)^{1/p},$$

because $(q-1)p = q$. Hence

$$\|f\mathbb{1}_{E_n}\|_q \leq \|\omega\|$$

for every n. Letting $n \to \infty$ and using the monotone convergence theorem, we obtain

$$\|f\|_q \leq \|\omega\| < \infty.$$

Thus $f \in L^q$.

If $E \subseteq X$ is measurable with $\mu(E) < \infty$, then

$$\ell(f)(\mathbb{1}_E) = \int \mathbb{1}_E f = \nu(E) = \omega(\mathbb{1}_E).$$

Since

$$\operatorname{span}\{\mathbb{1}_E : \mu(E) < \infty\}$$

is dense in L^p and both $\ell(f)$ and ω are continuous linear functionals, it follows that $\ell(f) = \omega$ on all of L^p. In particular, ℓ is surjective. Combining this with the previously established inequality $\|\ell(f)\| \leq \|f\|_q$ and the estimate $\|f\|_q \leq \|\omega\| = \|\ell(f)\|$, we conclude that

$$\|\ell(f)\| = \|f\|_q$$

for all $f \in L^q$. □

Remark 31.5 The same representation $(L^p)^* \cong L^q$ holds on σ-finite measure spaces for $1 \leq p < \infty$. Beyond the σ-finite setting, duality statements require additional structural hypotheses such as localizability. It is rarely true that $(L^\infty)^* \cong L^1$.

Theorem 31.6 There is no normed vector space V such that $V^* \cong L^1([0,1], m)$.

The proof requires the Krein–Milman theorem and the Banach–Alaoglu theorem from PMATH 453, so we will skip it.

Example 31.7 Let $X = \{x_0\}$, with $\mu(\emptyset) = 0$ and $\mu(\{x_0\}) = \infty$. Then $L^1(X, \mu) = \{0\}$, but $L^\infty(X, \mu) = \mathbb{C}$. This example shows why hypotheses such as semifiniteness matter in duality statements. The problems are paramount.

Lecture 32 — Thursday, March 24, 2016

Riesz Representation and Regularity

We need a few topological facts. Throughout this discussion, let X be a metric space.

Definition 32.1 If $A, B \subseteq X$ are nonempty, define

$$d(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\}.$$

Also, for $x \in X$, write $d(x, A) = d(\{x\}, A)$.

Lemma 32.2 If $A \subseteq X$ is nonempty, the function $X \to [0, \infty)$, $x \mapsto d(x, A)$ is continuous.

Proof. If $(x_n)_{n=1}^\infty \subseteq X$ and $x_n \to x$, for $a \in A$ and $n \geq 1$, $d(x_n, A) \leq d(x_n, a) \leq d(x_n, x) + d(x, a) \to d(x, a)$ as $x_n \to x$. Therefore $\limsup_{n\to\infty} d(x_n, A) \leq d(x, a)$. For $n \geq 1$, choose $a_n \in A$ with $d(x_n, a_n) \leq d(x_n, A) + 1/n$. Then

$$d(x, A) \leq d(x, a_n) \leq d(x, x_n) + d(x_n, a_n) \leq d(x, x_n) + d(x_n, A) + \frac{1}{n}.$$

Thus $d(x, A) \leq \liminf_{n\to\infty} d(x_n, A)$. Together with the reverse inequality for the limit superior, this gives $d(x_n, A) \to d(x, A)$. □

Lemma 32.3 If $A, B \subseteq X$ are closed, A is compact, and $A \cap B = \emptyset$, then $d(A, B) > 0$.

Proof. Fix $a_n \in A$ and $b_n \in B$ with $d(a_n, b_n) \to d(A, B)$. Since A is compact, there is a subsequence $(a_{n_k})_{k=1}^\infty$ converging to some $a \in A$. Now, $d(a, b_{n_k}) \leq d(a, a_{n_k}) + d(a_{n_k}, b_{n_k})$. If $d(A, B) = 0$, then letting $k \to \infty$ gives $d(a, b_{n_k}) \to 0$. Thus $b_{n_k} \to a$, and $a \in B$ because B is closed. This contradicts $A \cap B = \emptyset$. □

Remark 32.4 Compactness is necessary in the preceding lemma. For instance, the closed sets $A = \{1/n : n \in \mathbb{N}\}$ and $B = \{-1/n : n \in \mathbb{N}\}$ in the subspace $A \cup B \subseteq \mathbb{R}$ are disjoint but have distance 0.

Theorem 32.5 (Urysohn's lemma) If $A, B \subseteq X$ are closed and $A \cap B = \emptyset$, then there exists a continuous function $f : X \to [0, 1]$ such that $f(a) = 0$ for every $a \in A$ and $f(b) = 1$ for every $b \in B$.

Proof. If $A = \emptyset$, take $f \equiv 1$; if $B = \emptyset$, take $f \equiv 0$. Assume henceforth that both sets are nonempty. First note that if $x \in X$ and $d(x, A) = 0$, then $x \in \overline{A} = A$. Similarly, if $d(x, B) = 0$, then $x \in B$. Now define

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

The distance functions in this expression are continuous, and the denominator is never 0 because $A \cap B = \emptyset$. Thus f is continuous; moreover, $f(a) = 0$ for every $a \in A$ and $f(b) = 1$ for every $b \in B$. □

Definition 32.6 If $f : X \to \mathbb{C}$ is continuous, define the **support of f** by

$$\operatorname{supp}(f) = \overline{\{x \in X : f(x) \neq 0\}}.$$

Some people call this the **closed support of f**, but we shun those people.

Proposition 32.7 If X is a metric space and $K \subseteq V \subseteq X$, where K is compact and V is open, then there exists a continuous function $f : X \to [0, 1]$ such that $f|_K = 1$ and $\operatorname{supp}(f) \subseteq V$.

Proof. If $K = \emptyset$, take $f \equiv 0$. We may therefore assume $K \neq \emptyset$. If $K = V$, then K is both compact and open, and we may take $f = \mathbb{1}_V$. If $V = X$, we may take $f \equiv 1$. Otherwise, Lemma 32.3 gives $d(K, V^c) > 0$. Set $\varepsilon = \frac{1}{2} d(K, V^c)$ and define

$$U = \{x \in X : d(x, K) < \varepsilon\} = \bigcup_{x \in K} B_\varepsilon(x).$$

Then U is open and $K \subseteq U$. If $x \in \overline{U}$, choose $x_n \in U$ with $x_n \to x$. By continuity of $y \mapsto d(y, K)$, we get $d(x, K) \leq \varepsilon$. If $x \in V^c$, then $d(x, K) \geq d(K, V^c) = 2\varepsilon$, a contradiction. Hence $\overline{U} \subseteq V$.

The closed sets K and U^c are disjoint, so Urysohn's lemma gives a continuous function $f : X \to [0,1]$ such that $f|_K = 1$ and $f|_{U^c} = 0$. Then $\{x : f(x) \neq 0\} \subseteq U$, so $\operatorname{supp}(f) \subseteq \overline{U} \subseteq V$. $\square$

Proposition 32.8 Suppose X is a metric space, $K \subseteq X$ is compact, $V_1, \ldots, V_n \subseteq X$ are open, and $K \subseteq V_1 \cup \cdots \cup V_n$. Then there are continuous functions $h_j : X \to [0,1]$ such that $\operatorname{supp}(h_j) \subseteq V_j$ for $j = 1, \ldots, n$ and

$$\sum_{j=1}^{n} h_j(x) = 1$$

for every $x \in K$. The family $\{h_j\}_{j=1}^n$ is called a **partition of unity** subordinate to the cover $(V_j)_{j=1}^n$.

Proof. For each $x \in K$, choose an index $i(x) \in \{1, \ldots, n\}$ such that $x \in V_{i(x)}$. By Proposition 32.7, choose a continuous function $f_x : X \to [0,1]$ such that $f_x(x) = 1$ and $\operatorname{supp}(f_x) \subseteq V_{i(x)}$. The sets $\{y \in X : f_x(y) \neq 0\}$ cover K, so compactness gives a finite set $F \subseteq K$ such that

$$K \subseteq \bigcup_{x \in F} \{y \in X : f_x(y) \neq 0\}.$$

Set

$$g_j = \sum_{x \in F, i(x) = j} f_x$$

for $j = 1, \ldots, n$. Then $\operatorname{supp}(g_j) \subseteq V_j$ and $g_1(x) + \cdots + g_n(x) \neq 0$ for every $x \in K$.

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Proof continued. Set

$$L = \{x \in X : \sum_{j=1}^{n} g_j(x) = 0\} \subseteq X,$$

which is closed and disjoint from K. By Urysohn's lemma, there exists a continuous function $q : X \to [0,1]$ with $q|_K = 0$ and $q|_L = 1$. Then

$$g_1(x) + \cdots + g_n(x) + q(x) \neq 0$$

for every $x \in X$. Define

$$h_j(x) = \frac{g_j(x)}{g_1(x) + \cdots + g_n(x) + q(x)}, \quad j = 1, \ldots, n.$$

Then each h_j is continuous, $0 \leq h_j \leq 1$, and $\operatorname{supp}(h_j) \subseteq \operatorname{supp}(g_j) \subseteq V_j$. If $x \in K$, then $q(x) = 0$, so

$$\sum_{j=1}^{n} h_j(x) = \frac{g_1(x) + \cdots + g_n(x)}{g_1(x) + \cdots + g_n(x)} = 1.$$

□

Definition 33.1 A metric space X is **locally compact** if, for every point $x \in X$, there exists $\varepsilon > 0$ such that $\overline{B_\varepsilon(x)}$ is compact.

Example 33.2 The following are basic examples:

1. Every compact metric space is locally compact.
2. Every open or closed subset of $\mathbb{R}^n$ is locally compact.
3. A set $F \cap G$ where $F \subseteq \mathbb{R}^n$ is closed and $G \subseteq \mathbb{R}^n$ is open is locally compact.
4. The subspace $\mathbb{Q} \subseteq \mathbb{R}$ is not locally compact, since $\overline{B_\varepsilon(q)}$ contains irrational numbers for every $q \in \mathbb{Q}$ and $\varepsilon > 0$.

Definition 33.3 For a locally compact space X, define

$$C_c(X) = \{f : X \to \mathbb{C} : f \text{ is continuous and } \operatorname{supp}(f) \text{ is compact}\}.$$

Functions in $C_c(X)$ are called **continuous functions with compact support**.

Definition 33.4 Let μ be a positive Borel measure on a locally compact metric space X, and let $E \subseteq X$ be Borel. We say that μ is **inner regular on E** if

$$\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ is compact}\}.$$

We say that μ is **outer regular on E** if

$$\mu(E) = \inf\{\mu(U) : E \subseteq U,\ U \text{ is open}\}.$$

Definition 33.5 A **Radon measure** on X is a Borel measure μ such that the following hold:

1. We have $\mu(K) < \infty$ for every compact set $K \subseteq X$.
2. The measure μ is inner regular on every open set.
3. The measure μ is outer regular on every Borel set.

Warning This definition is not standard. Usually, a Radon measure is defined to be a Borel measure that is inner regular on every Borel set and outer regular on every open set. However, the definition given here is more convenient for our purposes, and we will show that it is equivalent to the usual definition when X is separable.

Definition 33.6 Let $\omega : C_c(X) \to \mathbb{C}$ be a linear functional. We say that ω is **positive** if $\omega(f) \geq 0$ for every $f \in C_c(X)$ with $f \geq 0$.

Theorem 33.7 (Riesz representation theorem) Let $\omega : C_c(X) \to \mathbb{C}$ be a positive linear functional. There is a unique Radon measure μ on X such that

$$\omega(f) = \int f \, \mathrm{d}\mu$$

for all $f \in C_c(X)$.

The correspondence in Figure 13 is the point of the theorem: a positive functional is not merely analogous to integration against a measure; it uniquely is integration against a Radon measure.

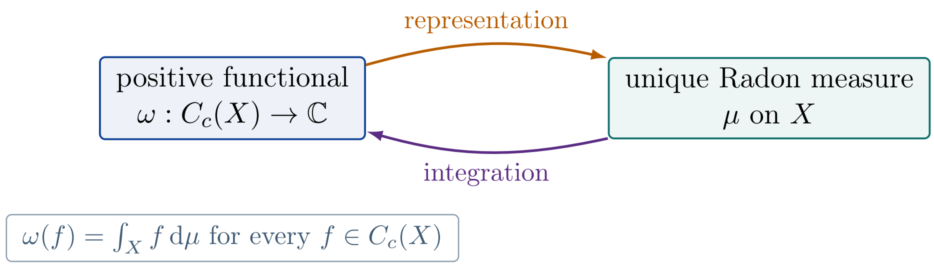

Figure 13

Remark 33.8 If μ is any Borel measure on X with $\mu(K) < \infty$ for every compact $K \subseteq X$, then the functional $C_c(X) \to \mathbb{C}$ given by

$$f \mapsto \int f \, \mathrm{d}\mu$$

is a positive linear functional. Indeed, if $f \in C_c(X)$, then there exists a compact set $K \subseteq X$ such that $f = 0$ on $X \setminus K$. Since f is continuous, it is bounded on K and hence on X. Thus there exists $r \geq 0$ such that $|f(x)| \leq r$ for every $x \in X$. Therefore,

$$\int |f| \, \mathrm{d}\mu \leq \int_K r \, \mathrm{d}\mu = r \cdot \mu(K) < \infty.$$

Hence $f \in L^1(X, \mu)$, so $\int f \, \mathrm{d}\mu$ is defined.

Proof of Theorem 33.7. This proof is somewhat long.

For every set $E \subseteq X$, define

$$\mu^*(E) = \inf \left\{ \sup\{\omega(f) : f \prec U\} : E \subseteq U, \ U \text{ open} \right\}.$$

If $U \subseteq X$ is open, we write $f \prec U$ to mean that $f \in C_c(X)$, $0 \leq f \leq 1$, and $\operatorname{supp}(f) \subseteq U$. The following facts follow immediately from the definition:

1. If $U \subseteq X$ is open, then $\mu^*(U) = \sup_{f \prec U} \omega(f)$.
2. We have $\mu^*(\emptyset) = 0$.
3. If $E \subseteq F$, then $\mu^*(E) \leq \mu^*(F)$.

We now verify that μ^* is an outer measure. Let $(E_n)_{n=1}^{\infty}$ be a sequence of subsets of X, and set

$$E = \bigcup_{n=1}^{\infty} E_n.$$

We must show that

$$\mu^*(E) \le \sum_{n=1}^{\infty} \mu^*(E_n).$$

If $\sum_{n=1}^{\infty} \mu^*(E_n) = \infty$, there is nothing to prove. Thus we may assume that this series converges. Fix $\varepsilon > 0$. For each n, choose an open set $U_n \supseteq E_n$ such that

$$\mu^*(U_n) \le \mu^*(E_n) + 2^{-n}\varepsilon.$$

Set

$$U = \bigcup_{n=1}^{\infty} U_n.$$

Then $E \subseteq U$, so by monotonicity,

$$\mu^*(E) \le \mu^*(U).$$

Fix $f \prec U$. Since $\operatorname{supp}(f)$ is compact and contained in $\bigcup_{n=1}^{\infty} U_n$, there exists $N \in \mathbb{N}$ such that

$$\operatorname{supp}(f) \subseteq \bigcup_{n=1}^{N} U_n.$$

By the partition-of-unity result proved earlier, there exist continuous functions $g_1, \ldots, g_N : X \to [0,1]$ such that

$$\operatorname{supp}(g_n) \subseteq U_n$$

for each n, and

$$\sum_{n=1}^{N} g_n(x) = 1$$

for all $x \in \operatorname{supp}(f)$. Hence

$$f = \sum_{n=1}^{N} f g_n.$$

Moreover, $0 \le f g_n \le 1$ and $\operatorname{supp}(f g_n) \subseteq U_n$, so $f g_n \prec U_n$ for each n. Therefore,

$$\omega(f) = \sum_{n=1}^{N} \omega(f g_n) \le \sum_{n=1}^{N} \mu^*(U_n) \le \sum_{n=1}^{\infty} \big(\mu^*(E_n) + 2^{-n}\varepsilon\big) = \sum_{n=1}^{\infty} \mu^*(E_n) + \varepsilon.$$

Taking the supremum over all $f \prec U$ gives

$$\mu^*(U) \leq \sum_{n=1}^{\infty} \mu^*(E_n) + \varepsilon.$$

Hence

$$\mu^*(E) \leq \sum_{n=1}^{\infty} \mu^*(E_n) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\mu^*(E) \leq \sum_{n=1}^{\infty} \mu^*(E_n).$$

Therefore μ^* is an outer measure.

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Proof continued. We now show that every open set is μ^*-measurable. Let $U \subseteq X$ be open, and let $E \subseteq X$ satisfy $\mu^*(E) < \infty$. It is enough to prove

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U).$$

Fix $\varepsilon > 0$. Choose an open set $V \supseteq E$ such that

$$\mu^*(V) \leq \mu^*(E) + \varepsilon.$$

Choose $f \prec U \cap V$ and $g \prec V \setminus \operatorname{supp}(f)$ such that

$$\mu^*(U \cap V) \leq \omega(f) + \varepsilon \qquad \text{and} \qquad \mu^*(V \setminus \operatorname{supp}(f)) \leq \omega(g) + \varepsilon.$$

Since $\operatorname{supp}(f) \subseteq U \cap V$, we have $E \setminus U \subseteq V \setminus \operatorname{supp}(f)$, and also $E \cap U \subseteq U \cap V$. Moreover, $f + g \prec V$. Hence

$$\begin{aligned}
\mu^*(E \cap U) + \mu^*(E \setminus U) &\leq \mu^*(U \cap V) + \mu^*(V \setminus \operatorname{supp}(f)) \\
&\leq \omega(f) + \varepsilon + \omega(g) + \varepsilon \\
&= \omega(f + g) + 2\varepsilon
\end{aligned}$$

$$\begin{aligned}&\le \mu^*(V) + 2\varepsilon\\ &\le \mu^*(E) + 3\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0^+$ proves the desired inequality. By Carathéodory's theorem, every Borel set is μ^*-measurable, and

$$\mu = \mu^*|_{\mathcal{B}_X}$$

is a Borel measure. By construction, for every Borel set $E \subseteq X$,

$$\mu(E) = \inf\{\mu(U) : E \subseteq U,\ U \text{ open}\},$$

so μ is outer regular on Borel sets.

Next, we prove the compact set formula

$$\mu(K) = \inf\{\omega(f) : f \in C_c(X), \mathbb{1}_K \le f \le 1\}$$

for every compact $K \subseteq X$. The condition $\mathbb{1}_K \le f \le 1$ means that $0 \le f \le 1$ and $f|_K = 1$.

Fix $\varepsilon > 0$. By outer regularity, choose an open set $U \supseteq K$ with $\mu(U) \le \mu(K) + \varepsilon$. Since X is locally compact and K is compact, there exists an open set V such that

$$K \subseteq V \subseteq \overline{V} \subseteq U$$

and $\overline{V}$ is compact. By Proposition 32.7, choose $f : X \to [0,1]$ continuous such that $f|_K = 1$ and $\operatorname{supp}(f) \subseteq V$. Then $f \in C_c(X)$, $\mathbb{1}_K \le f \le 1$, and $f \prec U$. Therefore

$$\omega(f) \le \mu(U) \le \mu(K) + \varepsilon.$$

It follows that

$$\inf\{\omega(f) : f \in C_c(X), \mathbb{1}_K \le f \le 1\} \le \mu(K).$$

Conversely, suppose $f \in C_c(X)$ and $\mathbb{1}_K \le f \le 1$. For $0 < \varepsilon < 1$, let

$$U_\varepsilon = \{x \in X : f(x) > 1 - \varepsilon\}.$$

Then U_ε is open and contains K. If $g \prec U_\varepsilon$, then $g \le (1-\varepsilon)^{-1} f$. Positivity of ω gives

$$\omega(g) \le \frac{1}{1-\varepsilon}\omega(f).$$

Taking the supremum over all $g \prec U_\varepsilon$ gives

$$\mu(U_\varepsilon) \leq \frac{1}{1-\varepsilon}\omega(f).$$

Since $K \subseteq U_\varepsilon$, we have

$$\mu(K) \leq \frac{1}{1-\varepsilon}\omega(f).$$

Letting $\varepsilon \to 0^+$ gives $\mu(K) \leq \omega(f)$. Therefore,

$$\inf\{\omega(f) : f \in C_c(X), \mathbb{1}_K \leq f \leq 1\} \geq \mu(K).$$

Combining the two inequalities yields

$$\inf\{\omega(f) : f \in C_c(X), \mathbb{1}_K \leq f \leq 1\} = \mu(K)$$

for every compact $K \subseteq X$.

We now verify that μ is a Radon measure. We already know that μ is a Borel measure and is outer regular on Borel sets. If $K \subseteq X$ is compact, then the compact set formula and the existence of a function $f \in C_c(X)$ with $\mathbb{1}_K \leq f \leq 1$ imply $\mu(K) \leq \omega(f) < \infty$.

It remains to prove inner regularity on open sets. Let $U \subseteq X$ be open and let $\alpha < \mu(U)$. By (1), choose $f \prec U$ with $\omega(f) > \alpha$. Set $K = \operatorname{supp}(f)$, which is compact and contained in U. If $W \supseteq K$ is open, then $f \prec W$, so $\mu(W) \geq \omega(f) > \alpha$. By outer regularity of μ on K,

$$\mu(K) = \inf_{W \supseteq K,\ W \text{ open}} \mu(W) \geq \omega(f) > \alpha.$$

Therefore $\mu(U) = \sup\{\mu(K) : K \subseteq U,\ K \text{ compact}\}$, and μ is Radon.

Lecture 35 — Friday, April 01, 2016

Proof continued. Next, we prove that

$$\int f \, \mathrm{d}\mu = \omega(f)$$

for every $f \in C_c(X)$. By decomposing into positive and negative real and imaginary parts and then rescaling, it suffices to prove the identity when $0 \leq f \leq 1$.

Fix $n \in \mathbb{N}$. Set $K_0 = \operatorname{supp}(f)$ and, for $1 \leq j \leq n$, set

$$K_j = \{x \in X : f(x) \geq j/n\}.$$

Define

$$f_j(x) = \begin{cases} 0, & x \notin K_{j-1}, \\ f(x) - (j-1)/n, & x \in K_{j-1} \setminus K_j, \\ 1/n, & x \in K_j. \end{cases}$$

Then

$$\frac{1}{n}\sum_{j=1}^{n} \mathbb{1}_{K_j} \leq f \leq \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{K_j}.$$

Integrating gives the Riemann sum estimate

$$\frac{1}{n}\sum_{j=1}^{n} \mu(K_j) \leq \int f \,\mathrm{d}\mu \leq \frac{1}{n}\sum_{j=0}^{n-1} \mu(K_j). \tag{35.1}$$

We next prove the analogous estimate for $\omega(f)$. Since $nf_j \in C_c(X)$, $0 \leq nf_j \leq 1$, and $(nf_j)|_{K_j} = 1$, the compact set formula gives

$$\omega(nf_j) \geq \mu(K_j).$$

On the other hand, if U is open and contains K_{j-1}, then $\operatorname{supp}(f_j) \subseteq K_{j-1} \subseteq U$ and $0 \leq nf_j \leq 1$, so $\omega(nf_j) \leq \mu(U)$. By outer regularity on K_{j-1},

$$\omega(f_j) \leq \frac{1}{n}\mu(K_{j-1}).$$

Thus

$$\frac{1}{n}\mu(K_j) \leq \omega(f_j) \leq \frac{1}{n}\mu(K_{j-1}).$$

Summing over j yields

$$\frac{1}{n}\sum_{j=1}^{n} \mu(K_j) \leq \omega(f) \leq \frac{1}{n}\sum_{j=0}^{n-1} \mu(K_j). \tag{35.2}$$

Subtracting (35.2) from (35.1), we get

$$\frac{1}{n}(\sum_{j=1}^{n}\mu(K_j)-\sum_{j=0}^{n-1}\mu(K_j))\leq\int f\,\mathrm{d}\mu-\omega(f)\leq\frac{1}{n}\left(\sum_{j=0}^{n-1}\mu(K_j)-\sum_{j=1}^{n}\mu(K_j)\right)$$

where all terms are finite because μ is finite on compact sets. Hence

$$\frac{1}{n}(\mu(K_n)-\mu(K_0))\leq\int f\,\mathrm{d}\mu-\omega(f)\leq\frac{1}{n}(\mu(K_0)-\mu(K_n)).$$

Since $0\leq\mu(K_n)\leq\mu(K_0)$,

$$\left|\int f\,\mathrm{d}\mu-\omega(f)\right|\leq\frac{1}{n}\mu(\mathrm{supp}(f))\to 0$$

as $n\to\infty$. Hence

$$\int f\,\mathrm{d}\mu=\omega(f).$$

It remains to prove uniqueness. Suppose μ and ν are two Radon measures on X and

$$\int f\,\mathrm{d}\mu=\int f\,\mathrm{d}\nu$$

for every $f\in C_c(X)$. Let $U\subseteq X$ be open, and fix a compact set $K\subseteq U$. Choose $f\in C_c(X)$ such that $0\leq f\leq 1$, $f|_K=1$, and $\mathrm{supp}(f)\subseteq U$. Then $\mathbb{1}_K\leq f\leq\mathbb{1}_U$, so

$$\mu(K)\leq\int f\,\mathrm{d}\mu=\int f\,\mathrm{d}\nu\leq\nu(U).$$

Taking the supremum over compact $K\subseteq U$ and using inner regularity of μ gives $\mu(U)\leq\nu(U)$. By symmetry, $\mu(U)=\nu(U)$ for every open set U.

Now let $E\subseteq X$ be Borel. By outer regularity,

$$\mu(E)=\inf_{E\subseteq U}\mu(U)=\inf_{E\subseteq U}\nu(U)=\nu(E).$$

Thus $\mu=\nu$. □

Definition 35.1 A metric space X is **separable** if there exists a countable set $X_0\subseteq X$ such that $\overline{X_0}=X$.

Example 35.2 $\mathbb{R}^n$ is separable since $\overline{\mathbb{Q}^n} = \mathbb{R}^n$.

Proposition 35.3 If X is separable and $Y \subseteq X$, then Y is separable (with respect to the same metric).

Proof. Let $(x_n)_{n=1}^{\infty}$ be a dense sequence in X. Consider the family

$$\mathcal{U} = \{B_q(x_n) : n \in \mathbb{N},\ q \in \mathbb{Q}_{>0},\ B_q(x_n) \cap Y \neq \emptyset\}.$$

Since both $\mathbb{N}$ and $\mathbb{Q}_{>0}$ are countable, the family $\mathcal{U}$ is countable. Enumerate it as

$$\mathcal{U} = \{U_j : j \in \mathbb{N}\}.$$

For each j, choose a point $y_j \in U_j \cap Y$.

We claim that $\{y_j : j \in \mathbb{N}\}$ is dense in Y. Fix $y \in Y$ and $\varepsilon > 0$. Since (x_n) is dense in X, there exists n such that

$$d(x_n, y) < \frac{\varepsilon}{2}.$$

Choose $q \in \mathbb{Q}$ such that

$$\frac{\varepsilon}{2} < q < \varepsilon - d(x_n, y).$$

Then $q > 0$ and $y \in B_q(x_n)$, so $B_q(x_n) \cap Y \neq \emptyset$. Hence $B_q(x_n) = U_j$ for some j, and by construction $y_j \in U_j \cap Y$. Moreover,

$$d(y_j, y) \leq d(y_j, x_n) + d(x_n, y) < q + d(x_n, y) < \varepsilon.$$

Thus every point of Y is a limit of points from the countable set $\{y_j : j \in \mathbb{N}\}$, so Y is separable. □

Theorem 35.4 If X is a locally compact, separable metric space and μ is a Borel measure on X with $\mu(K) < \infty$ for every compact set $K \subseteq X$, then, for every Borel set $E \subseteq X$,

$$\mu(E) = \sup\{\mu(K) : K \subseteq E,\ K \text{ compact}\}$$

and

$$\mu(E) = \inf\{\mu(U) : E \subseteq U,\ U \text{ open}\}.$$

In particular, μ is a Radon measure.

Proposition 35.5 If X is separable and locally compact, then there exists a sequence of compact sets $(K_n)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} K_n = X$. The same conclusion holds for every open or closed subset of X.

Proof. Let $(x_n)_{n=1}^{\infty}$ be a dense sequence in X. For each $y \in X$, choose $\varepsilon_y > 0$ such that $\overline{B_{\varepsilon_y}(y)}$ is compact. Since (x_n) is dense, there exists $n_y \in \mathbb{N}$ with

$$d(x_{n_y}, y) < \frac{\varepsilon_y}{2}.$$

Choose $q_y \in \mathbb{Q}$ such that

$$d(x_{n_y}, y) < q_y < \frac{\varepsilon_y}{2}.$$

If $z \in \overline{B_{q_y}(x_{n_y})}$, then

$$d(y,z) \leq d(y, x_{n_y}) + d(x_{n_y}, z) \leq d(y, x_{n_y}) + q_y < \varepsilon_y.$$

Thus

$$\overline{B_{q_y}(x_{n_y})} \subseteq \overline{B_{\varepsilon_y}(y)},$$

so $\overline{B_{q_y}(x_{n_y})}$ is compact. Also, $y \in B_{q_y}(x_{n_y}) \subseteq \overline{B_{q_y}(x_{n_y})}$.

Therefore

$$X = \bigcup_{y \in X} \overline{B_{q_y}(x_{n_y})}.$$

The collection on the right is countable, because its centers are chosen from the countable set $\{x_n\}$ and its radii are chosen from the countable set $\mathbb{Q}_{>0}$. Enumerating these compact sets gives the desired sequence.

Every open or closed subset of X is again separable and locally compact. Applying the first part to that subspace proves the final assertion. $\square$

Proof of Theorem 35.4. For $f \in C_c(X)$, the support of f is compact, so f is integrable with respect to μ. Define the positive linear functional

$$\omega(f) = \int_X f \, \mathrm{d}\mu.$$

By Theorem 33.7, there is a Radon measure ν such that

$$\int_X f \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\nu$$

for every $f \in C_c(X)$.

We first show that μ and ν agree on open sets. Let $U \subseteq X$ be open. By Proposition 35.5, there are compact sets $K_n \subseteq U$ whose union is U; replacing them by finite unions, assume that $K_n \subseteq K_{n+1}$. Local compactness and compactness give open sets V_n such that

$$K_n \subseteq V_n \subseteq \overline{V_n} \subseteq U \qquad \text{and} \qquad \overline{V_n} \text{ is compact.}$$

Choose $f_n \in C_c(X)$ with $0 \leq f_n \leq 1$, $f_n = 1$ on K_n, and $\operatorname{supp}(f_n) \subseteq V_n$. Replacing f_n by $\max\{f_1, \ldots, f_n\}$ makes the sequence increasing while preserving compact support inside U and the identity $f_n = 1$ on K_n. Then $f_n \uparrow \mathbb{1}_U$, so monotone convergence gives

$$\mu(U) = \lim_n \int f_n \,\mathrm{d}\mu = \lim_n \int f_n \,\mathrm{d}\nu = \nu(U).$$

Since ν is outer regular, for every Borel set E we have

$$\mu(E) \leq \inf_{U \supseteq E,\ U \text{ open}} \mu(U) = \inf_{U \supseteq E,\ U \text{ open}} \nu(U) = \nu(E).$$

Choose increasing open sets W_n with compact closures and $\bigcup_n W_n = X$; such a cover follows from the compact cover above and local compactness. Both measures are finite on each W_n. Applying the preceding inequality to $W_n \setminus E$ and subtracting from $\mu(W_n) = \nu(W_n)$ gives

$$\mu(E \cap W_n) \geq \nu(E \cap W_n).$$

The reverse inequality was already proved, so equality holds. Continuity from below now yields $\mu(E) = \nu(E)$. The inner and outer regularity of ν therefore transfers to μ, which proves the theorem. $\square$

Lecture 36 — Monday, April 04, 2016

Corollary 36.1 If X is a locally compact, separable metric space, and μ is a Radon measure on X, then μ is σ-finite.

Proof. By Proposition 35.5, there are compact sets $K_n \subseteq X$ such that $X = \bigcup_{n=1}^{\infty} K_n$. Since μ is Radon, $\mu(K_n) < \infty$ for every n. □

Proposition 36.2 If X is a locally compact metric space, μ is a σ-finite Radon measure on X, and $E \subseteq X$ is Borel, then for every $\varepsilon > 0$, there exist an open set $U \supseteq E$ and a closed set $F \subseteq E$ such that

$$\mu(U \setminus E) < \varepsilon \qquad \text{and} \qquad \mu(E \setminus F) < \varepsilon.$$

In particular, $\mu(U \setminus F) < 2\varepsilon$.

Proof. Write $E = \bigsqcup_{n=1}^{\infty} E_n$ where each E_n is Borel and $\mu(E_n) < \infty$. By outer regularity,

$$\mu(E_n) = \inf\{\mu(U) : E_n \subseteq U,\ U \text{ open}\}.$$

Hence there exists an open set $U_n \supseteq E_n$ such that $\mu(U_n \setminus E_n) < 2^{-n}\varepsilon$. Let $U = \bigcup_{n=1}^{\infty} U_n$. Then U is open, $E \subseteq U$, and

$$\mu(U \setminus E) \leq \sum_{n=1}^{\infty} \mu(U_n \setminus E_n) < \varepsilon.$$

Applying the same open approximation result to E^c, choose an open set $V \supseteq E^c$ such that $\mu(V \setminus E^c) < \varepsilon$. Set $F = V^c$. Then F is closed, $F \subseteq E$, and

$$V \setminus E^c = E \setminus F.$$

Therefore $\mu(E \setminus F) < \varepsilon$. □

Remark 36.3 By Theorem 35.4, if X is a separable locally compact metric space and μ is a Borel measure on X satisfying $\mu(K) < \infty$ for every compact set $K \subseteq X$, then for every

Borel set $E \subseteq X$ we have

$$\mu(E) = \inf\{\mu(U) : E \subseteq U,\ U \text{ open}\} = \sup\{\mu(K) : K \subseteq E,\ K \text{ compact}\}.$$

In particular, μ is a Radon measure.

Corollary 36.4 If μ is a finite Borel measure on a compact metric space X, then μ is regular: for every Borel set $E \subseteq X$,

$$\mu(E) = \inf\{\mu(U) : E \subseteq U,\ U \text{ open}\}$$

and

$$\mu(E) = \sup\{\mu(K) : K \subseteq E,\ K \text{ compact}\}.$$

Proof. Suppose $f \in C_c(X)$. Then $K = \operatorname{supp}(f)$ is compact and $f = 0$ on $X \setminus K$. Therefore there exists $M > 0$ such that $|f(x)| \leq M$ for every $x \in X$. Hence

$$\int_X |f| \, \mathrm{d}\mu = \int_K |f| \, \mathrm{d}\mu \leq M\mu(K) < \infty.$$

Thus $f \in L^1(X, \mu)$ for every $f \in C_c(X)$. Define $\omega : C_c(X) \to \mathbb{C}$ by

$$\omega(f) = \int_X f \, \mathrm{d}\mu$$

and note that ω is a positive linear functional. By Theorem 33.7, there is a unique Radon measure ν on X such that

$$\omega(f) = \int f \, \mathrm{d}\nu,$$

for every $f \in C_c(X)$. Hence

$$\int f \, \mathrm{d}\mu = \int f \, \mathrm{d}\nu$$

for every $f \in C_c(X)$.

Fix an open set $U \subseteq X$ and write

$$U = \bigcup_{n=1}^{\infty} K_n$$

with K_n compact and $K_1 \subseteq K_2 \subseteq \cdots$. Since $K_n \subseteq U$, the Urysohn-type proposition gives $f_n \in C_c(X)$ such that $0 \leq f_n \leq 1$, $f_n|_{K_n} = 1$, and $\operatorname{supp}(f_n) \subseteq U$. Replacing f_n by $\max\{f_1, \ldots, f_n\}$, we may assume $f_{n-1} \leq f_n$ for every $n \geq 2$. Then $f_n \to \mathbb{1}_U$ pointwise. By the monotone convergence theorem,

$$\mu(U) = \int \mathbb{1}_U \, \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \, \mathrm{d}\nu = \int \mathbb{1}_U \, \mathrm{d}\nu = \nu(U).$$

Thus $\mu(U) = \nu(U)$ for every open $U \subseteq X$.

Fix a Borel set $E \subseteq X$ and $\varepsilon > 0$. Since ν is Radon, the preceding proposition gives an open set $U \supseteq E$ and a closed set $F \subseteq E$ such that $\nu(U \setminus F) < \varepsilon$. Since $U \setminus F$ is open, $\mu(U \setminus F) = \nu(U \setminus F) < \varepsilon$. Hence

$$\mu(E) \leq \mu(U) = \mu(U \setminus F) + \mu(F) \leq \varepsilon + \mu(E).$$

This proves

$$\mu(E) = \inf\{\mu(U) : E \subseteq U,\ U \text{ open}\}.$$

Since X is compact and F is closed, F is compact. Also,

$$\mu(E) \leq \mu(F) + \mu(E \setminus F) \leq \mu(F) + \varepsilon.$$

Therefore,

$$\mu(E) - \varepsilon \leq \mu(F) \leq \sup\{\mu(K) : K \subseteq E,\ K \text{ compact}\} \leq \mu(E).$$

Letting $\varepsilon \to 0^+$ proves inner regularity. □

Remark 36.5 The preceding proof also shows that the original measure μ agrees with the Radon measure ν obtained from the Riesz representation theorem.

Appendix: Lecture and Tutorial Index